General Mathematics Vol. 18, No. 1 (2010), 3–17

Natural splines of Birkhoff type approximating the solution of differential equations [1]

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Abstract

Let us consider the following initial value problem

$$(1) \qquad \begin{cases} y^{(r)} = f(x, y, y', \dots, y^{(r-1)}) \\ y^{(j)}(x_1) = y_1^{(j)},\ j = \overline{0, r-1} \end{cases}$$

and suppose that $f : D \to \mathbb{R}$, $D \subset [a, b] \times \mathbb{R}^r$ satisfies all the conditions assuring the existence and uniqueness of the solution $y : [a, b] \to \mathbb{R}$ of the problem (1).

If y is the exact solution of the problem (1), denote by

$$\begin{aligned} \tilde{U}_y \ &= \ \Big\{u \in H^{m,2}[a, b] \,|\, u^{(j)}(x_1) = y^{(j)}(x_1), j = \overline{0, r-1}, \\ &\quad u^{(r)}(x_i) = y^{(r)}(x_i),\ i = \overline{2, n}\Big\}, \end{aligned}$$

where $x_1, x_2, \dots, x_n$ are distinct knots in the interval $[a, b]$, $a \le x_1 < x_2 < \dots < x_n \le b$.

[1] *Received 15 September, 2009*

Accepted for publication (in revised form) 14 October, 2009

We will give an algorithm to approximate the solution y of (1) by a Birkhoff-type natural spline interpolation function, s_y, corresponding to the interpolatory set $\tilde{U}_y$.

2000 Mathematics Subject Classification: 65D07, 65D05
Key words and phrases: Natural splines of Birkhoff type, differential equations, interpolation.

1 Introduction

Let us consider the following initial value problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{2}$$

and suppose that $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ satisfies all the conditions assuring the existence and uniqueness of the solution $y : [a, b] \to \mathbb{R}$ of the problem (2).

In paper [2] G. Micula and P. Blaga develop an algorithm to approximate the solution y of (2) by a spline function of even degree.

Using natural splines of Birkhoff type we will give an algorithm to approximate the solution of the differential equation problem

$$\begin{cases} y^{(r)} = f(x, y, y', \ldots, y^{(r-1)}) \\ y^{(j)}(x_1) = y_1^{(j)},\ j = \overline{0, r-1} \end{cases}$$

and suppose that $f : D \to \mathbb{R}$, $D \subset [a, b] \times \mathbb{R}^r$ satisfies all the conditions assuring the existence and uniqueness of the solution $y : [a, b] \to \mathbb{R}$ of the problem.

Let us consider an arbitrary finite interval $[a, b]$, $a < b$, on the real line and the Lebesgue space $L^2[a, b]$ with the corresponding norm

$$\|f\|_2^2 := \int_a^b f^2(x)dx.$$

Denote by

$$H^{m,2}[a,b] = \left\{ f : [a,b] \to \mathbb{R} \mid f \in C^{m-1}[a,b],\ f^{(m-1)} abs.cont.,\ f^{(m)} \in L^2[a,b] \right\}.$$

Let us take $x_1, x_2, \ldots, x_n$ as distinct knots in the interval $[a, b]$, $a \le x_1 < x_2 < \cdots < x_n \le b$, the numbers $\alpha_i \in \mathbb{N}$, where $1 \le \alpha_i \le m$, $i = \overline{1, n}$ and the set $I_i \subseteq \{0, 1, \ldots, \alpha_i - 1\}$, $i = \overline{1, n}$.

Denote the number of interpolation condition by $N := \sum_{i=1}^{n} card(I_i)$.

Definition 1 *The set*

$$\Lambda := \left\{ \lambda_{i,\nu_i} : H^{m,2}[a,b] \to \mathbb{R},\ \lambda_{i,\nu_i}(f) = f^{(\nu_i)}(x_i),\ i = \overline{1,n},\ \nu_i \in I_i \right\}$$

is named a set of Birkhoff-type functionals on $H^{m,2}[a, b]$.

Definition 2 *For each $y = (y_1, y_2, \cdots, y_N) \in \mathbb{R}^N$ we define the Birkhoff-type interpolatory set*

$$U_y := \left\{ u \in H^{m,2}[a,b] \mid u^{(\nu_i)}(x_i) = y_{i,\nu_i},\ i = \overline{1,n},\ \nu_i \in I_i \right\},$$

where $y := (y_1, y_2, \ldots, y_n) = \left((y_{1,\nu_1})_{\nu_1 \in I_1}, (y_{2,\nu_2})_{\nu_2 \in I_2}, \ldots, (y_{n,\nu_n})_{\nu_n \in I_n} \right).$

Definition 3 *The problem of finding function $s \in U_y$ which satisfy*

$$\left\| s^{(m)} \right\|_2 = \min_{u \in U_y} \left\| u^{(m)} \right\|_2 \tag{3}$$

is called Birkhoff-type natural spline interpolation problem, corresponding to the interpolatory set U_y.

Lemma 1 *For each $y \in \mathbb{R}^N$, if the set U_y is nonempty, then problem (3) has unique solution s_y given by*

$$s_y(x) = \sum_{k=0}^{m-1} a_k \frac{(b-x)^k}{k!} + \sum_{i=1}^{n} \sum_{\nu_i \in I_i} b_{i,\nu_i} \frac{(x-x_i)_+^{2m-1-\nu_i}}{(2m-1-\nu_i)!},$$

where the coefficients a_k, $k=\overline{0,m-1}$, b_{i,ν_i}, $i=\overline{1,n}$, $\nu_i \in I_i$, are given by the following linear system

$$\begin{cases} \sum\limits_{k=\mu_1}^{m-1} a_k (-1)^{\mu_1} \frac{(b-x_1)^{k-\mu_1}}{(k-\mu_1)!} = y_{1,\mu_1}, \ \mu_1 \in I_1 \\ \sum\limits_{k=\mu_j}^{m-1} a_k (-1)^{\mu_j} \frac{(b-x_j)^{k-\mu_j}}{(k-\mu_j)!} + \sum\limits_{i=1}^{j-1} \sum\limits_{\nu_i \in I_i} b_{i,\nu_i} \frac{(x_j-x_i)^{2m-1-\nu_i-\mu_i}}{(2m-1-\nu_i-\mu_j)!} = y_{j,\mu_j}, \\ j=\overline{2,n}, \ \mu_j \in I_j, \\ \sum\limits_{i=1}^{n} \sum\limits_{\nu_i \in I_i, \nu_i \le l} b_{i,\nu_i} \frac{(b-x_i)^{l-\nu_i}}{(l-\nu_i)!} = 0, \ l=\overline{0,m-1}. \end{cases}$$

Definition 4 *For each $j=\overline{1,n}$ and $\mu_j \in I_j$, let $y_{j,\mu_j} \in \mathbb{R}^N$ be defined by*

$$y_{j,\mu_j} := (\delta_{ij}\delta_{\nu_i,\mu_j})_{i=\overline{1,n},\nu_i \in I_i} = \left((\delta_{1,j}\delta_{\nu_1,\mu_j})_{\nu_1 \in I_1}, \ldots, (\delta_{n,j}\delta_{\nu_n,\mu_j})_{\nu_n \in I_n}\right)$$

and the corresponding interpolatory set

$$U_{j,\mu_j} := \left\{u \in H^{m,2}[a,b] \mid u^{(\nu_i)}(x_i) = \delta_{ij}\delta_{\nu_i,\mu_j}, i=\overline{1,n}, \nu_i \in I_i\right\}.$$

We assume that the sets U_{j,μ_j}, $j=\overline{1,n}$, $\mu_j \in I_j$ are nonempty. For each $j=\overline{1,n}$ and $\mu_j \in I_j$, denote by s_{j,μ_j} the unique solution of problem (3) corresponding to the interpolatory set U_{j,μ_j}. The functions s_{j,μ_j} are called the fundamental Birkhoff-type natural spline interpolation functions.

Lemma 2 *For each $y \in \mathbb{R}^N$ the set U_y is nonempty and the problem (3), corresponding to U_y, has unique solution given by*

$$s_y = \sum_{i=1}^{n} \sum_{\nu_i \in I_i} y_{i,\nu_i} s_{i,\nu_i},$$

where $y := (y_1, \dots, y_n) = \left((y_{1,\nu_1})_{\nu_1 \in I_1}, \dots, (y_{n,\nu_n})_{\nu_n \in I_n}\right).$

Remark 1 *For each $f \in H^{m,2}[a,b]$ can be considered*

$$y := \left(f^{(\nu_i)}(x_i)\right)_{i=\overline{1,n}, \nu_i \in I_i} = \left((f^{(\nu_1)}(x_1))_{\nu_1 \in I_1}, \dots, (f^{(\nu_n)}(x_n))_{\nu_n \in I_n}\right),$$

and the corresponding interpolatory set

$$U_f := \left\{u \in H^{m,2}[a,b] \mid u^{(\nu_i)}(x_i) = f^{(\nu_i)}(x_i), i = \overline{1,n}, \nu_i \in I_i\right\}.$$

Since $f \in U_f$, the interpolatory set U_f is nonempty and the problem (3) coresponding to U_f has unique solution for each $f \in H^{m,2}[a,b]$.

2 Main Results

Let us consider the following initial value problem

$$\text{(4)} \qquad \begin{cases} y^{(r)} = f(x, y, y', \dots, y^{(r-1)}) \\ y^{(j)}(x_1) = y_1^{(j)}, \ j = \overline{0, r-1} \end{cases}$$

and suppose that $f : D \subset \mathbb{R}^{r+1} \to \mathbb{R}$ satisfies all the conditions assuring the existence and uniqueness of the solution $y : [a,b] \to \mathbb{R}$ of the problem (4).

If y is the exact solution of the problem (4), denote by

$$\tilde{U}_y = \left\{u \in H^{m,2}[a,b] \mid u^{(j)}(x_1) = y^{(j)}(x_1), j = \overline{0, r-1}, u^{(r)}(x_i) = y^{(r)}(x_i), i = \overline{2,n}\right\}.$$

From Lemma 2 follow that there exist, and it is unique Birkhoff-type natural spline interpolation function, s_y, coresponding to the interpolatory set $\tilde{U}_y$, namely

$$\begin{aligned} s_y^{(j)}(x_1) &= y^{(j)}(x_1),\ j=\overline{0,r-1} \\ s_y^{(r)}(x_i) &= y^{(r)}(x_i),\ i=\overline{2,n}. \end{aligned}$$

The spline function s_y has the representation

$$s_y(x)=\sum_{j=0}^{r-1} y_1^{(j)} s_{1,j}(x)+\sum_{i=2}^{n} f(x_i,y_i^{(0)},y_i^{(1)},\dots,y_i^{(r-1)})s_{i,r}(x) \tag{5}$$

where the unknown values $y_i^{(0)},y_i^{(1)},\dots,y_i^{(r-1)}$ will be determined.

The functions $s_{j,\mu_j},\ j=\overline{1,n},\ \mu_j\in I_j$, where $I_1=\{0,1,\dots,r-1\}$, $I_i=\{r\}\,,\ i=\overline{2,n}$ are the fundamental Birkhoff-type natural spline interpolation functions and are given by

$$s_{j,\mu_j}(x)=\sum_{k=0}^{m-1} a_k^{(j)}\frac{(b-x)^k}{k!}+\sum_{i=0}^{r-1} b_{1,i}^{(j)}\frac{(x-x_1)_+^{2m-1-i}}{(2m-1-i)!}+\sum_{i=2}^{n} b_{i,r}^{(j)}\frac{(x-x_i)_+^{2m-1-r}}{(2m-1-r)!}$$

where the coefficients $a_k^{(j)}$, $k=\overline{0,m-1}$, $b_{1,i}^{(j)}$, $i=\overline{0,r-1}$, $b_{i,r}^{(j)}$, $i=\overline{2,n}$ are given by the following linear system

$$\begin{cases} \displaystyle\sum_{k=i}^{m-1} a_k^{(j)}(-1)^i\frac{(b-x_1)^{k-i}}{(k-i)!}=\delta_{1,j}\delta_{i,\mu_j},\ i=\overline{0,r-1} \\ \displaystyle\sum_{k=r}^{m-1} a_k^{(j)}(-1)^r\frac{(b-x_\nu)^{k-r}}{(k-r)!}+\sum_{i=0}^{r-1} b_{1,i}^{(j)}\frac{(x_\nu-x_1)^{2m-1-r-i}}{(2m-1-r-i)!} \\ \displaystyle+\sum_{i=2}^{\nu-1} b_{i,r}^{(j)}\frac{(x_\nu-x_i)_+^{2m-1-2r}}{(2m-1-2r)!}=\delta_{\nu,j}\delta_{r,\mu_j},\ \nu=\overline{2,n} \\ \displaystyle\sum_{i=0,i\le l}^{r-1} b_{1,i}^{(j)}\frac{(b-x_1)^{l-i}}{(l-i)!}+\sum_{i=2,r\le l}^{n} b_{i,r}^{(j)}\frac{(b-x_i)^{l-r}}{(l-r)!}=0,\ l=\overline{0,m-1}, \end{cases} \tag{6}$$

Theorem 1 *If y is the exact solution of the problem (4), $y \in H^{m+1,2}[a,b]$ and s_y is the spline approximating solution (5), then the following estimations*

$$\left\|y^{(k)}-s_y^{(k)}\right\|_\infty \leq \sqrt{m-r+1}(m-r)(m-r-1)\dots(k+1-r)\left\|\Delta_n\right\|^{m-k+1/2}\left\|y^{(m+1)}\right\|_2$$

hold, for $k=\overline{r,m}$, and $\|\Delta_n\| := \max_{i=\overline{0,n}} \{x_{i+1}-x_i\}$.

Proof. Because $y^{(r)}(x_i) - s^{(r)}(x_i) = 0,\ i = \overline{2,n}$, there exist the points $(x_i^{(1)}),\ x_i < x_i^{(1)} < x_{i+1}, i = \overline{2,n-1}$, such that

$$y^{(r+1)}(x_i^{(1)}) - s_y^{(r+1)}(x_i^{(1)}) = 0,\ i = \overline{2,n-1}$$

hold.

Now, because $y^{(r+1)}(x_i^{(1)}) - s_y^{(r+1)}(x_i^{(1)}) = 0,\ i = \overline{2,n-1}$, there exist a set of points $(x_i^{(2)})$, $x_i^{(1)} < x_i^{(2)} < x_{i+1}^{(1)}, i = \overline{2,n-2}$ such that

$$y^{(r+2)}(x_i^{(2)}) - s_y^{(r+2)}(x_i^{(2)}) = 0,\ i = \overline{2,n-2}.$$

Continuing in the same manner, from the relations $y^{(k)}(x_i^{(k-r)})-s_y^{(k)}(x_i^{(k-r)}) = 0$, for $i = \overline{2,n-k+r}$, there exist a set of points $(x_i^{(k-r+1)}),\ x_i^{(k-r)} < x_i^{(k-r+1)} < x_{i+1}^{(k-r)}$, $i = \overline{2,n-k+r-1}$, such that

$$y^{(k+1)}(x_i^{(k+1-r)}) - s_y^{(k+1)}(x_i^{(k+1-r)}) = 0,\ i = \overline{2,n-k-1+r}$$

hold. Finally, for $k = m-1$, there exist a set of points $(x_i^{(m-r)}),\ x_i^{(m-1-r)} < x_i^{(m-r)} < x_{i+1}^{(m-1-r)},\ i = \overline{2,n-m+r}$ such that the relation

$$y^{(m)}(x_i^{(m-r)}) - s_y^{(m)}(x_i^{(m-r)}) = 0,\ i = \overline{2,n-m+r}$$

hold.

Because $\left|x_{i+1}^{(k)}-x_i^{(k)}\right| \leq (k+1)\left\|\Delta_n\right\|$ for $k=\overline{0,m-r}$, $x_i^{(0)}=x_i$ it is clear that for any $x\in[a,b]$ there exist an index i_0 such that

$$\left|x-x_{i_0}^{(m-r)}\right| \leq (m-r+1)\left\|\Delta_n\right\|.$$

Thus

$$\begin{aligned}
\left|y^{(m)}(x)-s_y^{(m)}(x)\right| &= \left|\int_{x_{i_0}^{(m-r)}}^{x}\left[y^{(m+1)}(t)-s_y^{(m+1)}(t)\right]dt\right| \\
&\leq \left|\int_{x_{i_0}^{(m-r)}}^{x} dt\right|^{1/2}\left|\int_{x_{i_0}^{(m-r)}}^{x}\left[y^{(m+1)}(t)-s_y^{(m+1)}(t)\right]^2 dt\right|^{1/2} \\
&\leq \left\{(m-r+1)\left\|\Delta_n\right\|\right\}^{1/2}\left(\int_a^b\left[y^{(m+1)}(t)-s_y^{(m+1)}(t)\right]^2 dt\right)^{1/2} \\
&= \left\{(m-r+1)\left\|\Delta_n\right\|\right\}^{1/2}\left\|y^{(m+1)}-s_y^{(m+1)}\right\|_2 \\
&\leq \left((m-r+1)\left\|\Delta_n\right\|\right)^{1/2}\left\|y^{(m+1)}\right\|_2
\end{aligned}$$

and $\left\|y^{(m)}-s_y^{(m)}\right\|_\infty \leq \sqrt{m-r+1}\sqrt{\left\|\Delta_n\right\|}\left\|y^{(m+1)}\right\|_2$.

In the same manner, for any $x\in[a,b]$ there exists an index i_0 such that

$$\left|x-x_{i_0}^{(m-r-1)}\right| \leq (m-r)\left\|\Delta_n\right\|$$

holds, and

$$\begin{aligned}
&\left|y^{(m-1)}(x)-s_y^{(m-1)}(x)\right| = \left|\int_{x_{i_0}^{(m-r-1)}}^{x}\left[y^{(m)}(t)-s_y^{(m)}(t)\right]dt\right| \\
&\leq \left\|y^{(m)}-s_y^{(m)}\right\|_\infty\left|x-x_{i_0}^{(m-r-1)}\right| \\
&\leq \sqrt{m-r+1}\sqrt{\left\|\Delta_n\right\|}\cdot\left\|y^{(m+1)}\right\|_2(m-r)\cdot\left\|\Delta_n\right\|.
\end{aligned}$$

That means $\left\|y^{(m-1)}-s_y^{(m-1)}\right\|_\infty \leq \sqrt{m-r+1}(m-r)\left\|\Delta_n\right\|^{1+1/2}\left\|y^{(m+1)}\right\|_2$.

For any $x\in[a,b]$ there exist an index i_0 such that

$$\left|x-x_{i_0}^{(m-r-2)}\right| \leq (m-r-1)\left\|\Delta_n\right\|$$

holds, and

$$\left|y^{(m-2)}(x)-s_y^{(m-2)}(x)\right| = \left|\int_{x_{i_0}^{(m-r-2)}}^{x}\left[y^{(m-1)}(t)-s_y^{(m-1)}(t)\right]dt\right|$$
$$\leq\left\|y^{(m-1)}-s_y^{(m-1)}\right\|_{\infty}\left|x-x_{i_0}^{(m-r-2)}\right|\leq\sqrt{m-r+1}(m-r)\left\|\Delta_n\right\|^{1+1/2}\left\|y^{(m+1)}\right\|_2$$
$$\cdot(m-r-1)\left\|\Delta_n\right\| = \sqrt{m-r+1}(m-r)(m-r-1)\left\|\Delta\right\|^{2+1/2}\cdot\left\|y^{(m+1)}\right\|_2,$$

namely

$$\left\|y^{(m-2)}-s_y^{(m-2)}\right\|_{\infty}\leq\sqrt{m-r+1}(m-r)(m-r-1)\left\|\Delta_n\right\|^{2+1/2}\left\|y^{(m+1)}\right\|_2.$$

In generally

$$\begin{aligned}\left\|y^{(k)}-s_y^{(k)}\right\|_{\infty} &\leq \sqrt{m-r+1}(m-r)(m-r-1)\dots(k+1-r)\\ &\cdot \left\|\Delta_n\right\|^{m-k+1/2}\left\|y^{(m+1)}\right\|_2,\ k=\overline{r,m}.\end{aligned}$$

Corollary 1 *If the exact solution of the problem (4) $y\in H^{m+1,2}[a,b]$ and s_y is the spline approximating solution, then*
$$\left\|y^{(k)}-s_y^{(k)}\right\|_{\infty}\leq(b-a)^{r-k}\sqrt{m-r+1}(m-r)!\left\|\Delta_n\right\|^{m-r+1/2}\left\|y^{(m+1)}\right\|_2,$$
$k=\overline{0,r-1}$.

Proof. We have

$$\left|y^{(r-1)}(x)-s_y^{(r-1)}(x)\right| = \left|\int_{x_1}^{x}\left[y^{(r)}(t)-s_y^{(r)}(t)\right]dt\right|\leq|x-x_1|\cdot\left\|y^{(r)}-s_y^{(r)}\right\|_{\infty}$$
$$\leq(b-a)\sqrt{m-r+1}(m-r)!\left\|\Delta_n\right\|^{m-r+1/2}\left\|y^{(m+1)}\right\|_2.$$

From the above relation we obtain

$$\left\|y^{(r-1)}-s_y^{(r-1)}\right\|_{\infty}\leq(b-a)\sqrt{m-r+1}(m-r)!\left\|\Delta_n\right\|^{m-r+1/2}\left\|y^{(m+1)}\right\|_2.$$

In generally, for $k=\overline{0,r-1}$

$$\left\|y^{(k)}-s_y^{(k)}\right\|_{\infty}\leq(b-a)^{r-k}\sqrt{m-r+1}(m-r)!\left\|\Delta_n\right\|^{m-r+1/2}\left\|y^{(m+1)}\right\|_2.$$

Corollary 2 *If the exact solution of the problem (4) $y \in H^{m+1,2}[a,b]$ and s_y is the spline approximating solution then*

$$\lim_{\|\Delta_n\|\to 0} \left\| y^{(k)} - s_y^{(k)} \right\|_\infty = 0 \quad \textit{for} \quad k = \overline{0,m}$$

hold and the order of convergence is

$$\begin{cases} m-k+1/2 & \textit{for} \quad k = \overline{r,m}, \\ m-r+1/2 & \textit{for} \quad k = \overline{0,r-1}. \end{cases}$$

Now the problem is to construct effectively the approximating spline function (5), i.e. to find an algorithm for determination the value $s_y^{(j)}(x_i)$, $i = \overline{2,n},\ j = \overline{0,r-1}$, considering that $s_y(x) \approx y(x)$, for any $x \in [a,b]$.

Let denote the error of the method, as usual by

$$e(x) = y(x) - s_y(x),\ x \in [a,b].$$

Directly from the Corollary 1, if $y \in H^{m+1,2}[a,b]$, we have

$$\left| e^{(j)}(x) \right| \le (b-a)^{r-j} \sqrt{m-r+1}(m-r)! \, \|\Delta_n\|^{m-r+1/2} \cdot \left\| y^{(m+1)} \right\|_2,$$

namely $e^{(j)}(x) = O\left(\|\Delta_n\|^{m-r+1/2} \right)$ $(\forall)\ x \in [a,b]$.

Writing again

$$s_y(x) = \sum_{j=0}^{r-1} y_1^{(j)} s_{1,j}(x) + \sum_{k=2}^{n} f\left(x_k, y_k^{(0)}, y_k^{(1)}, \ldots, y_k^{(r-1)} \right) s_{k,r}(x)$$

and denoting $s_y^{(j)}(x_i) =: w_i^j,\ e^{(j)}(x_i) =: e_i^j,\ i = \overline{2,n},\ j = \overline{0,r-1}$, it follows

(7)
$$w_i^j = \sum_{k=0}^{r-1} y_1^{(k)} s_{1,k}^{(j)}(x_i) + \sum_{k=2}^{n} s_{k,r}^{(j)}(x_i) f\left(x_k, e_k^0 + w_k^0, e_k^1 + w_k^1, \ldots, e_k^{r-1} + w_k^{r-1} \right),$$

$i=\overline{2,n},\ j=\overline{0,r-1}$.

We suppose that in (4) the function $f: D\subset \mathbb{R}^{r+1}\to\mathbb{R},\ D\subset [a,b]\times\mathbb{R}^r$ is a continuous function, and also that derivatives $\dfrac{\partial f(x,u_0,u_1,\ldots,u_{r-1})}{\partial u_i}$, $i=\overline{0,r-1}$ are continuous function on the domain D. Thus

$$f(x_k,y_k^{(0)},y_k^{(1)},\ldots,y_k^{(r-1)})=f\left(x_k,e_k^0+w_k^0.e_k^1+w_k^1,\ldots,e_k^{r-1}+w_k^{r-1}\right)$$
$$=f\left(x_k,w_k^0,w_k^1,\ldots,w_k^{r-1}\right)+\sum_{i=0}^{r-1}e_k^i\frac{\partial f(x_k,\xi_k^0,\xi_k^1,\xi_k^{r-1})}{\partial u_i},$$

where $\min(w_k^i,w_k^i+e_k^i)<\xi_k^i<\max(w_k^i,w_k^i+e_k^i)$.

We can write the system (7) in the form

$$w_i^j=\sum_{k=0}^{r-1}y_1^{(k)}s_{1,k}^{(j)}(x_i)+\sum_{k=2}^{n}s_{k,r}^{(j)}(x_i)f\left(x_k,w_k^0,w_k^1,\ldots,w_k^{r-1}\right)+E_i^j,\text{ where}$$

$$E_i^j=\sum_{k=2}^{n}s_{k,r}^{(j)}(x_i)\sum_{i=0}^{r-1}e_k^i\frac{\partial f(x_k,\xi_k^0,\xi_k^1,\ldots,\xi_k^{r-1})}{\partial u_i},\ i=\overline{2,n},\ j=\overline{0,r-1}$$

supposing that $\left|\dfrac{\partial f(t,u_0,u_1,\ldots,u_{r-1})}{\partial u_i}\right|\le M,\ i=\overline{0,r-1}$ on D. Obviously $E_i^j\to 0$ for $\|\Delta_n\|\to 0$.

We have o solve the following nonlinear system:

(8)
$$w_i^j=\sum_{k=0}^{r-1}y_1^{(k)}s_{1,k}^{(j)}(x_i)+\sum_{k=2}^{n}s_{k,r}^{(j)}(x_i)f(x_k,w_k^0,w_k^1,\ldots,w_k^{r-1}),\ i=\overline{2,n},\ j=\overline{0,r-1},$$

Denote by

$$\begin{aligned}
w^0&=(w_2^0\ w_3^0\ \cdots\ w_n^0)\\
w^1&=(w_2^1\ w_3^1\ \cdots\ w_n^1)\\
&\vdots\\
w^{r-1}&=(w_2^{r-1}\ w_3^{r-1}\ \cdots\ w_n^{r-1})
\end{aligned}$$

$$W = (w^0 \ w^1 \ \dots \ w^{r-1})$$
$$H_i^j(W) = \sum_{k=0}^{r-1} y_1^{(k)} s_{1,k}^{(j)}(x_i) + \sum_{k=2}^{n} s_{k,r}^{(j)}(x_i) f(x_k, w_k^0, w_k^1, \dots, w_k^{r-1}),$$
$$H(W) = \big(H_2^0(W) \dots H_n^0(W), H_2^1(W) \dots H_n^1(W), H_2^{r-1}(W) \dots H_n^{r-1}(W)\big)$$
$$= \big(H_i^j(W)\big)_{j=\overline{0,r-1}, i=\overline{2,n}}$$

and

$$A = \begin{pmatrix} \frac{\partial H_2^0(W)}{\partial w_2^0} & \dots & \frac{\partial H_2^0(W)}{\partial w_n^0} & \dots & \frac{\partial H_2^0(W)}{\partial w_2^{r-1}} & \dots & \frac{\partial H_2^0(W)}{\partial w_n^{r-1}} \\ \vdots & & & & & & \\ \frac{\partial H_n^0(W)}{\partial w_2^0} & \dots & \frac{\partial H_n^0(W)}{\partial w_n^0} & \dots & \frac{\partial H_n^0(W)}{\partial w_2^{r-1}} & \dots & \frac{\partial H_n^0(W)}{\partial w_n^{r-1}} \\ \vdots & & & & & & \\ \frac{\partial H_2^{r-1}(W)}{\partial w_2^0} & \dots & \frac{\partial H_2^{r-1}(W)}{\partial w_n^0} & \dots & \frac{\partial H_2^{r-1}(W)}{\partial w_2^{r-1}} & \dots & \frac{\partial H_2^{r-1}(W)}{\partial w_n^{r-1}} \\ \vdots & & & & & & \\ \frac{\partial H_n^{r-1}(W)}{\partial w_2^0} & \dots & \frac{\partial H_n^{r-1}(W)}{\partial w_n^0} & \dots & \frac{\partial H_n^{r-1}(W)}{\partial w_2^{r-1}} & \dots & \frac{\partial H_n^{r-1}(W)}{\partial w_n^{r-1}} \end{pmatrix}$$

We remark that $A = SF$, where

$$S = \begin{pmatrix} s_{2,r}(x_2) & \dots & s_{n,r}(x_2) & \dots & s_{2,r}(x_2) & \dots & s_{n,r}(x_2) \\ \vdots & & & & & & \\ s_{2,r}(x_n) & \dots & s_{n,r}(x_n) & \dots & s_{2,r}(x_n) & \dots & s_{n,r}(x_n) \\ \vdots & & & & & & \\ s_{2,r}^{(r-1)}(x_2) & \dots & s_{n,r}^{(r-1)}(x_2) & \dots & s_{2,r}^{(r-1)}(x_2) & \dots & s_{n,r}^{(r-1)}(x_2) \\ \vdots & & & & & & \\ s_{2,r}^{(r-1)}(x_n) & \dots & s_{n,r}^{(r-1)}(x_n) & \dots & s_{2,r}^{(r-1)}(x_n) & \dots & s_{n,r}^{(r-1)}(x_n) \end{pmatrix}$$

and

$$F=\begin{pmatrix} \frac{\partial f(x_2,w_2^0,w_2^1,\ldots,w_2^{r-1})}{\partial w_2^0} & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \frac{\partial f(x_n,w_n^0,w_n^1,\ldots,w_n^{r-1})}{\partial w_n^0} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & \ldots & \frac{\partial f(x_n,w_n^0,w_n^1,\ldots,w_n^{r-1})}{\partial w_n^{r-1}} \end{pmatrix}$$

To obtain our results we will use the following theorem

Theorem 2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $H : \Omega \to \Omega$ be a vector function defined by $(w_1, \ldots, w_n) \to (H_1(w_1, \ldots, w_n), \ldots, H_n(w_1, \ldots, w_n))$. Denote by $A(w)$ the following matric*

$$A(w) = A(w_1, w_2, \ldots, w_n) := \left(\frac{\partial H_i(w)}{\partial w_j}\right)_{i,j=\overline{1,n}}.$$

If the functions H and $\frac{\partial H_i}{\partial w_j}, i, j = \overline{1,n}$ are continuous in Ω, then there exist in Ω a fix-point w^* of the application H, i.e $w^* = H(w^*)$, which can be found by iteration $w^* = \lim_{n\to\infty} w^{(n)}$, $w^{(k)} := H(w^{(k-1)})$, $k = 1, 2, \ldots$, $w^{(0)} \in \Omega$ (arbitrary).

If in addition $\|A\| \leq q < 1$, for any iteration $w^{(k)}$, the following estimation holds

$$\left\|w^{(k)} - w^*\right\| \leq \frac{q^k}{1-q}\left\|w^{(1)} - w^{(0)}\right\|.$$

Theorem 3 *Suppose that*

$$\left\|\frac{\partial f(x, u_0, u_1, \ldots, u_{r-1})}{\partial u_i}\right\| \leq M,\ i = \overline{0, r-1},$$
$$|f(x, u_0, u_1, \ldots, u_{r-1})| \leq N\,(\forall)\,(x, u_0, u_1, \ldots, u_{r-1}) \in D.$$

If $M < \|S\|^{-1}$, then the system (8) has a solution which can be determined by iteration.

Proof. From (8) we can write

$$\begin{aligned} \left|w_i^j\right| &\leq \sum_{k=0}^{r-1} \left|y_1^{(k)}\right| \cdot \left|s_{1,k}^{(j)}(x_i)\right| + \sum_{k=2}^{n} \left|s_{k,r}^{(j)}(x_i)\right| \cdot \left|f(x_k, w_k^0, \ldots, w_k^{r-1})\right| \\ &\leq \sum_{k=0}^{r-1} \left|y_1^{(k)}\right| \cdot \left|s_{1,k}^{(j)}(x_i)\right| + N \sum_{k=2}^{n} \left|s_{k,r}^{(j)}(x_i)\right|, \ i = \overline{2,n}, \ j = \overline{0,r-1}. \end{aligned}$$

Now we can apply the Theorem 2, where the domain Ω is defined by

$$\Omega = \Big\{(w_2^0, \ldots, w_n^0, w_2^1, \ldots, w_n^1, \ldots, w_2^{r-1}, \ldots, w_n^{r-1}) \subset \mathbb{R}^{r(n-1)} \mid \\ \left|w_i^j\right| \leq \sum_{k=0}^{r-1} \left|y_1^{(k)}\right| \cdot \left|s_{1,k}^{(j)}(x_i)\right| + N \sum_{k=2}^{n} \left|s_{k,r}^{(j)}(x_i)\right|, \ j = \overline{0,r-1}, \ i = \overline{2,n} \Big\}.$$

It is clear that

$$\|A\| = \|SF\| \leq \|S\| \cdot \|F\| \leq \|S\| \cdot M \leq 1,$$

and therefore all the conditions of Theorem 2 are satisfied. There exist a solution w^* of (8), which can be found by iteration.

Remark 2 *In the above considerations, for the matrix $A = (a_{i,j})_{i,j=\overline{1,n}}$ we can use one of the following norms*

$$\|A\|_r := \max_{i=\overline{1,n}} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}, \ \|A\|_c := \max_{j=\overline{1,n}} \left\{ \sum_{i=1}^{n} |a_{ij}| \right\}.$$

The algorithm for obtaining of numerical solution of the differential equation problem (4) has the following steps:

1. The determination of fundamental spline functions $s_{1,j}$, $s_{i,r}$, $j = \overline{0,r-1}$,

$i = \overline{2, n}$ which are obtained from system (6).

2. The starting iteration point $W^{(0)}$ can be chosen using a numerical method for differential equation.

3. From Theorem 2.3 follow that the solution of differential equation (4) can be found by iteration $W^* = \lim_{n\to\infty} W^{(n)}$, where $W^{(k)} = H(W^{(k-1)})$.

References

[1] P. Blaga, G. Micula , *Natural spline functions of even degree*, Studia Univ. Babeş-Bolyai Cluj-Napoca, Series Math., 38, 1993, 31-40.

[2] P. Blaga, G. Micula, *Polynomial spline functions of even degree approximating the solution of differential equations*, (I), ZAMM, 76, 1996, Suppl.1, 477-478.

[3] P. Blaga, G. Micula, *Polynomial spline functions of even degree approximating the solution of (delay) diferential equations* , PAMM, Vol. 1, No.1, 2002, 516-517.

[4] B.D. Bojanov, H.A. Hakopian and A.A. Sahakian, *Spline Functios and Multivariate Interpolations*, Kluwer Academic Publishers, 1993.

[5] A. Branga, *Natural splines of Birkhoff type and optimal approximation*, Journal of Computational Analysis and Applications, Vol. 7, No.1, 2005, 81-88.

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