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Some evaluations of the remainder term [1]

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Abstract

We present some representations of the remainder $f(x)-(L_nf)(x)$, where L_n is defined in (1). Using Lupaş operators (4), we prove Theorem 3 and one finds a lower-bound for $(L_ne_2)(x)$ (see Theorem 4).

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1. Let $f:[0,1]\to\mathbb{R}$. The *Bernstein polynomials* of f is

$$(B_nf)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),\quad \text{with}\ \ n=1,2,\dots,\ .$$

We consider the operator

$$(L_nf)(x)=(B_nf)(x)-\alpha_n(x)(B_n^{''}f)(x),$$

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where $\alpha(x) = \dfrac{x(1-x)}{2(n-1)}$ results from the following condition $(L_n e_0) = e_0$, $(L_n e_1) = e_1$, $(L_n e_2) = e_2$, with $e_j(t) = t^j$, $j = 0, 1, \ldots$. We obtain the next operator (see [2])

$$(L_n f)(x) = (B_n f)(x) - \frac{x(1-x)}{2(n-1)}(B_n'' f)(x), \tag{1}$$

where $(B_n f)(x)$ is Bernstein polynomials.

We use the notations

$$K = [a, b], \quad \infty < a < b < +\infty,$$

$$\Omega_j(t, x) = \Omega_j(t) = |t - x|^j, \quad j = 0, 1, \ldots, \quad x \in K, \tag{2}$$

$$\omega(f; \delta) = \sup_{|t-x|<\delta} |f(t) - f(x)|, \quad t, x \in K, \ \delta \geq 0.$$

From [3] we obtain

Theorem 1 *If $L : C(K) \to C(K_1)$, $K_1 = [a_1, b_1] \subseteq K$, is a linear positive operator, then for all $f \in C(K)$ and $\delta > 0$ we have*

$$||f - Lf||_{K_1} \leq ||f|| \cdot ||e_0 - Le_0||_{K_1} + \inf_{m=1,2,\ldots} \{||Le_0||_{K_1} + \delta^{-m} ||L\Omega_m||_{K_1}\} \omega(f; \delta),$$

where $|| \cdot || = \max_K | \cdot |$ and Ω_m are defined in (2).

Theorem 2 *Let $f \in C[0, 1]$. If L_n are linear operator defined as in (1), then*

$$||f - L_n f|| \leq \frac{19}{16} \omega\left(f; \frac{1}{\sqrt{n}}\right) + \frac{n}{4} \omega\left(f; \frac{1}{n}\right)$$

Proof. We consider $m = 4$. Results

$$\Omega_4(t;x) = |t-x|^4 = (t-x)^4.$$

But for Bernstein operator we know

$$(B_n\Omega_4)(x) = \frac{1}{n^3}\left[3(n-2)x^2(1-x)^2 + x(1-x)\right]$$

(3)

$$(B_n''f)(x) = \frac{2n(n-1)}{n^2}\sum_{k=0}^{n-2}\binom{n-2}{k}x^k\cdot(1-x)^{n-2-k}\left[\frac{k}{n},\frac{k+1}{n},\frac{k+2}{n};f\right]$$

and using theorem 1 we obtain

$$||f - B_nf|| \le \frac{19}{16}\omega\left(f;\frac{1}{\sqrt{n}}\right)$$

We have

$$\begin{aligned} ||f-L_nf|| &= \left|\left|f - L_nf - \frac{x(1-x)}{2(n-1)}B_n''f + \frac{x(1-x)}{2(n-1)}B_n''f\right|\right| \\ &\le ||f - B_nf|| + \left|\left|\frac{x(1-x)}{2(n-1)}B_n''f\right|\right|. \end{aligned}$$

But from (3) the theorem is proved because

$$\begin{aligned} \frac{x(1-x)}{2(n-1)}B_n''f &= \frac{x(1-x)}{2(n-1)}\frac{2!n(n-1)}{n^2}\sum_{k=0}^{n-2}\binom{n-2}{k}x^k\cdot \\ \cdot\quad & (1-x)^{n-2-k}\frac{\left[\frac{k+1}{n},\frac{k+2}{n};f\right]-\left[\frac{k}{n},\frac{k+1}{n};f\right]}{\frac{2}{n}} \\ = \quad & \frac{x(1-x)}{2}\sum_{k=0}^{n-2}\binom{n-2}{k}x^k(1-x)^{n-2-k} \end{aligned}$$

$$\cdot \quad \left(\frac{f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right)}{\frac{1}{n}} - \frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{\frac{1}{n}} \right)$$

$$\leq \quad \frac{x(1-x)}{2} n \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{n-2-k} \cdot \left(\omega\left(f; \frac{1}{n}\right) + \omega\left(f; \frac{1}{n}\right) \right)$$

$$= \quad nx(1-x) \cdot \omega\left(f; \frac{1}{n}\right) \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{n-2-k}$$

$$= \quad nx(1-x)\omega\left(f; \frac{1}{n}\right) \leq \frac{n}{4}\omega\left(f; \frac{1}{n}\right).$$

On the other hand we observe

$$\left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; f\right] = \frac{f\left(\frac{k}{n}\right)}{\frac{2}{n^2}} - \frac{f\left(\frac{k+1}{n}\right)}{\frac{1}{n^2}} + \frac{f\left(\frac{k+2}{n}\right)}{\frac{2}{n^2}}$$

$$= \frac{n^2}{2}\left[f\left(\frac{k}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k+2}{n}\right)\right] = \frac{n^2}{2}\Delta^2_{\frac{1}{n}}\left(f; \frac{k}{n}\right),$$

$$\Delta_h^r(f;x) = \sum_{k=0}^{r} \binom{r}{k} (-r)^{r-k} f(x+kh),$$

with $r = 1, 2, \ldots,$ and $h \in \mathbb{R}$.

Using the following definition $\omega_r(f,\delta) = \sup_{0<h\leq\delta} ||\Delta_h^r(f;\cdot)||$, with $\delta > 0$, results

$$\left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; f\right] \leq \frac{n^2}{2}\omega_2\left(f; \frac{1}{n}\right),$$

and

$$\frac{x(1-x)}{2(n-1)} B_n'' f \leq \frac{nx(1-x)}{2}\omega_2\left(f; \frac{1}{n}\right) \leq \frac{n}{8}\omega_2\left(f; \frac{1}{n}\right).$$

from above formula the following proposition is proved

Corollary 1 *Let $f \in C[0,1]$. If L_n are linear operator defined as in (1), then*

$$||f(x) - (L_n f)(x)|| \leq \frac{19}{16}\omega\left(f; \frac{1}{\sqrt{n}}\right) + \frac{n}{8}\omega_2\left(f; \frac{1}{n}\right)$$

2. For approximation of the continuous functions $f : [0,\infty) \to \mathbb{R}$ and which satisfy an inequality like

$$|f(x)| \le Ae^{Bx} \quad , \quad (x > 0),\ A > 0,\ B > 0$$

on $[0,\infty)$ with A and B independent of f (that is f is of exponential type) we use linear positive operators as (see [4])

$$(L_n f)(x) = \sum_{k=0}^{\infty} a_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty) \tag{4}$$

where $a_{n,k} : [0,\infty) \to [0,\infty)$, and

i) the series $\sum_{k=0}^{\infty} a_{n,k}(x) z^k$ are convergent for $|z| < r$, where $r > 1$

$$\left(\forall\ n \ge n_0 := 1 + \left[\frac{B}{\ln r}\right], \quad x \in [0,\infty) \right)$$

ii) $\sum_{k=0}^{\infty} a_{n,k}(x) = 1$ iii) $\sum_{k=0}^{\infty} k \cdot a_{n,k}(x) = nx$

Examples:

I. $a_{n,k} = e^{-nx} \dfrac{(nx)^k}{k!}$ - L_n = Favard – Szasz operators

II. $a_{n,k} = \dbinom{n+k}{n} \dfrac{x^k}{(1+x)^{n+k}}$ - L_n = Lupaş - Baskakov operators

$$(L_n e_i = e_i,\ i = 0, 1)$$

Our purpose: a representation of the remainder

$$(R_n f)(x) := f(x) - (L_n f)(x).$$

Known methods:

(A) D.D. Stancu - with the help of divided differences.

(B) Peano′s method. We suppose that $f \in C^2[0,\infty)$.

(C) The present method witch is actually a following of those in (A) which where studied on particular cases. The idea is to use A. Lupa's operators defined as

$$(S_n f)(x) = \sum_{k=0}^{\infty} E_{n,k}(f;x) f\left(\frac{k}{n}\right)$$
$$(S_n f)\left(\frac{j}{n}\right) = \frac{2}{n}\left[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; |t-x|\right]_t f\left(\frac{k}{n}\right),$$

with

$$E_{n,k}(f;x) = \frac{2}{n}\left[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; |t-x|_+\right] = \begin{cases} 0 & , \quad k = 0, \ldots, k_0 \\ 1 - \{nx\} & , \quad k = k_0 \\ \{nx\} & , \quad k = k_0 - 1 \\ 0 & , \quad k \geq k_0 + 2 \end{cases}$$

where $k_0 = [nx]$.

Properties:

- They are positive linear operator ($h(t) = |t-x|$ convex);
- $L_n S_n f = L_n f$ (*)

Proof. We have

$$(L_n S_n f)(x) = \sum_{k=0}^{\infty} a_{n,k} (S_n f)\left(\frac{k}{n}\right),$$

with (5) we obtain

$$(L_nS_nf)(x) = \sum_{k=0}^{\infty} a_{n,k} f\left(\frac{k}{n}\right) = L_n f$$

(5) $$(S_nf)\left(\frac{j}{n}\right) = f\left(\frac{j}{n}\right), \quad j = 0, 1, \dots$$

(6) $$S_n e_i = e_i, \quad i \in \{0, 1\}, \quad e_0(t) = 1, \quad e_1(t) = t$$

We write successive

$$(S_nf)\left(\frac{j}{n}\right) = \sum_{k=0}^{\infty} E_{n,k}\left(f; \frac{k}{n}\right) \overset{k=k_0=[nx],\ x=\frac{j}{n}}{=}$$

$$= \left(1 - \left\{n \cdot \frac{j}{n}\right\}\right) + \left\{n \cdot \frac{j}{n}\right\} f\left(\frac{1 + \left[n \cdot \frac{j}{n}\right]}{n}\right)$$

but $n \cdot \frac{j}{n} = 0$, and from that results

$$(S_nf)\left(\frac{j}{n}\right) = f\left(\frac{j}{n}\right)$$

we have

$$(S_nf)(x) = (1 - \{nx\}) f\left(\frac{[nx]}{n}\right) + \{nx\} f\left(\frac{1 + [nx]}{n}\right).$$

But

$$|t - x|_+ = \frac{1}{2}(|t - x| + t - x)[x_1, x_2, x_3; |t - x|_+]_t =$$

$$\frac{1}{2}[x_1, x_2, x_3; |t - x|]_t + \frac{1}{2}\underbrace{[x_1, x_2, x_3; t - x]_t}_{0}$$

Theorem 3 *We have the following representation:*

$$(R_n f)(x) = -\frac{z_n(x)(1 - z_n(x))}{n^2}\left[\frac{[nx]}{n}, x\frac{1 + [nx]}{n}; f\right] +$$

$$+\frac{2}{n}\sum_{k=2}^{\infty}\left[\frac{k-2}{n}, \frac{k-1}{n}, \frac{k}{n}; f\right]\varphi_{k,n}(x)\ ,$$

where $z_n = \{nx\}$ and $\varphi_{k,n}(x) = \Omega_{k,n}(x) - (L_n\Omega_{k,n})(x)$ and $\Omega_{k,n}(t) = \left|\frac{k-1}{n} - t\right|$.

Proof. Using the next representation:

$$(7)\quad (S_n f)(x) = a_{0,n}(f)x + b_{0,n}(f) + \frac{2}{n}\sum_{k=2}^{\infty}\left[\frac{k-2}{n}, \frac{k-1}{n}, \frac{k}{n}; f\right]\cdot\left|\frac{k-1}{n} - x\right|$$

and

$$(8)\quad (S_n f)(t) = a_{0,n}(f)t + b_{0,n}(f) + \frac{2}{n}\sum_{k=2}^{\infty}\left[\frac{k-2}{n}, \frac{k-1}{n}, \frac{k}{n}; f\right]\cdot\left|\frac{k-1}{n} - t\right|.$$

We calculate (7)-(8):

$$(S_n f)(x) - (S_n f)(t) = a_{0,n}(f)(x - t) +$$

$$(9)\quad +\frac{2}{n}\sum_{k=2}^{\infty}\left[\frac{k-2}{n}, \frac{k-1}{n}, \frac{k}{n}; f\right]\cdot\left(\left|\frac{k-1}{n} - x\right| - \left|\frac{k-1}{n} - t\right|\right)$$

Applied on (9) linear operator L_n, refer to t and we take the result on x.

From that result:

$$(S_nf)(x) - (L_nS_nf)(x) = a_{0,n}(f)(\underbrace{x - (L_ne_1)(x))}_{0}$$

$$+\frac{2}{n}\sum_{k=2}^{\infty}\left[\frac{k-2}{n}, \frac{k-1}{n}, \frac{k}{n}; f\right] \cdot \left(\left|\frac{k-1}{n} - x\right| - (L_n\Omega_{k,n})(x)\right).$$

$$f(x) - f(x) + (S_nf)(x) - (L_nf)(x)$$

$$= \frac{2}{n}\sum_{k=2}^{\infty}\left[\frac{k-2}{n}, \frac{k-1}{n}, \frac{k}{n}; f\right] \cdot (\Omega_{k,n}(x) - (L_n\Omega_{k,n})(x))$$

From (*) result

$$f(x) - (L_nf)(x) = [f(x) - (S_nf)(x)] + \frac{2}{n}\sum_{k=2}^{\infty}\left[\frac{k-2}{n}, \frac{k-1}{n}, \frac{k}{n}; f\right]\varphi_{k,n}(x)$$

where

$$\varphi_{k,n}(x) = \Omega_{k,n}(x) - (L_n\Omega_{k,n})(x).$$

But $f(x) - (S_nf)(x)$ is a representation introduce by A. Lupaş in ([3], p23). And now we have

$$(R_n)(x) = -\frac{z_n(x)(1 - z_n(x))}{n^2}\left[\frac{[nx]}{n}, x\frac{1 + [nx]}{n}; f\right]$$

$$+\frac{2}{n}\sum_{k=2}^{\infty}\left[\frac{k-2}{n}, \frac{k-1}{n}, \frac{k}{n}; f\right]\varphi_{k,n}(x)$$

where $z_n(x) = \{nx\}$, and

$$\varphi_{k,n}(x) = \Omega_{k,n}(x) - (L_n\Omega_{k,n})(x).$$

If $\Omega_{k,n}$ are convergent, and if L is linear operator and positive with $Le_j = e_j$, $j = 0, 1, \ldots$ then $\forall$ h convex

$$h(x) \leq (Lh)(x) \ \Rightarrow\ \varphi_{k,n}(x) \leq 0 \quad \text{(see [3])}.$$

Theorem 4 *If L_n, verifies the hypothesis, then*

$$(L_ne_2)(x) \geq \frac{nx[nx] + \{nx\}(1 + [nx])}{n^2} \quad , \quad x \geq 0$$

with $e_2(t) = t^2$.

Proof. Let $f(t) = e_2(t)$. From Theorem 3 we have:

$$(R_ne_2)(x) = -\frac{z_n(1 - z_n)}{n^2} + \frac{2}{n}\sum_{k=2}^{\infty} \varphi_{k,n}(x).$$

But

$$(R_ne_2)(x) = x^2 - (L_ne_2)(x),$$

then

$$\frac{2}{n}\sum_{k=2}^{\infty} \varphi_{k,n}(x) = x^2 - (L_ne_2)(x) + \frac{z_n(1 - z_n)}{n^2},$$

and because $\varphi_{k,n} \leq 0$, we have

$$\sum_{k=2}^{\infty} \varphi_{k,n}(x) = \frac{n}{2}\left[x^2 - (L_ne_2(x) + \frac{\{nx\}(1 - \{nx\})}{n^2}\right],$$

and

$$\underbrace{\sum_{k=2}^{\infty} \varphi_{k,n}(x)}_{\leq 0} = \frac{n}{2}\left[\frac{([nx] + \{nx\})^2 + \{nx\} - \{nx\}^2}{n^2} - (L_ne_2)(x)\right]$$

$$= \frac{n}{2}\left[\frac{([nx]^2 + 2[nx]\{nx\}) + \{nx\}}{n^2} - (L_ne_2)(x)\right] \leq 0$$

From above we have

$$(L_ne_2)(x) \geq \frac{nx[nx] + \{nx\}(1 + [nx])}{n^2}, \quad x \geq 0$$

and $e_2(t) = t^2$.

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