General Mathematics Vol. 18, No. 1 (2010), 131–137

# Some properties from Laguerre polynomials [1]

Ioan Ţincu

**Abstract**

In this paper, we demonstrate some properties of Laguerre polynomials. Using the Beta function we prove a projection inequality and an inequality of the form $|L_n^{(\alpha)}(x)| < a_n e^{\frac{x}{2}},\ x \geq 0$. The last result consist of the representation of polynomial $L_n^{(\alpha)}$ depending on Hermite polynomials.

**2000 Mathematics Subject Classification:** 33C05, 33C45

**Key words and phrases:** Laguerre polynomials, Hermite polynomials, Feldheim formula, Beta function, Mehler-Hermite quadrature formula.

[1] *Received 11 October, 2009*

*Accepted for publication (in revised form) 22 November, 2009*

# 1 Introduction

Let $L_n^{(\alpha)}(x)$, $\alpha > -1$, $x \geq 0$, $n \in \mathbb{N}$ Laguerre polynomials on degree $n$ and order $\alpha$.

The polynomial $L_n^{(\alpha)}(x)$ verifies the following equalities:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \cdot \frac{(-x)^k}{k!}, \tag{1}$$

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_k^{(\alpha)}(x) L_{n-k}^{(\beta)}(y), \tag{2}$$

$$L_n^{(\alpha)}(tx) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{(n-k)!\Gamma(\alpha+k+1)} t^k (1-t)^{n-k} L_k^{(\alpha)}(x), \tag{3}$$

$$L_n^{(-\frac{1}{2})}(x) = \frac{(-1)^n}{n! 4^n} H_{2n}(\sqrt{x}),\ L_n^{(\frac{1}{2})}(x) = \frac{(-1)^n}{n! 2^{2n+1}\sqrt{x}} H_{2n+1}(\sqrt{x}), \tag{4}$$

$$L_n^{(\alpha)}(x) = \frac{(-1)^n \Gamma(n+\alpha+1)}{\sqrt{\pi}(2n)!\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} H_{2n}(t\sqrt{x}) dt,\ \alpha > -\frac{1}{2}, \tag{5}$$

$H_n(x)$ being Hermite polynomial on degree $n$.

From (1) for $x = 0$ we obtain

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$$

We denote $l_n^{(\alpha)}(x) = \dfrac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}$ Laguerre polynomial of $\alpha$ order normalized through the condition $l_n^{(\alpha)}(0) = 1$

Using $l_n^{(\alpha)}(x)$ and using properties (1)-(5), we proof that $l_n^{(\alpha)}(x)$ verifies:

$$l_n^{(\alpha)}(x) = \frac{1}{\begin{pmatrix} n+\alpha \\ n \end{pmatrix}} \sum_{k=0}^{n} \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix} \frac{(-x)^k}{k!}, \tag{6}$$

$$l_n^{\alpha+\beta+1}(x+y) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{B(k+\alpha+1, n-k+\beta+1)}{B(\alpha+1,\beta+1)} l_k^{(\alpha)}(x) l_{n-k}^{(\beta)}(y), \tag{7}$$

$$l_n^{(\alpha)}(tx) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} t^k (1-t)^{n-k} l_k^{(\alpha)}(x), \tag{8}$$

$$l_n^{(-\frac{1}{2})}(x) = \frac{(-1)^n \cdot n!}{(2n)!} H_{2n}(\sqrt{x}), \tag{8'}$$

$$l_{m+n}^{(\alpha+\beta+1)}(x) = \frac{1}{B(\alpha+1,\beta+1)} \int_0^1 l_m^{(\alpha)}(xt) l_n^{(\beta)}((1-t)x) t^{\alpha} (1-t)^{\beta} \mathrm{dt} \tag{8''}$$ (Feldheim)

## 2 Main results

**Theorem 1.** *If $\beta > \alpha > -1$ and $x \geq 0$ then*

$$l_n^{(\beta)}(x) = \frac{1}{B(\alpha+1,\beta-\alpha)} \int_0^1 t^{\alpha}(1-t)^{\beta-\alpha-1} l_n^{(\alpha)}(tx) dt, \tag{9}$$

*$B(x,y)$ being beta function.*

**Proof.** From (8) we have:

$$t^{\alpha}(1-t)^{\beta-\alpha-1} l_n^{(\alpha)}(tx) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} t^{\alpha+k} (1-t)^{n-k+\beta-\alpha-1} l_k^{(\alpha)}(x),$$

(10) $$\int_0^1 t^\alpha(1-t)^{\beta-\alpha-1} l_n^{(\alpha)}(tx)dt = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) l_k^{(\alpha)}(x) \int_0^1 t^{\alpha+k} \cdot$$

$$\cdot (1-t)^{n-k+\beta-\alpha-1} dt = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) B(\alpha+k+1, n-k+\beta-\alpha) l_k^{(\alpha)}(x).$$

In (7) we consider $y = 0$ and $\beta = \beta - \alpha - 1$ and we obtain

(11) $$l_n^{(\beta)}(x) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{B(k+\alpha+1), n-k+\beta-\alpha}{B(\alpha+1, \beta-\alpha)} l_k^{(\alpha)}(x)$$

From (10) and (11) we write

$$l_n^{(\beta)}(x) \frac{1}{B(\alpha+1, \beta-\alpha)} \int_0^1 t^\alpha (1-t)^{\beta+1} l_n^{(\alpha)}(tx)dt,$$

for $\beta > \alpha > -1$, and the proof is complete.

**Theorem 2.** *Let* $\beta > \alpha > -1$ *and* $P_n(x;\alpha) = \sum_{k=0}^{n} a_k l_k^{(\alpha)}(x)$, $x \geq 0$. *If* $P_n(x;\alpha) \geq 0, (\forall)x \geq 0$ *then*

(12) $$P_n(x;\beta) \geq 0,\ (\forall)x \geq 0 \text{ and } \beta > \alpha > -1$$

**Proof.** From (9) we write:
$a_k l_k^{(\beta)}(x) = \dfrac{1}{B(\alpha+1, \beta-\alpha)} \displaystyle\int_0^1 t^\alpha (1-t)^{\beta-\alpha-1} a_k l_k^{(\alpha)}(tx)dt$, we make sum over the $k$, and we obtain:

$$P_n(x;\beta) = \frac{1}{B(\alpha+1, \beta-\alpha)} \int_0^1 t^\alpha (1-t)^{\beta-\alpha-1} P_n(xt;\alpha)dt$$

From $P_n(x;\alpha) \geq 0\, (\forall)x \geq 0$ results $P_n(x;\beta) \geq 0$, $(\forall)x \geq 0$.

**Theorem 3.** *If $l_n^{(\beta)}(x)$ is Laguerre normalized polynomial through $l_n^{(\alpha)}(0) = 1$, then for $\beta > -\frac{1}{2}$ and $k_0 = 1,086435$ we have:*

$$|l_n^{(\beta)}(x)| < \frac{k_0 \cdot 2^n \cdot n!}{\sqrt{(2n)!}} \cdot e^{\frac{x}{2}}, (\forall)x \geq 0 \tag{13}$$

**Proof.** In (9) let $\alpha = -\frac{1}{2}$,

$$l_n^{(\beta)}(x) = \frac{1}{B(\frac{1}{2}, \beta + \frac{1}{2})} \int_0^1 t^{-\frac{1}{2}} (1-t)^{\beta - \frac{1}{2}} l_n^{(-\frac{1}{2})}(tx) dt.$$

Using $(8')$ we obtain

$$l_n^{(\beta)}(x) = \frac{(-1)^n \cdot n!}{(2n)! B(\frac{1}{2}, \beta + \frac{1}{2})} \int_0^1 t^{-\frac{1}{2}} (1-t)^{\beta - \frac{1}{2}} H_{2n}(\sqrt{tx}) dt.$$

We know that

$$|H_{2n}(\sqrt{x})| < k_0 \sqrt{(2n)!} 2^n e^{\frac{x}{2}},\ k_0 = 1,086435.$$

It results,

$$|l_n^{(\beta)}(x)| < \frac{k_0 2^n n! e^{\frac{x}{2}}}{\sqrt{(2n)!}},\ (\forall)x \geq 0.$$

**Theorem 4.** *If $n, m, p \in \mathbb{N}$ such that $m + n \leq 2p - 1$ then*

$$l_{m+n}(x) = l_{m+n}^{(0)}(x) = \frac{(-1)^{n+m} n! m!}{(2n)!(2m)! p} \sum_{k=1}^{p} H_{2m}(\sqrt{x} \cos \frac{2k-1}{4p} \pi) \cdot \tag{14}$$

$$\cdot H_2 n(\sqrt{x} \cdot \sin \frac{2k-1}{4p} \pi),\ (\forall)x > 0.$$

**Proof.** In Feldheim formula we consider $\alpha = \beta = -\frac{1}{2}$,

$$l_{m+n}(x) = \frac{1}{\pi} \int_0^1 l_m^{(-\frac{1}{2})}(xt) l_n^{(-\frac{1}{2})}((1-t)x) \frac{dt}{\sqrt{t(1-t)}}.$$

From (8′) we have

$$(15)\quad l_{m+n}(x)=\frac{(-1)^{n+m}n!m!}{(2n)!(2m)!}\cdot\frac{1}{\pi}\int_0^1 H_{2m}(\sqrt{xt})H_{2n}(\sqrt{(1-t)x})\frac{dt}{\sqrt{t(1-t)}}$$

We observe that in integral we have a polynomial of $m+n$ degree in $t$. In the following we use the Mehler-Hermite cuadrature formula:

$$\int_{-1}^1 f(t)\frac{dt}{\sqrt{1-t^2}}=\frac{\pi}{p}\sum_{k=1}^p f(x_k)+\frac{\pi}{2^{2p-1}(2p)!}f^{(2p)}(\xi),\ x_k=\cos\frac{2k-1}{2p},\xi\in[-1,1].$$

For $f\in\Pi_{2p-1}$ we obtain:

$$\int_{-1}^1 f(t)\frac{dt}{\sqrt{1-t^2}}=\frac{\pi}{p}\sum_{k=1}^p f(x_k).$$

Making variable substitution: $t=2u-1$, it results

$$\int_0^1 f(2u-1)\frac{du}{\sqrt{u(1-u)}}=\frac{\pi}{p}\sum_{k=1}^p f(x_k).$$

We put: $f(2u-1)=g(u)$.

For $u=\frac{1+y}{2}$ we obtain $f(y)=g(\frac{1+y}{2})$,

$$\int_0^1 f(2u-1)\frac{du}{\sqrt{u(1-u)}}=\int_0^1 g(u)\frac{du}{\sqrt{u(1-u)}}=\frac{\pi}{p}\sum_{k=1}^p g\left(\frac{1+x_k}{2}\right),$$

$$\int_0^1 g(t)\frac{dt}{\sqrt{t(1-t)}}=\frac{\pi}{p}\sum_{k=1}^p g\left(\cos^2\frac{2k-1}{4p}\pi\right),\ g\in\Pi_{2p-1}.$$

Using (15) we have

$$l_{m+n}(x)=\frac{(-1)^{n+m}n!m!}{(2n)!(2m)!}\cdot\frac{1}{p}\sum_{k=1}^p H_{2m}\left(\sqrt{x\cos^2\frac{2k-1}{4p}\pi}\right)$$
$$\cdot H_{2n}\left(\sqrt{x\sin^2\frac{2k-1}{4p}\pi}\right)=\frac{(-1)^{n+m}n!m!}{(2n)!(2m)!}$$

$$\cdot\frac{1}{p}\sum_{k=1}^{p} H_{2m}\left(\sqrt{x}\cos\frac{2k-1}{4p}\pi\right) H_{2n}\left(\sqrt{x}\sin\frac{2k-1}{4p}\pi\right)$$

# References


[1] Richard Askey, George Gasper, *Positive Jacobi polynomial sums*, II, Amer. J. Math. 98, 1976, 709-737.

[2] E. Feldheim, *Développment en série de polynomes d'Hermite et de Laguerre a l'aide des transformations de Gauss et de Hankel*, I, II, Konin. Neder. Akad. van Weten. 43, 1940, 224-248.

[3] G. Palamă, *Sulli unlimite inferioare della distanza di due zeri consecutivi di* $H_n(x)$ *e su di una limitazione di* $H_n^2(x) - H_{n_1}(x)H_{n+1}(x)$, Boll. Un. Mat. Hol., VII, 3, 1952, 311-315.

[4] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publications, 23, 1978.



**Ioan Ţincu**
"Lucian Blaga" University
Department of Mathematics
Sibiu, Romania
e-mail: tincuioan@yahoo.com