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On modified Noor iterations for strongly pseudocontractive mappings[1]

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Abstract

In this paper, we analyze a three-step iterative scheme for three strongly pseudocontractive mappings in a uniformly smooth Banach space. Our results can be viewed as an extension of three-step and two-step iterative schemes of Glowinski and Le Tallec [3], Noor [12-15] and Ishikawa [7], Liu [10] and Xu [20].

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1 Introduction

Form now onward, we assume that E is a real uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \text{ and } ||f^*|| = ||x||\},$$

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where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E is uniformly smooth, then J is single-valued and is uniformly continuous on bounded subsets of E. We shall denote the single-valued duality map by j.

Definition 1. *A map $T : E \to E$ is called strongly accretive if there exists a constant $0 < k < 1$ such that, for each $x, y \in E$, there is a $j(x-y) \in J(x-y)$ satisfying*

$$\langle Tx - Ty, j(x-y) \rangle \geq k||x-y||^2. \tag{1}$$

Definition 2. *An operator T with domain $D(T)$ and range $R(T)$ in E is called strongly pseudocontractive if for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a constant $0 < k < 1$ such that*

$$\langle Tx - Ty, j(x-y) \rangle \leq (1-k)||x-y||^2. \tag{2}$$

It is known that T is strongly pseudocontractive if and only if $(I - T)$ is strongly accretive.

The concept of accretive mapping was at first introduced independently by Browder [2] and Kato [9] in 1967. An early fundamental result in the theory of accretive mapping, due to Browder, states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0, \qquad u(0) = u_0$$

is solvable if T is locally Lipschitzian and accretive on E.

In recent year, much attention has been given to solve the nonlinear operator equations in Banach spaces by using the two-step and the one-step iterative schemes, see [2-10, 20]. Noor [12-13] has suggested and analyzed three-step iterative methods for finding the approximate solutions of the variational inclusions (inequalities) in a Hilbert space by using the techniques of updating the solution and the auxiliary principle. These three-step schemes are similar to those of the so-called θ-schemes of Glowinski and

Le Tallec [3] for finding a zero of the sum of two (more) maximal monotone operators, which they have suggested by using the Lagrange multiplier method. Glowinski and Le Tallec [3] used these three-step iterative schemes for solving elastoviscoplasticity, liquid crystal and eigen-value problems. They have shown that the three-step approximations perform better than the two-step and one-step iterative methods. Haubruge et all [6] have studied the convergence analysis of the three-step schemes of Glowinski and Le Tallec [3] and applied these three-step iterations to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They have also proved that three-step iterations lead also to highly parallelized algorithms under certain conditions. It has been shown in [6, 12-13] that three-step schemes are a natural generalization of the splitting methods for solving partial differential equations (inclusions). For the applications of the splitting and decomposition methods, see [1, 3, 6, 12-14] and the references therein. Thus we conclude that three-step schemes play an important and significant part in solving various problems, which arise in pure and applied sciences.

In 2002, Noor, Rassias and Huang [15] suggested the following three-step iteration process for solving the nonlinear equations $Tu = 0$.

Let E is a real normed space and K be a nonempty closed convex subset of E.

Algorithm NRH. Let $T : K \to K$ be a mapping. For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$
\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\
y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\
z_n &= (1-\gamma_n)x_n + \gamma_n T x_n,\ n \geq 0,
\end{aligned} \tag{3}
$$

which is called the three-step iterative process, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in [0,1] satisfying some certain conditions.

If $\gamma_n = 0$ and $\beta_n = 0$, then Algorithm NRH reduces to:

Algorithm M. For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative scheme

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, n \geq 0, \tag{4}$$

which is called the Mann iterative process, see [11].

For $\gamma_n = 0$, Algorithm NRH becomes:

Algorithm I. Let K be a nonempty convex subset of E and let $T : K \to K$ be a mapping. For any given $x_0 \in K$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, n \geq 0, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, n \geq 0, \end{aligned} \tag{5}$$

which is called the two-step Ishikawa iterative process, and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two real sequences in [0,1] satisfying some certain conditions.

These facts motivated us to introduce and analyze a class of three-step iterative scheme for three strongly pseudocontractive mappings . This scheme defined as follows.

Algorithm A. Let $T_1, T_2, T_3 : K \to K$ be three given mappings. For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative scheme

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2 z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T_3 x_n, \ n \geq 0, \end{aligned} \tag{6}$$

which is called the modified three-step iterative process, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in [0,1] satisfying some certain conditions.

It may be noted that the iteration schemes (3-5) may be viewed as the special case of (6).

In this paper, we establish the strong convergence for a modified three-step iterative scheme for three strongly pseudocontractive mappings in a uniformly smooth Banach space. Our results can be viewed as an extension of three-step and two-step iterative schemes of Glowinski and Le Tallec [3], Noor [12-15] and Ishikawa [7], Liu [10] and Xu [20]. We also study the convergence analysis of the iterative method.

2 Main Results

We will use the following results.

Lemma 1. *[19] Let E be a real uniformly smooth Banach space and let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \text{for all } \ j(x+y) \in J(x+y). \tag{7}$$

Lemma 2. *[2] E is a uniformly smooth Banach space if and only if J is single valued and uniformly continuous on any bounded subset of E.*

The following lemma is proved in [17].

Lemma 3. *If there exists a positive integer N such that for all $n \geq N$, $n \in \mathbb{N}$,*

$$\rho_{n+1} \leq (1-\alpha_n)\rho_n + b_n,$$

then

$$\lim_{n\to\infty} \rho_n = 0,$$

where $\alpha_n \in [0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $b_n = o(\alpha_n)$.

Theorem 1. *Let E be a real uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let T_1, T_2, T_3 be strongly pseudocontractive self maps of K with $T_i(K)$ bounded; $i = 1, 2, 3$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2 z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T_3 x_n, \ n \geq 0, \end{aligned} \tag{8}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ *and* $\{\gamma_n\}_{n=0}^{\infty}$ *are three real sequences in* $[0,1]$ *satisfying the conditions:*

$$\lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\gamma_n \text{ and } \sum_{n=0}^{\infty}\alpha_n = \infty. \tag{9}$$

If $F(T_1)\cap F(T_1)\cap F(T_1)\neq\varphi$, *then the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the common fixed point of* T_1, T_2, T_3.

Proof. Since each T_i; $i = 1,\ 2,\ 3$ is strongly pseudocontractive, then there exists $k_i \in (0,1)$; $i = 1,\ 2,\ 3$ such that

$$\langle T_ix - T_iy, j(x-y)\rangle \leq (1-k_i)||x-y||^2,\ i = 1,\ 2,\ 3.$$

Let $k = \min_{1\leq i\leq N}\{k_i\}$. Then

$$\langle T_ix - T_iy, j(x-y)\rangle \leq (1-k)||x-y||^2,\ i = 1,\ 2,\ 3.$$

Let $p \in F := F(T_1)\cap F(T_2)\cap F(T_3)$.We will show that p is the unique fixed point of F. Let $p \in F(T_1)$. Suppose there exists $q_1 \in F(T_1)$. Then

$$||p-q_1||^2 = \langle p-q_1, j(p-q_1)\rangle = \langle T_1p - T_1q_1, j(p-q_1)\rangle \leq (1-k)||p-q_1||^2.$$

Since $k \in (0,1)$, it follows that $||p-q_1||^2 \leq 0$, which implies $p = q_1$. Hence $F(T_1) = \{p\}$. Similarly we can prove that p is the unique fixed point of T_2 and T_3 respectively. Thus $p \in F$.

Since each T_i; $i = 1,2,3$ has bounded range, we set

$$M_1 = ||x_0 - p|| + \sup_{x,y\in K}||T_ix - T_iy||;\ i = 1,2,3.$$

Obviously $M_1 < \infty$.

It is clear that $||x_0 - p|| \leq M_1$. Let $||x_n - p|| \leq M_1$. Next we will prove that $||x_{n+1} - p|| \leq M_1$.

Consider

$$\begin{aligned} ||x_{n+1} - p|| &= ||(1-\alpha_n)x_n + \alpha_n T_1 y_n - p|| \\ &= ||(1-\alpha_n)(x_n - p) + \alpha_n (T_1 y_n - p)|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||T_1 y_n - p|| \\ &\leq (1-\alpha_n)M_1 + M_1 \alpha_n = M_1. \end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n=0}^{\infty}$ is bounded.

Let $M_2 = \sup_{n\geq 0} ||x_n - p||$.

Since

$$\begin{aligned} ||x_n - y_n|| &= ||x_n - (1-\beta_n)x_n - \beta_n T_2 z_n|| \\ &= \beta_n ||x_n - T_2 z_n|| \\ &\leq \beta_n ||x_n - p|| + \beta_n ||T_2 z_n - p|| \leq \beta_n M_2 + \beta_n M_1 \\ &= (M_2 + M_1)\beta_n \longrightarrow 0, \text{ as } \ n \to \infty, \end{aligned}$$

implies $\{x_n - y_n\}_{n=0}^{\infty}$ is bounded. Let $M_3 = \sup_{n\geq 0} ||x_n - y_n||$. Since $||y_n - p|| \leq ||x_n - p|| + ||x_n - y_n||$, so $\{y_n - p\}_{n=0}^{\infty}$ is also bounded. Let $M_4 = \sup_{n\geq 0} ||y_n - p||$. In a similar way, we can prove that the sequence $\{||z_n - p||\}_{n=0}^{\infty}$ is bounded. Let $M_5 = \sup_{n\geq 0} ||z_n - p||$.

Denote $M = M_1 + M_2 + M_4 + M_5$. Obviously $M < \infty$.

From Lemma 1 for all $n \geq 0$, and by taking $A_n = \langle T_3 x_n - p, j(z_n - p) - j(x_n - p)\rangle$, we have

$$\begin{aligned} ||z_n - p||^2 &= ||(1-\gamma_n)x_n + \gamma_n T_3 x_n - p||^2 \\ &= ||(1-\gamma_n)(x_n - p) + \gamma_n (T_3 x_n - p)||^2 \\ &\leq (1-\gamma_n)^2 ||x_n - p||^2 + 2\gamma_n \langle T_3 x_n - p, j(z_n - p)\rangle \end{aligned} \tag{10}$$

$$\begin{aligned}
&= (1-\gamma_n)^2||x_n-p||^2 + 2\gamma_n\langle T_3x_n - p, j(x_n-p)\\
&- j(x_n-p) + j(z_n-p)\rangle\\
&= (1-\gamma_n)^2||x_n-p||^2 + 2\gamma_n\langle T_3x_n - p, j(x_n-p)\rangle + 2\gamma_n A_n\\
&\le (1-\gamma_n)^2||x_n-p||^2 + 2\gamma_n(1-k)||x_n-p||^2 + 2\gamma_n A_n\\
&= [1+\gamma_n(\gamma_n-2k)]||x_n-p||^2 + 2\gamma_n A_n.
\end{aligned}$$

Now by $\lim_{n\to\infty}\gamma_n = 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, we have $\gamma_n \le 2k$. From (10), we get

$$||z_n-p||^2 \le ||x_n-p||^2 + 2\gamma_n A_n. \tag{11}$$

CLAIM 1: $\lim_{n\to\infty} A_n = 0$.

Indeed, from Lemma 2, since E is uniformly smooth Banach space, J is single valued and uniformly continuous on any bounded subset of E. Observe that

$$\begin{aligned}
(z_n-p)-(x_n-p) &= z_n - x_n\\
&= (1-\gamma_n)x_n + \gamma_n T_3x_n - x_n\\
&= \gamma_n(T_3x_n - x_n),
\end{aligned}$$

so as $n\to\infty$, we have

$$\begin{aligned}
||(z_n-p)-(x_n-p)|| &= \|x_n - T_3x_n\|\\
&\le \gamma_n\left(||x_n-p|| + ||T_3x_n-p||\right)\\
&\le M\gamma_n + M\gamma_n = 2M\gamma_n \longrightarrow 0.
\end{aligned}$$

Since we have shown that the sequences $\{x_n-p\}_{n=0}^{\infty}$ and $\{z_n-p\}_{n=0}^{\infty}$ are all bounded sets, it follows that as $n\to\infty$,

$$||j(z_n-p) - j(x_n-p)|| \longrightarrow 0,$$

and hence,

$$A_n = \langle T_3x_n - p, j(z_n-p) - j(x_n-p)\rangle \longrightarrow 0.$$

Also from Lemma 1 for all $n \geq 0$, and by taking $B_n = \langle T_2 z_n - p, j(y_n - p) - j(z_n - p)\rangle$, we have

$$\begin{aligned} ||y_n - p||^2 &= ||(1-\beta_n)x_n + \beta_n T_2 z_n - p||^2 \\ &= ||(1-\beta_n)(x_n - p) + \beta_n(T_2 z_n - p)||^2 \\ &\leq (1-\beta_n)^2||x_n - p||^2 + 2\beta_n\langle T_2 z_n - p, j(y_n - p)\rangle \\ &= (1-\beta_n)^2||x_n - p||^2 \\ &\quad +2\beta_n\langle T_2 z_n - p, j(z_n - p) - j(z_n - p) + j(y_n - p)\rangle \\ &= (1-\beta_n)^2||x_n - p||^2 \\ &\quad +2\beta_n\langle T_2 z_n - p, j(z_n - p)\rangle + 2\beta_n B_n \\ &\leq (1-\beta_n)^2||x_n - p||^2 + 2\beta_n(1-k)||z_n - p||^2 + 2\beta_n B_n. \end{aligned} \tag{12}$$

Substituting (11) in (12), we get

$$\begin{aligned} ||y_n - p||^2 &\leq (1-\beta_n)^2||x_n - p||^2 \\ &\quad +2\beta_n(1-k)\left[||x_n - p||^2 + 2\gamma_n A_n\right] + 2\beta_n B_n \\ &= \left[(1-\beta_n)^2 + 2\beta_n(1-k)\right]||x_n - p||^2 \\ &\quad +4(1-k)\beta_n\gamma_n A_n + 2\beta_n B_n \\ &= [1 + \beta_n(\beta_n - 2k)]||x_n - p||^2 \\ &\quad +4(1-k)\beta_n\gamma_n A_n + 2\beta_n B_n. \end{aligned} \tag{13}$$

Now by $\lim_{n\to\infty} \beta_n = 0$, implies there exists a natural number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $\beta_n \leq 2k$. From (13), we get

$$||y_n - p||^2 \leq ||x_n - p||^2 + 4(1-k)\beta_n\gamma_n A_n + 2\beta_n B_n. \tag{14}$$

CLAIM 2: $\lim_{n\to\infty} B_n = 0$.

Indeed, from Lemma 2, since E is uniformly smooth Banach space, J is single valued and uniformly continuous on any bounded subset of E.

Observe that

$$\begin{aligned}(y_n - p) - (z_n - p) &= y_n - z_n \\ &= (1-\beta_n)x_n + \beta_n T_2 z_n - (1-\gamma_n)x_n - \gamma_n T_3 x_n \\ &= \beta_n(T_2 z_n - x_n) + \gamma_n(x_n - T_3 x_n),\end{aligned}$$

so as $n \to \infty$, we have

$$\begin{aligned}||(y_n - p) - (z_n - p)|| &= \|\beta_n(T_2 z_n - x_n) + \gamma_n(x_n - T_3 x_n)\| \\ &\leq \beta_n \|T_2 z_n - x_n\| + \gamma_n \|x_n - T_3 x_n\| \\ &\leq \beta_n (\|T_2 z_n - p\| + \|x_n - p\|) \\ &\quad + \gamma_n (\|x_n - p\| + \|T_3 x_n - p\|) \\ &\leq 2M\beta_n + 2M\gamma_n = 2M(\beta_n + \gamma_n) \longrightarrow 0.\end{aligned}$$

Since we have shown that the sequences $\{y_n - p\}_{n=0}^{\infty}$ and $\{z_n - p\}_{n=0}^{\infty}$ are all bounded sets, it follows that as $n \to \infty$,

$$||j(y_n - p) - j(z_n - p)|| \longrightarrow 0,$$

and hence,

$$B_n = \langle T_2 z_n - p, j(y_n - p) - j(z_n - p)\rangle \to 0.$$

Thus, from Lemma 1 for all $n \geq 0$, and by taking $C_n = \langle T_1 y_n - p, j(x_{n+1} - p) - j(y_n - p)\rangle$, we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= ||(1-\alpha_n)x_n + \alpha_n T_1 y_n - p||^2 \\ &= ||(1-\alpha_n)(x_n - p) + \alpha_n(T_1 y_n - p)||^2 \\ &\leq (1-\alpha_n)^2||x_n - p||^2 + 2\alpha_n\langle T_1 y_n - p, j(x_{n+1} - p)\rangle \\ &= (1-\alpha_n)^2||x_n - p||^2 \\ &\quad + 2\alpha_n\langle T_1 y_n - p, j(y_n - p) - j(y_n - p) + j(x_{n+1} - p)\rangle \\ &= (1-\alpha_n)^2||x_n - p||^2 + 2\alpha_n\langle T_1 y_n - p, j(y_n - p)\rangle \\ &\quad + 2\alpha_n C_n \\ &\leq (1-\alpha_n)^2||x_n - p||^2 + 2\alpha_n(1-k)||y_n - p||^2 + 2\alpha_n C_n. \qquad (15)\end{aligned}$$

Substituting (14) in (15), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n)^2||x_n-p||^2\\ &\quad +2\alpha_n(1-k)\left[||x_n-p||^2+4(1-k)\beta_n\gamma_nA_n+2\beta_nB_n\right]\\ &\quad +2\alpha_nC_n\\ &= \left[(1-\alpha_n)^2+2\alpha_n(1-k)\right]||x_n-p||^2\\ &\quad +8(1-k)^2\alpha_n\beta_n\gamma_nA_n+4(1-k)\alpha_n\beta_nB_n+2\alpha_nC_n\\ &= [1+\alpha_n(\alpha_n-2k)]||x_n-p||^2\\ &\quad +8(1-k)^2\alpha_n\beta_n\gamma_nA_n+4(1-k)\alpha_n\beta_nB_n+2\alpha_nC_n. \qquad (16)\end{aligned}$$

Now by $\lim_{n\to\infty}\alpha_n=0$, implies there exists a natural number $n_0\in\mathbb{N}$ such that for all $n\ge n_0$, we have $\alpha_n\le k$. From (16), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-k\alpha_n)||x_n-p||^2+8(1-k)^2\alpha_n\beta_n\gamma_nA_n\\ &\quad +4(1-k)\alpha_n\beta_nB_n+2\alpha_nC_n\\ &= (1-k\alpha_n)||x_n-p||^2+\delta_n\alpha_n; \qquad (17)\end{aligned}$$

$$\delta_n=4(1-k)\beta_n[2(1-k)\gamma_nA_n+B_n]+2C_n.$$

CLAIM 3: $\lim_{n\to\infty}C_n=0$.

Indeed, from Lemma 2, since E is uniformly smooth Banach space, J is single valued and uniformly continuous on any bounded subset of E. Observe that

$$\begin{aligned}(x_{n+1}-p)-(y_n-p) &= x_{n+1}-y_n\\ &= (1-\alpha_n)x_n+\alpha_nT_1y_n-(1-\beta_n)x_n-\beta_nT_2z_n\\ &= \alpha_n(T_1y_n-x_n)+\beta_n(x_n-T_2z_n),\end{aligned}$$

so as $n \to \infty$, we have

$$\begin{aligned}||(x_{n+1}-p)-(y_n-p)|| &= \|\alpha_n(T_1y_n-x_n)+\beta_n(x_n-T_2z_n)\| \\ &\leq \alpha_n\|T_1y_n-x_n\|+\beta_n\|x_n-T_2z_n\| \\ &\leq \alpha_n(\|T_1y_n-p\|+\|x_n-p\|) \\ &\quad +\beta_n(\|x_n-p\|+\|T_2z_n-p\|) \\ &\leq 2M\alpha_n+2M\beta_n=2M(\alpha_n+\beta_n)\longrightarrow 0.\end{aligned}$$

Since we have shown that the sequences $\{x_{n+1}-p\}_{n=0}^{\infty}$ and $\{y_n-p\}_{n=0}^{\infty}$ are all bounded sets, it follows that as $n \to \infty$,

$$||j(x_{n+1}-p)-j(y_n-p)|| \longrightarrow 0,$$

and hence,

$$C_n = \langle T_1y_n-p, j(x_{n+1}-p)-j(y_n-p)\rangle \longrightarrow 0.$$

Now applying Lemma 3 on (17), we obtain that

$$\lim_{n\to\infty} ||x_n-p|| = 0,$$

completing the proof.

As a special case of theorem 1, the following corollary can be deduced by taking $T_1 = T_2 = T_3$.

Corollary 1. *Let E be a real uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let T be strongly pseudocontractive self map of K with $T(K)$ bounded. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by*

$$\begin{aligned}x_{n+1} &= (1-\alpha_n)x_n+\alpha_nTy_n, \\ y_n &= (1-\beta_n)x_n+\beta_nTz_n, \\ z_n &= (1-\gamma_n)x_n+\gamma_nTx_n,\ n\geq 0,\end{aligned} \tag{18}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying the conditions:

$$\lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\gamma_n \text{ and } \sum_{n=0}^{\infty}\alpha_n=\infty. \tag{19}$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point p of T.

Remark 1. *Corollary 1 is the theorem 2.1 of [15] due to Noor, Rassias and Huang.*

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