General Mathematics Vol. 17, No. 4 (2009), 211–220

Univalence Criterion for Analytic Functions

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Abstract

In this paper, we obtain a new univalence criterion for analytic functions defined outside of the unit disk. Relevant connections of the results, which are presented in this paper with various known results are also considered.

2000 Mathematics Subject Classification: Primary 30C45.

Key words and phrases: Analytic function, univalence condition, Loewner chain.

1 Introduction

We denote by U_r the disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \leq 1$, by $U = U_1$ the open unit disk of the complex plane and by I the interval $[0, \infty)$.

Let A denote the class of analytic functions in the open unit disk U which satisfy the usual normalization condition:

$$g(0) = g'(0) - 1 = 0.$$

We denote by S the subclass of A consisting of functions which are also univalent in U.

Closely related to S is the class Σ_0 of the functions

$$f(z) = z + \sum_{k=0}^{\infty} b_k z^{-k} \tag{1}$$

analytic in the domain $U' := \{\xi \in \mathbb{C} : |\xi| > 1\}$ exterior to U, except for a simple pole at the infinity residue 1.

2 Preliminary results

In proving our results, we will need the following theorem due to Ch. Pommerenke [6,7].

Theorem 1 *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + ...,\ a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I, and locally uniform with respect to U_r. For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \forall z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies the condition $\Re\left(p(z,t)\right) > 0$ for all $z \in U,\ t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t) / a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

The following univalence criterion is due to Aksentév [1]. Later, Krzyz [4] gave quasiconformal extension for the functions.

Theorem 2 *(Aksentév, Krzyz). Let* $0 \le k \le 1$. *If* $f \in \sum_0$ *satisfies the inequality*

$$|f'(\xi) - 1| \le k, \quad \xi \in U',$$

then f *univalent. Furthermore, if* $k < 1$, *then* f *extends to a* $k-$*quasiconformal mapping of the extended complex plane. The radii* 1 *and* k *are best possible.*

In this paper we shall consider univalence conditions for functions $f \in \Sigma_0$ analytic in the domain $U' := \{\xi \in \mathbb{C} : \ |\xi| > 1\}$.

3 Main results

Making use of the Theorem 1 we can prove now, our main results.

Theorem 3 *Let* $s = \alpha + i\beta$ *and* c *be complex numbers such that* $\alpha > 0$ *and* $c \neq 1$, $|c| < 1$, *respectively. Suppose that* $f \in \Sigma_0$, $f'(\xi) \neq 0$ *and* $g(\xi) = 1 + c_2\xi^{-2} + \ldots$ *are two analytic in* U'. *If the following inequalities*

$$\left|(1-c)\left(\frac{\xi f'(\xi)}{f(\xi)}\frac{1}{g(\xi)}\right) - \frac{s}{\alpha}\right| < \frac{|s|}{\alpha} \tag{2}$$

and

$$\left|\frac{(|\xi|^{2/\alpha} - c)^2}{|\xi|^{2/\alpha}(1-c)}\frac{\xi f'(\xi)}{f(\xi)}\frac{1}{g(\xi)} - \frac{(|\xi|^{2/\alpha} - c)(|\xi|^{2/\alpha} - 1)}{|\xi|^{2/\alpha}(1-c)}\left[\frac{\xi f'(\xi)}{f(\xi)} + s\frac{\xi g'(\xi)}{g(\xi)}\right] - \frac{s}{\alpha}\right| \le \frac{|s|}{\alpha} \tag{3}$$

are satisfied for all $\xi \in U'$, *then the function* f *is univalent in* U'.

Proof. We prove that there exists a real number $r \in (0,1]$ such that the function $L : U_r \times I \to \mathbb{C}$, defined formally by

$$L(z,t) = \frac{1}{f(e^{st}/z)} \left\{ 1 - \frac{(e^{2t}-1)}{(e^{2t}-c)} g(e^{st}/z) \right\}^{-s} \tag{4}$$

is analytic in U_r for all $t \in I$.

Let us consider the function $\varphi_1(z,t)$ given by

$$\varphi_1(z,t) = g(e^{st}/z). \tag{5}$$

For all $t \in I$ and $z \in U$, the function $\varphi_1(z,t)$ is analytic in U and $\varphi_1(0,t) = 1$. Then there exist a disc U_{r_1}, $0 < r_1 < 1$, in which $\varphi_1(z,t) \neq 0$ for all $t \in I$ and $z \in U_{r_1}$.

For the function

$$\varphi_2(z,t) = \left\{ 1 - \frac{(e^{2t}-1)}{(e^{2t}-c)} \varphi_1(z,t) \right\}^{-s} \tag{6}$$

it can be easily shown that $\varphi_2(z,t)$ is analytic in U_{r_1} and $\varphi_2(0,t) = e^{2st} \left\{ \frac{1-ce^{-2t}}{1-c} \right\}^s$ for all $t \in I$. From these considerations it follows that the function

$$L(z,t) = \frac{1}{f(e^{st}/z)} \varphi_2(z,t) \tag{7}$$

is analytic in U_{r_1} for all $t \in I$ and has an following form

$$L(z,t) = a_1(t)z + \ldots.$$

Furthermore $|L'(0,t)| = \left| e^{st} \left\{ \frac{1-ce^{-2t}}{1-c} \right\}^s \right| = e^{\alpha t} \left| \left\{ \frac{1-ce^{-2t}}{1-c} \right\}^s \right|$ which is nonvanishing in I and tends to infinity for $t \to \infty$ once we have chosen a fixed branch for these numbers.

Thus $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\in I}$ forms a normal family of analytic functions in U_{r_2}, $0<r_2<r_1$. From the analyticity of $\frac{\partial L(z,t)}{\partial t}$, we obtain that for all fixed numbers $T>0$ and $r_3,\ 0<r_3<r_2$, there exists a constant $K>0$ (that depends on T and r_3) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right|<K, \forall z\in U_{r_3},\ t\in[0,T]\,.$$

Therefore, the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_3}.

The function $p(z,t)$ defined by

$$p(z,t)=z\frac{\partial L(z,t)}{\partial z}\Big/\frac{\partial L(z,t)}{\partial t}$$

is analytic in a disk $U_r,\ 0<r<r_3$, for all $t\in I$.
In order to prove that the function $p(z,t)$ has an analytic extension in U and $\Re p(z,t)>0$ for all $t\in I$, we will show that the function $w(z,t)$ given by

$$w(z,t)=\frac{p(z,t)-1}{p(z,t)+1} \tag{8}$$

has an analytic extension in U and $|w(z,t)|<1$, for all $z\in U$ and $t\in I$.
From equality (8) we have

$$w(z,t)=\frac{(1+s)\Omega(\xi,t)-2}{(1-s)\Omega(\xi,t)+2}, \tag{9}$$

where $\xi=\frac{1}{z}$and

$$\Omega(\xi,t)=\frac{1}{s}\frac{(e^{2t}-c)^2}{e^{2t}(1-c)}\frac{e^{st}\xi f'(e^{st}\xi)}{f(e^{st}\xi)}\frac{1}{g(e^{st}\xi)} \tag{10}$$

$$-\frac{(e^{2t}-c)(e^{2t}-1)}{e^{2t}(1-c)}\left(\frac{1}{s}\frac{e^{st}\xi f'(e^{st}\xi)}{f(e^{st}\xi)}+\frac{e^{st}\xi g'(e^{st}\xi)}{g(e^{st}\xi)}\right)$$

for $\xi \in U'$ and $t \in I$.

The inequality $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$, where $w(z,t)$ is defined by (9), is equivalent to

$$\left|\Omega(\xi,t)-\frac{1}{\alpha}\right| < \frac{1}{\alpha}, \quad \alpha = \Re(s),\ \forall \xi \in U',\ t \in I. \tag{11}$$

Define:

$$B(\xi,t)=\Omega(\xi,t)-\frac{1}{\alpha}, \quad \forall \xi \in U',\ t \in I.$$

From (2) and (10) we have

$$|B(\xi,0)| = \left|(1-c)\left(\frac{\xi f'(\xi)}{f(\xi)}\frac{1}{g(\xi)}\right)-\frac{s}{\alpha}\right| < \frac{|s|}{\alpha}. \tag{12}$$

Inequality (2) from the hypothesis, yields

$$|w(z,0)| < 1 \qquad (z \in U).$$

Let $t > 0$. Since $\left|\frac{e^{st}}{z}\right| \geq |e^{st}| = e^{\alpha t} > 1$ for all $z \in \overline{U} = \{z \in \mathbb{C} :\ |z| \leq 1\}$ and $t > 0$, it follows that $B(\xi,t)$ is an analytic function in $\overline{U'}$. Making use of the maximum modulus principle we obtain that for each $t > 0$ arbitrarily fixed there exists $\theta = \theta(t) \in \mathbb{R}$ such that:

$$|B(\xi,t)| < \max_{|\xi|=1}|B(\xi,t)| = \left|B(e^{i\theta},t)\right|, \tag{13}$$

for all $\xi \in U'$ and $t \in I$.

Denote $u = e^{st}e^{-i\theta}$. Then $|u| = e^{\alpha t}$, $e^{2t} = |u|^{2/\alpha}$ and from (10) we have

$$\left|B(e^{i\theta},t)\right| = \frac{1}{|s|}\left|\frac{(|u|^{2/\alpha}-c)^2}{|u|^{2/\alpha}(1-c)}\frac{uf'(u)}{f(u)}\frac{1}{g(u)}\right.$$

$$-\frac{(|u|^{2/\alpha}-c)(|u|^{2/\alpha}-1)}{|u|^{2/\alpha}(1-c)}\left[\frac{uf'(u)}{f(u)}+s\frac{ug'(u)}{g(u)}\right]-\frac{s}{\alpha}\Bigg|$$

Because $u \in U'$, the inequality (3) implies that

$$\left|B(e^{i\theta},t)\right| \leq \frac{1}{\alpha},$$

and from (12) and (13), we conclude that

$$|B(\xi,t)| = \left|\Omega(\xi,t)-\frac{1}{\alpha}\right| < \frac{1}{\alpha}$$

for all $\xi \in U'$ and $t \in I$. Therefore $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.

Since all the conditions of Theorem 1 are satisfied, we obtain that the function $L(z,t)$ has an analytic and univalent extension to the whole unit disk U, for all $t \in I$ and so is f because $L(z,0) = \frac{1}{f(z^{-1})}$ is analytic and univalent in U'.The proof of Theorem 3 has been completed.

The univalence criteria obtained by Becker and Ruscheweyh are contained in their expressions $|\xi|^2$, it is important that from Theorem 3 we obtain new results with $|\xi|^2$ instead of $|\xi|^{2/\alpha}$. If we set $\alpha \geq 1$ in Theorem 3, we obtain following theorem.

Theorem 4 *Let $s = \alpha + i\beta$ and c be complex numbers such that $\alpha \geq 1$ and $c \neq 1$, $|c| < 1$, respectively. Suppose that $f \in \Sigma_0$, $f'(\xi) \neq 0$ and $g(\xi) = 1 + c_2\xi^{-2} + ...$ are two analytic in U'. If the following inequalities*

$$\left|(1-c)\left(\frac{\xi f'(\xi)}{f(\xi)}\frac{1}{g(\xi)}\right)-\frac{s}{\alpha}\right| < \frac{|s|}{\alpha} \tag{14}$$

and

$$\left|\frac{(|\xi|^2-c)^2}{|\xi|^2(1-c)}\frac{\xi f'(\xi)}{f(\xi)}\frac{1}{g(\xi)}\right. \tag{15}$$

$$-\frac{(|\xi|^2-c)(|\xi|^2-1)}{|\xi|^2(1-c)}\left[\frac{\xi f'(\xi)}{f(\xi)}+s\frac{\xi g'(\xi)}{g(\xi)}\right]-\frac{s}{\alpha}\Bigg|\le\frac{|s|}{\alpha}$$

are satisfied for all $\xi\in U'$,*then the function* f *is univalent in* U'.

Next we will give another Theorem which contain some results.

If we take $g(\xi)=\frac{\xi f'(\xi)}{f(\xi)}$, in Theorem 4, then we have the following result.

Theorem 5 *Let* $s=\alpha+i\beta$ *and* c *be complex numbers such that* $\alpha\ge 1$ *and* $c\neq 1,\ |c|<1$, *respectively. Suppose that* $f\in\Sigma_0,\ f'(\xi)\neq 0$ *be analytic in* U'. *If the following inequalities*

$$|c\alpha+i\beta|<|s| \tag{16}$$

and

$$\Bigg|i\beta+\alpha\left(1-\frac{(|\xi|^2-c)^2}{|\xi|^2(1-c)}\right) \tag{17}$$

$$+\alpha\frac{(|\xi|^2-c)(|\xi|^2-1)}{|\xi|^2(1-c)}\left[(1-s)\frac{\xi f'(\xi)}{f(\xi)}+s(1+\frac{\xi f''(\xi)}{f'(\xi)})\right]\Bigg|\le|s|$$

are satisfied for all $\xi\in U'$,*then the function* f *is univalent in* U'.

Now we will give important results which are obtained by earlier authors.

For $c=0$, $(f\in\Sigma_0, b_0=0)$ in Theorem 5, we obtain closely related to Ruscheweyh's univalence criterion [5].

Corollary 1 *Let* $s=\alpha+i\beta$ *be complex number such that* $\alpha\ge 1$. *Suppose that* $f\in\Sigma_0$ *be analytic in* U'. *If the following inequality*

$$\left|i\beta+\alpha(1-|\xi|^2)\left[(1-s)(1-\frac{\xi f'(\xi)}{f(\xi)})-s(\frac{\xi f''(\xi)}{f'(\xi)})\right]\right|\le|s|$$

is satisfied for all $\xi\in U'$,*then the function* f *is univalent in* U'.

For $s = 1$ in Theorem 5 we obtain Becker's univalence criterion [3].

Corollary 2 *Suppose that $f(\xi) \in \Sigma_0$ is analytic in U' and for some $c \neq 1$, $|c| < 1$, it satisfies the condition*

$$\left| \frac{(|\xi|^2 - c)(|\xi|^2 - 1)}{(1-c)} \frac{\xi f''(\xi)}{f'(\xi)} + c \right| \leq |\xi|^2$$

then the function f is univalent in U'.

For $s = 1$ and $c = 0$ in Theorem 5 we obtain Becker's another univalence criterion [2].

Corollary 3 *Let $f(\xi) \in \Sigma_0$ be analytic in U'.If the following inequality*

$$(|\xi|^2 - 1) \left| \frac{\xi f''(\xi)}{f'(\xi)} \right| \leq 1$$

is satisfied for all $\xi \in U'$,then the function f is univalent in U'.

For $s = 1$, $c = 0$ and $g(\xi) = \frac{\xi}{f(\xi)}$ in Theorem 4 we obtain Theorem 2 (for $k = 1$)

Corollary 4 *Let $f \in \Sigma_0$ be analytic in U'. If the following inequality*

$$|f'(\xi) - 1| < 1$$

is satisfied for all $\xi \in U'$,then the function f is univalent in U'.

For $c = 0$ and $g(\xi) = 1$ in Theorem 4 then we obtain a simple univalence condition.

Corollary 5 *Let $s = \alpha + i\beta$ be a complex number such that $\alpha \geq 1$. Let $f \in \Sigma_0$ be analytic in U'. If the following inequality*

$$\left| \frac{\xi f'(\xi)}{f(\xi)} - \frac{s}{\alpha} \right| \leq \frac{|s|}{\alpha}$$

is satisfied for all $\xi \in U'$,then the function f is univalent in U'.

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