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On a general class of modified gamma approximating operators [1]

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Abstract

By using the generalized gamma distribution we shall define the general modified gamma transform $\Gamma^{(a,b)}_{\alpha,\beta,\gamma}$, $a, b \in \mathbb{R}$ from which we obtain as a special case both general modified gamma operators of the first and second kind. We obtain generalization of a several positive linear operator, as a special case of this general gamma operators.

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1 Introduction

In this paper we continue our earlier investigations [5], [6], [7], [8], [9] concerning to use Euler's gamma distribution for constructing linear positive operators.

In probability theory and statistics, the gamma distribution (G) is a two parameters family of continuous probability distribution. The probability density function (p.d.f.) of the gamma distribution can be expressed in terms of the gamma function parametrized in terms of a shape parameter α and an inverse scale parameter $\beta = 1/\theta$, called a rate parameter

$$G(t;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} \text{ for } t > 0 \text{ and } \alpha, \beta > 0. \tag{1}$$

The Weibull distribution (named after Naloddi Weibull) is a continuous probability distribution given by

$$W(t;\beta,\gamma) = \gamma \beta^{\gamma} t^{\gamma-1} e^{-(\beta t)^{\gamma}} \text{ for } t > 0, \tag{2}$$

where $\gamma > 0$ is the shape parameter and $\beta > 0$ is a rate parameter.

The most general form of the gamma distribution is the generalized gamma distribution (GG). It was introduced by Stacy and Mihran [12], [13], in order to combine the power of two distribution: the gamma distribution (1) and the Weibull distribution (2).

The generalized gamma distribution is a three-parameter distribution, with the probability density function (p.d.f.) given by

$$GG(t;\alpha,\beta,\gamma) = \frac{\gamma \beta^{\alpha\gamma}}{\Gamma(\alpha)} t^{\gamma\alpha-1} e^{-(\beta t)^{\gamma}} \tag{3}$$

for $t > 0$, where $\alpha > 0$ and $\gamma > 0$ are shape parameter and $\beta > 0$ is rate parameter.

The moments of (3) can be shown to be

$$E_{GG}(t^k) = \beta^k \frac{\Gamma(\alpha + k/\gamma)}{\Gamma(\alpha)}. \tag{4}$$

The generalized gamma distribution is a flexible distribution and it includes as special cases several distributions: the exponential distribution, the gamma distribution, the half normal distribution, the Levy distribution, the Weibull distribution and the log-normal distribution in limit case (α tends to infinity). For more details for the generalized gamma distribution see also [1], [2], [3].

The generalized beta distribution (GB) was introduced by J.B. McDonald and Y.J. Xu [4]. It is five-parameter distribution, with the probability density function (p.d.f.) given by

$$GB(t;\gamma,c,d,p,q) = \frac{t^{\gamma p-1}\left(1-(1-c)(t/d)^{\gamma}\right)^{q-1}}{d^{\gamma p}B(p,q)\left(1+c\,(t/d)^{\gamma}\right)^{p+q}} \tag{5}$$

for $0 < t^{\gamma} < d^{\gamma}/(1-c)$ and zero otherwise, with $0 \le c \le 1$ and γ, d, p, q, positive, $\gamma \in \mathbb{R}^*$.

The moments of (5) can be shown to be [4]

$$E_{GB}(t^k) = d^k \frac{B(p + k/\gamma, q)}{B(p,q)} \; {}_2F_1\left(\begin{matrix} p+\frac{k}{\gamma}, \frac{k}{\gamma} \\ p+q+\frac{k}{\gamma} \end{matrix} ; c\right) \tag{6}$$

where ${}_2F_1$ denotes the hypergeometric series which converges for all k if $c < 1$, or for $k\gamma < q$ if $c = 1$. Substituting $k = 0$ into (6) verifies that (5) integrates to one.

The generalized beta distribution (GB) includes the generalized beta of the first kind ($GB1$) and the generalized beta of the second kind ($GB2$), corresponding to $c = 0$ and $c = 1$, (see [4]).

The generalized gamma is a limiting case of GB, a.e.

$$GG(t; \alpha, \beta, \gamma) = \lim_{q\to\infty} GB\left(t; \gamma, c, d = \frac{1}{\beta} q^{\frac{1}{\gamma}}, \alpha, q\right). \tag{7}$$

Hence, the generalized beta includes generalized gamma as a limiting case for all admissible values of c. We obtain by (6)

$$E_{GG}(t^k) = \lim_{q\to\infty} E_{GB}(t^k) \tag{8}$$

$$= \lim_{q\to\infty} \frac{q^{\frac{k}{\gamma}} \beta^k B\left(\alpha + k/\gamma, q\right)}{B(\alpha, q)} = \beta^k \frac{\Gamma\left(\alpha + k/\gamma\right)}{\Gamma(\alpha)}.$$

By using the generalized gamma distribution we shall define the general modified gamma transform $\Gamma^{(a,b)}_{\alpha,\beta,\gamma}$, $a, b \in \mathbb{R}$ from which we obtain as a special case both general modified gamma operators of the first and second kind. We obtain generalization of a several positive linear operator, as a special case of this general gamma operators.

2 The general modified gamma transform

By using (3) we define the general modified gamma transform of a function f

$$\Gamma^{(a,b)}_{\alpha,\beta,\gamma} f = \gamma \frac{\beta^{\alpha\gamma}}{\Gamma(\alpha)} \int_0^\infty t^{\alpha\gamma - 1} e^{-(\beta t)^\gamma} f(ct^a e^{-bt^\gamma}) dt \tag{9}$$

where $\alpha, \beta, \gamma > 0$; $a, b \in \mathbb{R}$ and $f \in L_{1,loc}(0, \infty)$ such that $\Gamma^{(a,b)}_{\alpha,\beta,\gamma} |f| < \infty$.

We determine $c \in \mathbb{R}$ such that $\Gamma^{(a,b)}_{\alpha,\beta,\gamma} e_1 = e_1$, that is

$$c = \frac{(\beta^\gamma + b)^{\alpha + a/\gamma}}{\beta^{\alpha\gamma}} \cdot \frac{\Gamma(\alpha) x}{\Gamma(\alpha + a/\gamma)}$$

and we obtain from (9) the (a, b)-general modified gamma operators

$$(\Gamma^{(a,b)}_{\alpha,\beta,\gamma} f)(x) \tag{10}$$

$$= \gamma \frac{\beta^{\alpha\gamma}}{\Gamma(\alpha)} \int_0^\infty t^{\alpha\gamma - 1} e^{-(\beta t)^\gamma} f\left(\frac{(\beta^\gamma + b)^{\alpha + a/\gamma}}{\beta^{\alpha\gamma}} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha + a/\gamma)} t^a e^{-bt^\gamma} x \right) dt.$$

One observe that $\Gamma^{(a,b)}_{\alpha,\beta,\gamma}$ is a positive linear operator.

Theorem 1 *The moment of order* k *of the operator* $\Gamma^{(a,b)}_{\alpha,\beta,\gamma}$ *has the following value*

$$(\Gamma^{(a,b)}_{\alpha,\beta,\gamma} e_k)(x) = \frac{(\beta^\gamma + b)^{k(\alpha + a/\gamma)}}{(\beta^\gamma + kb)^{\alpha + ka/\gamma}} \cdot \frac{\Gamma(\alpha + ka/\gamma)\, \Gamma^{k-1}(\alpha)}{\Gamma^k(\alpha + a/\gamma)} \cdot \frac{x^k}{\beta^{\alpha\gamma(k-1)}}. \tag{11}$$

Proof. We have

$$\begin{aligned}
&(\Gamma^{(a,b)}_{\alpha,\beta,\gamma} e_k)(x) \\
= {}& \gamma \frac{\beta^{\alpha\gamma}}{\Gamma(\alpha)} \int_0^\infty t^{\gamma\alpha - 1} e^{-(\beta t)^\gamma} \left(\frac{(\beta^\gamma + b)^{\alpha + a/\gamma} \Gamma(\alpha)}{\beta^{\alpha\gamma}} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha + a/\gamma)} t^a e^{-bt^\gamma} x \right)^k dt \\
= {}& \gamma \frac{\beta^{\alpha\gamma}}{\Gamma(\alpha)} \int_0^\infty t^{\gamma\alpha - 1} e^{-(\beta t)^\gamma} \frac{(\beta^\gamma + b)^{k(\alpha + a/\gamma)}}{\beta^{k\alpha\gamma}} \cdot \frac{\Gamma^k(\alpha)}{\Gamma^k(\alpha + a/\gamma)} t^{ka} e^{-kbt^\gamma} x^k dt \\
= {}& \gamma \frac{\beta^{\alpha\gamma}}{\Gamma(\alpha)} \cdot \frac{(\beta^\gamma + b)^{k(\alpha + a/\gamma)}}{\beta^{k\alpha\gamma}} \cdot \frac{\Gamma^k(\alpha) x^k}{\Gamma^k(\alpha + a/\gamma)} \int_0^\infty t^{\gamma\alpha + ka - 1} e^{-(\beta t)^\gamma - kbt^\gamma} dt \\
= {}& \gamma \frac{\beta^{\alpha\gamma}}{\Gamma(\alpha)} \cdot \frac{(\beta^\gamma + b)^{k(\alpha + a/\gamma)}}{\beta^{k\alpha\gamma}} \cdot \frac{\Gamma^k(\alpha) x^k}{\Gamma^k(\alpha + a/\gamma)} \int_0^\infty t^{\gamma\left(\alpha + \frac{ka}{\gamma}\right) - 1} e^{-((\beta^\gamma + (kb)^{1/\gamma})t)^\gamma} dt \\
= {}& \gamma \frac{(\beta^\gamma + b)^{k(\alpha + a/\gamma)}}{\beta^{\alpha\gamma(k-1)}} \cdot \frac{\Gamma^{k-1}(\alpha) x^k}{\Gamma^k(\alpha + a/\gamma)} \cdot \frac{\Gamma(\alpha + ka/\gamma)}{(\beta^\gamma + kb)^{\alpha + \frac{ka}{\gamma}}} \cdot \frac{1}{\gamma} \\
= {}& \frac{(\beta^\gamma + b)^{\alpha + a/\gamma}}{(\beta^\gamma + kb)^{\alpha + \frac{ka}{\gamma}}} \cdot \frac{\Gamma(\alpha + ka/\gamma)\, \Gamma^{k-1}(\alpha)}{\Gamma^k(\alpha + a/\gamma)} \cdot \frac{x^k}{\beta^{\alpha\gamma(k-1)}}.
\end{aligned}$$

Consequently, we obtain

$$(\Gamma^{(a,b)}_{\alpha,\beta,\gamma}e_2)(x) = \frac{(\beta^\gamma + b)^{2(\alpha+a/\gamma)}}{(\beta^\gamma + 2b)^{\alpha+\frac{2a}{\gamma}}} \cdot \frac{\Gamma\left(\alpha + 2a/\gamma\right)}{\Gamma^2\left(\alpha + a/\gamma\right)} \cdot \frac{\Gamma(\alpha)x^k}{\beta^{\alpha\gamma}} \tag{12}$$

and

$$\Gamma^{(a,b)}_{\alpha,\beta,\gamma}((t-x)^2;x) \tag{13}$$

$$= \frac{(\beta^\gamma + b)^{2(\alpha+a/\gamma)}\Gamma(\alpha)\Gamma\left(\alpha + 2a/\gamma\right) - \beta^{\alpha\gamma}(\beta^\gamma + 2b)^{\alpha+\frac{2a}{\gamma}}\Gamma^2\left(\alpha + a/\gamma\right)}{\beta^{\alpha\gamma}(\beta^\gamma + 2b)\alpha + 2a/\gamma\Gamma^2\left(\alpha + a/\gamma\right)} \cdot x^2.$$

3 The modified gamma first kind operators

If we put in (10) $b = 0$ we obtain the general modified gamma first kind operators

$$(\Gamma^{(a)}_{\alpha,\beta,\gamma}f)(x) = \gamma\frac{\beta^{\alpha\gamma}}{\Gamma(\alpha)}\int_0^\infty t^{\alpha\gamma-1}e^{-(\beta t)^\gamma} f\left(\frac{\Gamma(\alpha)(\beta t)^a}{\Gamma\left(\alpha + a/\gamma\right)}x\right) dt \tag{14}$$

or equivalent

$$(\Gamma^{(a)}_{\alpha,\beta,\gamma}f)(x) = \frac{\gamma}{\Gamma(\alpha)}\int_0^\infty u^{\alpha\gamma-1}e^{-u^\gamma} f\left(\frac{\Gamma(\alpha)u^a x}{\Gamma\left(\alpha + a/\gamma\right)}\right) du \tag{15}$$

where $f \in L_{1,loc}(0,\infty)$ such that $\Gamma^{(a)}_{\alpha,\beta,\gamma}|f| < \infty$.

We observe that $\Gamma^{(a)}_{\alpha,\beta,\gamma}$ does not depend on β and we may consider $\beta = 1$.

Corollary 1 *The moment of order k of the operator $\Gamma^{(a)}_{\alpha,\beta,\gamma}$ has the following value*

$$(\Gamma^{(a)}_{\alpha,\beta,\gamma}e_k)(x) = \frac{\Gamma^{k-1}(\alpha)\Gamma\left(\alpha + ka/\gamma\right)}{\Gamma^k\left(\alpha + a/\gamma\right)}x^k. \tag{16}$$

Proof. The result follows from (11) for $b = 0$.

Consequently, we obtain

$$\Gamma_{\alpha,\gamma}^{(a)}((t-x)^2) = \frac{\Gamma(\alpha)\Gamma\left(\alpha + 2a/\gamma\right)}{\Gamma^2\left(\alpha + a/\gamma\right)} x^2. \tag{17}$$

For $\gamma = 1$ we obtain the modified gamma first kind operators (see [9]) and for $\alpha = 1$ we obtain the modified Weibull first kind operators (see [11]).

Special cases

Case 1. If we consider $a = 1$ in (15) we obtain the modified gamma first kind operator

$$(\Gamma_{\alpha,\gamma} f)(x) = \frac{\gamma}{\Gamma(\alpha)} \int_0^\infty t^{\alpha\gamma-1} e^{-t^\gamma} f\left(\frac{\Gamma(\alpha)tx}{\Gamma\left(\alpha + 1/\gamma\right)}\right) dt. \tag{18}$$

Corollary 2 *The moment of order k of the operator $\Gamma_{\alpha,\gamma}$ has the following value*

$$(\Gamma_{\alpha,\gamma} e_k)(x) = \frac{\Gamma^{k-1}(\alpha)\Gamma\left(\alpha + k/\gamma\right)}{\Gamma^k\left(\alpha + 1/\gamma\right)} x^k. \tag{19}$$

Proof. The result follows from (16) for $a = 1$.

We deduce

$$(\Gamma_{\alpha,\gamma} e_2)(x) = \frac{\Gamma(\alpha)\Gamma\left(\alpha + 2/\gamma\right)}{\Gamma^2\left(\alpha + 1/\gamma\right)} x^2, \tag{20}$$

$$\Gamma_{\alpha,\gamma}((t-x)^2; x) = \frac{\Gamma(\alpha)\Gamma\left(\alpha + 2/\gamma\right) - \Gamma^2\left(\alpha + 1/\gamma\right)}{\Gamma^2\left(\alpha + 1/\gamma\right)} x^2.$$

If we choose $\alpha = n$, $n \in \mathbb{N}$ in (18) then we obtain the generalization of the Post-Wider positive linear operator defined for $f \in L_{1,loc}(0,\infty)$ by (see [9])

$$(P_{n,\gamma} f)(x) = \frac{\gamma}{\Gamma(n)} \int_0^\infty t^{\gamma n-1} e^{-t^\gamma} f\left(\frac{\Gamma(n)}{\Gamma\left(n + 1/\gamma\right)} tx\right) dt. \tag{21}$$

If we replace $\alpha = nx$, $n \in \mathbb{N}$ in (18) then we obtain the generalization of the Rathore positive linear operator defined for $f \in L_{1,loc}(0,\infty)$ by (see [9])

$$(22) \qquad (R_{n,\gamma}f)(x) = \frac{\gamma}{\Gamma(nx)} \int_0^\infty t^{\gamma nx-1} e^{-t^\gamma} f\left(\frac{\Gamma(nx)tx}{\Gamma\left(nx+1/\gamma\right)}\right) dt$$

Case 2. If we replace $a = -1$ in (15) we obtain the modified gamma operator

$$(23) \qquad (\widetilde{\Gamma}_{\alpha,\gamma}f)(x) = \frac{\gamma}{\Gamma(\alpha)} \int_0^\infty t^{\alpha\gamma-1} e^{-t^\gamma} f\left(\frac{\Gamma(\alpha)}{\Gamma\left(\alpha-1/\gamma\right)} \cdot \frac{x}{t}\right) dt.$$

Corollary 3 *The moment of order k of the operator $\widetilde{\Gamma}_{\alpha,\gamma}$ has the following value*

$$(24) \qquad (\widetilde{\Gamma}_{\alpha,\gamma}e_k)(x) = \frac{\Gamma^{k-1}(\alpha)\Gamma\left(\alpha-k/\gamma\right)}{\Gamma^k\left(\alpha-1/\gamma\right)} x^k, \quad 0 \le k < \alpha\gamma.$$

Proof. The result follows from (16) for $a = -1$.

We obtain

$$(\widetilde{\Gamma}_{\alpha,\gamma}e_2)(x) = \frac{\Gamma(\alpha)\Gamma\left(\alpha-2/\gamma\right)}{\Gamma^2\left(\alpha-1/\gamma\right)} x^2$$

$$\widetilde{\Gamma}_{\alpha,\gamma}((t-x)^2;x) = \frac{\Gamma(\alpha)\Gamma\left(\alpha-2/\gamma\right) - \Gamma^2\left(\alpha-1/\gamma\right)}{\Gamma^2\left(\alpha-1/\gamma\right)} x^2.$$

For $\alpha = n+1$, $n \in \mathbb{N}$ we obtain the generalization of the operator introduced and studied by A. Lupaş and M. Müller

$$(G_{n,\gamma}f)(x) = \frac{\gamma}{\Gamma(n+1)} \int_0^\infty t^{(n+1)\gamma-1} e^{-t^\gamma} f\left(\frac{\Gamma(n+1)}{\Gamma\left(n+1-1/\gamma\right)} \cdot \frac{x}{t}\right) dt.$$

4 The modified gamma second kind operators

If we choose in (10) $a = 0$ then we obtain the modified gamma second kind operators

$$(25) \qquad (\Gamma^{(b)}_{\alpha,\beta,\gamma} f)(x) = \gamma \frac{\beta^{\alpha\gamma}}{\Gamma(\alpha)} \int_0^\infty t^{\alpha\gamma-1} e^{-(\beta t)^\gamma} f\left(\left(\frac{\beta^\gamma + b}{\beta^\gamma}\right)^\alpha e^{-bt^\gamma} x\right) dt$$

where $f \in L_{1,loc}(0,\infty)$ such that $\Gamma^{(b)}_{\alpha,\beta,\gamma}|f| < \infty$.

Corollary 4 *The moment of order k of the operator $\Gamma^{(b)}_{\alpha,\beta,\gamma}$ has the following value*

$$(26) \qquad (\Gamma^{(b)}_{\alpha,\beta,\gamma} e_k)(x) = \frac{(\beta^\gamma + b)^{k\alpha}}{(\beta^\gamma + kb)^\alpha} \cdot \frac{x^k}{\beta^{\alpha\gamma(k-1)}}.$$

Proof. The result follows from (11) for $a = 0$.

Consequently, we obtain

$$(27) \qquad (\Gamma^{(b)}_{\alpha,\beta,\gamma}(e_2)(x) = \frac{(\beta^\gamma + b)^{2\alpha}}{(\beta^\gamma + 2b)^\alpha} \cdot \frac{x^2}{\beta^{\alpha\gamma}}$$

$$(28) \qquad \Gamma^{(b)}_{\alpha,\beta,\gamma}((t-x)^2; x) = \frac{(\beta^\gamma + b)^{2\alpha} - \beta^{\alpha\gamma}(\beta^\gamma + 2b)^\alpha}{\beta^{\alpha\gamma}(\beta^\gamma + 2b)^\alpha} \cdot x^2.$$

For $\gamma = 1$ we obtain the modified gamma second kind operators (see [9]) and for $\alpha = 1$ we obtain the modified Weibull second kind operators (see [11]).

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