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Some sufficient problems for Strongly Close-to-Convex of order μ

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Abstract

For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ with $f(0) = f'(0) - 1 = 0$, a class $\mathcal{STC}(\mu)$ is defined. The object of the present paper is to discuss some sufficient problems for $f(z)$ to be strongly close-to-convex of order μ in $\mathbb{U}$.

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1 Introduction

Let $\mathcal{A}_{n_j}$ denote the class of functions

$$f(z) = z + a_{n_j+1}z^{n_j+1} + a_{n_j+2}z^{n_j+2} + \dots \qquad (n = 1, 2, 3, \dots \ ; \ j = 1, 2)$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A} = \mathcal{A}_1$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}_n$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$.

Let $\mathcal{S}^*(\alpha)$ be defined by

$$\mathcal{S}^*(\alpha) = \left\{ f(z) \in \mathcal{A}_n : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \leqq {}^{\exists}\alpha < 1 \right\}.$$

We denote by $\mathcal{S}^* = \mathcal{S}^*(0)$.

Also, let $\mathcal{STC}(\mu)$ be defined by

$$\mathcal{STC}(\mu) = \left\{ f(z) \in \mathcal{A}_n : \mathrm{Re}\left(\left(\frac{zf'(z)}{g(z)}\right)^{\frac{1}{\mu}}\right) > 0,\ 0 < {}^{\exists}\mu \leqq 1,\ {}^{\exists}g(z) \in \mathcal{S}^* \right\}.$$

A function $f(z) \in \mathcal{STC}(\mu)$ is said to be strongly close-to-convex of order μ in $\mathbb{U}$.

The basic tool in proving our results is the following lemma due to Jack [1] (also, due to Miller and Mocanu [2]).

Lemma 1. *Let the function $w(z)$ defined by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots \qquad (n = 1, 2, 3, \dots)$$

be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $k \geqq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k.$$

2 Main results

Applying Lemma 1, we drive the following results for $\mathcal{STC}(\mu)$.

Theorem 1. *If* $f(z) \in \mathcal{A}_{n_1}$ *satisfies*

$$\left|\left(\frac{zf'(z)}{g(z)}\right)^{\frac{1}{\mu}} - 1\right|^{\beta} \left(\mathrm{Re}\left(\delta + 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right)\right)^{\gamma} < \left(\mathrm{Re}(\delta) + \frac{\mu n}{2}\right)^{\gamma}$$

for some real $\beta \geqq 0$, $\gamma \geqq 0$ *such that* $\beta + \gamma > 0$, *some real* $0 < \mu \leqq 1$, *some complex* δ *with* $\mathrm{Re}(\delta) > -\dfrac{\mu n}{2}$, *and for some* $g(z) \in \mathcal{A}_{n_2} \cap \mathcal{S}^*$ *where* $n = \min\{n_1, n_2\}$, *then*

$$\left|\left(\frac{zf'(z)}{g(z)}\right)^{\frac{1}{\mu}} - 1\right| < 1 \qquad (z \in \mathbb{U}).$$

This means that $f(z) \in \mathcal{STC}(\mu)$.

Proof. Let us define $w(z)$ by

$$\begin{aligned} w(z) &= \left(\frac{zf'(z)}{g(z)}\right)^{\frac{1}{\mu}} - 1 \qquad (z \in \mathbb{U}) \\ &= b_n z^n + b_{n+1} z^{n+1} + \dots \end{aligned} \tag{1}$$

where $n = \min\{n_1, n_2\}$.

Then, clearly, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Differentiating both sides in (1), we obtain

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} = \frac{\mu z w'(z)}{w(z) + 1},$$

and therefore,

$$\left|\left(\frac{zf'(z)}{g(z)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left(\operatorname{Re}\left(\delta+1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right)\right)^{\gamma}$$
$$=|w(z)|^{\beta}\left(\operatorname{Re}\left(\delta+\frac{\mu zw'(z)}{w(z)+1}\right)\right)^{\gamma}<\left(\operatorname{Re}(\delta)+\frac{\mu n}{2}\right)^{\gamma}\qquad(z\in\mathbb{U}).$$

If there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=1,$$

then Lemma 1 gives us that $w(z_0)=e^{i\theta}$ and $z_0w'(z_0)=kw(z_0)$ $(k\geqq n)$.

For such a point z_0, we have

$$\begin{aligned}
&\left|\left(\frac{z_0f'(z_0)}{g(z_0)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left(\operatorname{Re}\left(\delta+1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0g'(z_0)}{g(z_0)}\right)\right)^{\gamma}\\
&=|w(z_0)|^{\beta}\left(\operatorname{Re}\left(\delta+\frac{\mu z_0w'(z_0)}{w(z_0)+1}\right)\right)^{\gamma}\\
&=\left(\operatorname{Re}\left(\delta+\frac{\mu kw(z_0)}{w(z_0)+1}\right)\right)^{\gamma}\\
&=\left(\operatorname{Re}\left(\delta+\frac{\mu k}{2}\left(1+i\tan\frac{\theta}{2}\right)\right)\right)^{\gamma}\\
&=\left(\operatorname{Re}(\delta)+\frac{\mu k}{2}\right)^{\gamma}\\
&\geqq\left(\operatorname{Re}(\delta)+\frac{\mu n}{2}\right)^{\gamma}.
\end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0\in\mathbb{U}$ such that $|w(z_0)|=1$. This means that $|w(z)|<1$ for all $z\in\mathbb{U}$. It follows that

$$\left|\left(\frac{zf'(z)}{g(z)}\right)^{\frac{1}{\mu}}-1\right|<1\qquad(z\in\mathbb{U})$$

so that $f(z)\in\mathcal{STC}(\mu)$.

We also derive

Theorem 2. *If $f(z) \in \mathcal{A}_{n_1}$ satisfies*

$$\left| \left(\frac{zf'(z)}{g(z)} \right)^{\frac{1}{\mu}} - 1 \right|^{\beta} \left(\mathrm{Re} \left(\delta + 1 + \frac{zf''(z)}{f'(z)} \right) \right)^{\gamma} \leqq \left(\mathrm{Re}(\delta) + \alpha + \frac{\mu n}{2} \right)^{\gamma}$$

for some real $\beta \geqq 0$, $\gamma \geqq 0$ such that $\beta + \gamma > 0$, some real $0 < \mu \leqq 1$, $0 \leqq \alpha < 1$, some complex δ with $\mathrm{Re}(\delta) > -\frac{\mu n}{2} - \alpha$, and for some $g(z) \in \mathcal{A}_{n_2} \cap \mathcal{S}^*(\alpha)$ where $n = \min\{n_1, n_2\}$, then

$$\left| \left(\frac{zf'(z)}{g(z)} \right)^{\frac{1}{\mu}} - 1 \right| < 1 \qquad (z \in \mathbb{U}),$$

which shows that $f(z) \in \mathcal{STC}(\mu)$.

Proof. Define $w(z)$ in $\mathbb{U}$ by

$$w(z) = \left(\frac{zf'(z)}{g(z)} \right)^{\frac{1}{\mu}} - 1 \qquad (z \in \mathbb{U}) \tag{2}$$

$$= b_n z^n + b_{n+1} z^{n+1} + \dots$$

where $n = \min\{n_1, n_2\}$.

Evidently, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Differentiating (2) logarithmically and simplyfing, we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{\mu z w'(z)}{w(z) + 1},$$

and hence,

$$\begin{aligned}&\left|\left(\frac{zf'(z)}{g(z)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left(\mathrm{Re}\left(\delta+1+\frac{zf''(z)}{f'(z)}\right)\right)^{\gamma}\\&\quad=|w(z)|^{\beta}\left(\mathrm{Re}\left(\delta+\frac{zg'(z)}{g(z)}+\frac{\mu zw'(z)}{w(z)+1}\right)\right)^{\gamma}\leqq\left(\mathrm{Re}(\delta)+\alpha+\frac{\mu n}{2}\right)^{\gamma}.\end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=1,$$

then Lemma 1 gives us that $w(z_0)=e^{i\theta}$ and $z_0w'(z_0)=kw(z_0)$ $(k \geqq n)$.

For such a point z_0, we have

$$\begin{aligned}&\left|\left(\frac{z_0f'(z_0)}{g(z_0)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left(\mathrm{Re}\left(\delta+1+\frac{z_0f''(z_0)}{f'(z_0)}\right)\right)^{\gamma}\\&=|w(z_0)|^{\beta}\left(\mathrm{Re}\left(\delta+\frac{z_0g'(z_0)}{g(z_0)}+\frac{\mu z_0w'(z_0)}{w(z_0)+1}\right)\right)^{\gamma}\\&=\left(\mathrm{Re}\left(\delta+\frac{z_0g'(z_0)}{g(z_0)}+\frac{\mu kw(z_0)}{w(z_0)+1}\right)\right)^{\gamma}\\&=\left(\mathrm{Re}\left(\delta+\frac{z_0g'(z_0)}{g(z_0)}+\frac{\mu k}{2}\left(1+i\tan\frac{\theta}{2}\right)\right)\right)^{\gamma}\\&=\left(\mathrm{Re}(\delta)+\mathrm{Re}\left(\frac{z_0g'(z_0)}{g(z_0)}\right)+\frac{\mu k}{2}\right)^{\gamma}\\&>\left(\mathrm{Re}(\delta)+\alpha+\frac{\mu n}{2}\right)^{\gamma}.\end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)|=1$. This means that $|w(z)|<1$ for all $z \in \mathbb{U}$. This implies that

$$\left|\left(\frac{zf'(z)}{g(z)}\right)^{\frac{1}{\mu}}-1\right|<1 \qquad (z\in\mathbb{U})$$

so that $f(z) \in \mathcal{STC}(\mu)$.

We consider a new appplication for Lemma 1. Our new application is follows.

Theorem 3. *If $f(z) \in \mathcal{A}_{n_1}$ satisfies*

$$\left| \left(\frac{zf'(z)}{g(z)} \right)^{\frac{1}{\mu}} - 1 \right|^{\beta} \left| \delta + 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right|^{\gamma} < \rho^{\beta} \left(\delta + \frac{\mu \rho n}{1+\rho} \right)^{\gamma}$$

for some real $\beta \geqq 0$, $\gamma \geqq 0$ such that $\beta + \gamma > 0$, some real $0 < \mu \leqq 1$, $\delta > 0$, ρ with $\rho > \sqrt{\dfrac{\delta}{\delta + \mu n}}$, and for some $g(z) \in \mathcal{A}_{n_2} \cap \mathcal{S}^*$ where $n = \min\{n_1, n_2\}$, then

$$\left| \left(\frac{zf'(z)}{g(z)} \right)^{\frac{1}{\mu}} - 1 \right| < \rho \qquad (z \in \mathbb{U}).$$

In addition for $\rho < 1$, we have $f(z) \in \mathcal{STC}(\mu)$.

Proof. Defining the function $w(z)$ by

$$\begin{aligned} w(z) &= \left(\frac{zf'(z)}{g(z)} \right)^{\frac{1}{\mu}} - 1 \qquad (z \in \mathbb{U}) \\ &= b_n z^n + b_{n+1} z^{n+1} + \dots \end{aligned}$$

where $n = \min\{n_1, n_2\}$, we have that $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$. Since,

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} = \frac{\mu z w'(z)}{w(z)+1},$$

we obtain that

$$\left|\left(\frac{zf'(z)}{g(z)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left|\delta+1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right|^{\gamma}=|w(z)|^{\beta}\left|\delta+\frac{\mu zw'(z)}{w(z)+1}\right|^{\gamma}$$
$$<\rho^{\beta}\left(\delta+\frac{\mu\rho n}{1+\rho}\right)^{\gamma}.$$

If there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=\rho,$$

then Lemma 1 gives us that $w(z_0)=\rho e^{i\theta}$ and $z_0w'(z_0)=kw(z_0)$ $(k\geqq n)$.

Thus we have

$$\begin{aligned}
&\left|\left(\frac{z_0f'(z_0)}{g(z_0)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left|\delta+1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0g'(z_0)}{g(z_0)}\right|^{\gamma}\\
&=|w(z_0)|^{\beta}\left|\delta+\frac{\mu z_0w'(z_0)}{w(z_0)+1}\right|^{\gamma}\\
&=\rho^{\beta}\left|\delta+\frac{\mu kw(z_0)}{w(z_0)+1}\right|^{\gamma}\\
&=\rho^{\beta}\left|\delta+\frac{\mu\rho k(\rho+\cos\theta)}{\rho^2+1+2\rho\cos\theta}+i\frac{\mu\rho k\sin\theta}{\rho^2+1+2\rho\cos\theta}\right|^{\gamma}\\
&=\rho^{\beta}\left(\delta^2+\mu\delta k+\frac{\mu\delta k(\rho^2-1)+\mu^2\rho^2k^2}{\rho^2+1+2\rho\cos\theta}\right)^{\frac{\gamma}{2}}\\
&\geqq\rho^{\beta}\left(\delta^2+\mu\delta k+\frac{\mu\delta k(\rho^2-1)+\mu^2\rho^2k^2}{\rho^2+1+2\rho}\right)^{\frac{\gamma}{2}}\\
&=\rho^{\beta}\left(\delta+\frac{\mu\rho k}{1+\rho}\right)^{\gamma}\\
&\geqq\rho^{\beta}\left(\delta+\frac{\mu\rho n}{1+\rho}\right)^{\gamma}.
\end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0\in\mathbb{U}$ such that $|w(z_0)|=\rho$. This means that $|w(z)|<\rho$ for all $z\in\mathbb{U}$.

We also consider a new aplocstion for Lemma 1.

Theorem 4. *If $f(z) \in \mathcal{A}_{n_1}$ satisfies*

$$\left|\left(\frac{g(z)}{zf'(z)}\right)^{\frac{1}{\mu}} - 1\right|^{\beta} \left(\mathrm{Re}\left(\delta - 1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)}\right)\right)^{\gamma} < \left(\mathrm{Re}(\delta) + \frac{\mu n}{2}\right)^{\gamma}$$

for some real $\beta \geqq 0$, $\gamma \geqq 0$ such that $\beta + \gamma > 0$, some real $0 < \mu \leqq 1$, some complex δ with $\mathrm{Re}(\delta) > -\dfrac{\mu n}{2}$, and for some $g(z) \in \mathcal{A}_{n_2} \cap \mathcal{S}^*$ where $n = \min\{n_1, n_2\}$, then

$$\left|\left(\frac{g(z)}{zf'(z)}\right)^{\frac{1}{\mu}} - 1\right| < 1 \qquad (z \in \mathbb{U}).$$

This means that $f(z) \in \mathcal{STC}(\mu)$.

Proof. Let us define the function $w(z)$ by

$$w(z) = \left(\frac{g(z)}{zf'(z)}\right)^{\frac{1}{\mu}} - 1 \qquad (z \in \mathbb{U}) \tag{3}$$
$$= b_n z^n + b_{n+1} z^{n+1} + \dots$$

where $n = \min\{n_1, n_2\}$.

Clearly, $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$. We want to prove that $|w(z)| < 1$ in $\mathbb{U}$. Since,

$$-1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)} = \frac{\mu z w'(z)}{w(z) + 1},$$

we see that

$$\left|\left(\frac{g(z)}{zf'(z)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left(\mathrm{Re}\left(\delta-1-\frac{zf''(z)}{f'(z)}+\frac{zg'(z)}{g(z)}\right)\right)^{\gamma}$$
$$=|w(z)|^{\beta}\left(\mathrm{Re}\left(\delta+\frac{\mu zw'(z)}{w(z)+1}\right)\right)^{\gamma}<\left(\mathrm{Re}(\delta)+\frac{\mu n}{2}\right)^{\gamma}\qquad(z\in\mathbb{U}).$$

If there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=1,$$

then Lemma 1 gives us that $w(z_0)=e^{i\theta}$ and $z_0w'(z_0)=kw(z_0)$ $(k\geqq n)$.

Thus we have

$$\begin{aligned}
&\left|\left(\frac{g(z_0)}{z_0f'(z_0)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left(\mathrm{Re}\left(\delta-1-\frac{z_0f''(z_0)}{f'(z_0)}+\frac{z_0g'(z_0)}{g(z_0)}\right)\right)^{\gamma}\\
&=|w(z_0)|^{\beta}\left(\mathrm{Re}\left(\delta+\frac{\mu z_0w'(z_0)}{w(z_0)+1}\right)\right)^{\gamma}\\
&=\left(\mathrm{Re}\left(\delta+\frac{\mu kw(z_0)}{w(z_0)+1}\right)\right)^{\gamma}\\
&=\left(\mathrm{Re}\left(\delta+\frac{\mu k}{2}\left(1+i\tan\frac{\theta}{2}\right)\right)\right)^{\gamma}\\
&=\left(\mathrm{Re}(\delta)+\frac{\mu k}{2}\right)^{\gamma}\\
&\geqq\left(\mathrm{Re}(\delta)+\frac{\mu n}{2}\right)^{\gamma}.
\end{aligned}$$

This contradicts the condition in the theorem. Therefore, there is no $z_0\in\mathbb{U}$ such that $|w(z_0)|=1$. This means that $|w(z)|<1$ for all $z\in\mathbb{U}$. We conclude that

$$\left|\left(\frac{g(z)}{zf'(z)}\right)^{\frac{1}{\mu}}-1\right|<1\qquad(z\in\mathbb{U})$$

which implies that $f(z)\in\mathcal{STC}(\mu)$.

Finally, we derive

Theorem 5. *If $f(z) \in \mathcal{A}_{n_1}$ satisfies*

$$\left| \left(\frac{g(z)}{zf'(z)} \right)^{\frac{1}{\mu}} - 1 \right|^{\beta} \left| \delta - 1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)} \right|^{\gamma} < \rho^{\beta} \left(\delta + \frac{\mu \rho n}{1+\rho} \right)^{\gamma}$$

for some real $\beta \geqq 0$, $\gamma \geqq 0$ such that $\beta + \gamma > 0$, some real $0 < \mu \leqq 1$, $\delta > 0$, ρ with $\rho > \sqrt{\dfrac{\delta}{\delta + \mu n}}$, and for some $g(z) \in \mathcal{A}_{n_2} \cap \mathcal{S}^*$ where $n = \min\{n_1, n_2\}$, then

$$\left| \left(\frac{g(z)}{zf'(z)} \right)^{\frac{1}{\mu}} - 1 \right| < \rho \qquad (z \in \mathbb{U}).$$

In addition, if $\rho < 1$, then we have $f(z) \in \mathcal{STC}(\mu)$.

Proof. Let us define $w(z)$ by

$$w(z) = \left(\frac{g(z)}{zf'(z)} \right)^{\frac{1}{\mu}} - 1 \qquad (z \in \mathbb{U}) \tag{4}$$

$$= b_n z^n + b_{n+1} z^{n+1} + \dots$$

where $n = \min\{n_1, n_2\}$.

Then, we have that $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Differenciating (4) in both side logarithmically and simplifying, we obtain

$$-1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)} = \frac{\mu z w'(z)}{w(z)+1},$$

and hence,

$$\left|\left(\frac{g(z)}{zf'(z)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left|\delta-1-\frac{zf''(z)}{f'(z)}+\frac{zg'(z)}{g(z)}\right|^{\gamma}=|w(z)|^{\beta}\left|\delta+\frac{\mu zw'(z)}{w(z)+1}\right|^{\gamma}$$
$$<\rho^{\beta}\left(\delta+\frac{\mu\rho n}{1+\rho}\right)^{\gamma}.$$

If there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=\rho,$$

then Lemma 1 gives us that $w(z_0)=\rho e^{i\theta}$ and $z_0w'(z_0)=kw(z_0)$ $(k\geqq n)$.

Therefore, we have

$$\begin{aligned}
&\left|\left(\frac{g(z_0)}{z_0f'(z_0)}\right)^{\frac{1}{\mu}}-1\right|^{\beta}\left|\delta-1-\frac{z_0f''(z_0)}{f'(z_0)}+\frac{z_0g'(z_0)}{g(z_0)}\right|^{\gamma}\\
&=|w(z_0)|^{\beta}\left|\delta+\frac{\mu z_0w'(z_0)}{w(z_0)+1}\right|^{\gamma}\\
&=\rho^{\beta}\left|\delta+\frac{\mu kw(z_0)}{w(z_0)+1}\right|^{\gamma}\\
&=\rho^{\beta}\left|\delta+\frac{\mu\rho k(\rho+\cos\theta)}{\rho^2+1+2\rho\cos\theta}+i\frac{\mu\rho k\sin\theta}{\rho^2+1+2\rho\cos\theta}\right|^{\gamma}\\
&=\rho^{\beta}\left(\delta^2+\mu\delta k+\frac{\mu\delta k(\rho^2-1)+\mu^2\rho^2k^2}{\rho^2+1+2\rho\cos\theta}\right)^{\frac{\gamma}{2}}\\
&\geqq\rho^{\beta}\left(\delta^2+\mu\delta k+\frac{\mu\delta k(\rho^2-1)+\mu^2\rho^2k^2}{\rho^2+1+2\rho}\right)^{\frac{\gamma}{2}}\\
&=\rho^{\beta}\left(\delta+\frac{\mu\rho k}{1+\rho}\right)^{\gamma}\\
&\geqq\rho^{\beta}\left(\delta+\frac{\mu\rho n}{1+\rho}\right)^{\gamma}.
\end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0\in\mathbb{U}$ such that $|w(z_0)|=\rho$. This means that $|w(z)|<\rho$ for all $z\in\mathbb{U}$.

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