General Mathematics Vol. 18, No. 1 (2010), 139–149

A negative result on the linear precision of certain rational Schoenberg splines [1]

Michael Wozniczka

Abstract

We investigate the degree of polynomial exactness of the rational Schoenberg-type operator introduced by Gonska et al. [4]. As presumed by Tachev [11, Conjecture 2.6], linear polynomials are generally reproduced if and only if all weights coincide. Consequently, the convex hull property and geometric variation-diminution do not extend to the truly rational case.

2000 Mathematics Subject Classification: 41A15, 41A20, 41A36, 65D07, 65D15

Key words and phrases: Schoenberg operator, rational splines, NURBS, linear precision

[1] *Received 2 September, 2009*

Accepted for publication (in revised form) 18 January, 2010

1 Introduction

Despite the remarkable success of *parametric* curve and surface techniques on the basis of Non-Uniform Rational B(asis)-Splines (NURBS) [5, 12] in Computer Aided Geometric Design (CAGD) [1], correlating rational generalizations of Schoenberg's variation-diminishing spline *functions* [10] have not been systematically studied from an approximation theoretical point of view until recently.

In [4], Gonska et al. propose a rational Schoenberg-type operator basically as follows.

Definition 1. *Let* $(t_i)_{i=0}^m$, $m \in \mathbb{N}$, *be a strictly increasing sequence of real numbers partitioning the* basic interval $[t_0, t_m] = [0, 1]$ *into* m *segments* $T_i := [t_i, t_{i+1})$, $0 \le i \le m-2$, *and* $T_{m-1} := [t_{m-1}, t_m]$. *Given* $d \in \mathbb{N}$, *let*

$$\mathbf{t} := (t_{-d} = \cdots = t_0 < \cdots < t_m = \cdots = t_{m+d}) \tag{1}$$

denote the corresponding knot sequence. *Considering* Greville's abscissae

$$\xi_{i,d,\mathbf{t}} := \frac{1}{d} \sum_{j=1}^{d} t_{i+j}, \quad -d \le i \le m-1, \tag{2}$$

and a sequence of strictly positive weights $\mathbf{w} := (w_i)_{i=-d}^{m-1}$, *the* rational Schoenberg operator $\mathcal{R}_{d,\mathbf{t},\mathbf{w}}$ *is defined by*

$$\mathcal{R}_{d,\mathbf{t},\mathbf{w}} : \mathbb{R}^{[t_0,t_m]} \ni f \mapsto \frac{\mathcal{W}_{d,\mathbf{t},\mathbf{w}} f}{\mathcal{W}_{d,\mathbf{t},\mathbf{w}} e_0} \in \mathbb{R}^{[t_0,t_m]}, \tag{3}$$

where $e_0 : [t_0, t_m] \ni x \mapsto 1 \in \mathbb{R}$, *and* $\mathcal{W}_{d,\mathbf{t},\mathbf{w}}$ *denotes the* weighted Schoenberg operator, *specified by*

$$\mathcal{W}_{d,\mathbf{t},\mathbf{w}} : \mathbb{R}^{[t_0,t_m]} \ni f \mapsto \sum_{i=-d}^{m-1} w_i f(\xi_{i,d,\mathbf{t}}) N_{i,d,\mathbf{t}} \in \mathbb{R}^{[t_0,t_m]}. \tag{4}$$

Here, $N_{i,d,\mathbf{t}}$*,* $-d \le i \le m-1$*, is the classical* (normalized) B-spline *of* degree d *with respect to the knots* $t_i, \dots, t_{i+d+1}$*, such that it is polynomial on each segment* T_j*,* $0 \le j \le m-1$*, and, particularly,* $N_{m-1,d,\mathbf{t}}(t_m) = 1$*.*

Remark 1.

(a) For $m = 1$*, i.e., if we have no* interior *knots, the rational Schoenberg operator* $\mathcal{R}_{d,\mathbf{t},\mathbf{w}}$ *is also called* rational Bernstein operator*.*

(b) The weighted Schoenberg operator $\mathcal{W}_{d,\mathbf{t},\mathbf{w}}$ *reduces to the* classical Schoenberg operator $\mathcal{S}_{d,\mathbf{t}}$ *if all weights are equal to* 1*.*

(c) Definition 1 readily generalizes to basic intervals $[t_0, t_m] \ne [0, 1]$*.*

We recall some basic properties of the *classical* Schoenberg operator. Proofs of these and further characteristics are collected in de Boor's eminent work [3].

Remark 2.

(a) $\mathcal{S}_{d,\mathbf{t}}$ *is* discretely defined*,* linear*, and* positive*.*

(b) $\mathcal{S}_{d,\mathbf{t}} L = L$ *for all* linear *polynomials* $L \in \Pi_1[t_0, t_m]$*.*

(c) $\mathcal{S}_{d,\mathbf{t}}$ *possesses the* (strong) convex hull property*, i.e., for all* $f \in \mathbb{R}^{[t_0,t_m]}$ *and* $x \in T_j$*,* $0 \le j \le m-1$*, the point* $(x, \mathcal{S}_{d,\mathbf{t}}(f;x))$ *is a* convex *linear combination of* $\{(\xi_{i,d,\mathbf{t}}, f(\xi_{i,d,\mathbf{t}}))\}_{i=j-d}^{j}$*.*

(d) $\mathcal{S}_{d,\mathbf{t}}$ *is* geometrically variation-diminishing*, i.e., it holds*

$$\mathrm{S}^-_{[t_0,t_m]}(\mathcal{S}_{d,\mathbf{t}} f - L) \le \mathrm{S}^-_{[t_0,t_m]}(f - L) \tag{5}$$

for all $f \in \mathbb{R}^{[t_0,t_m]}$ *and* $L \in \Pi_1[t_0,t_m]$, *where* $\mathrm{S}_A^- g$ *denotes the number of* (strict) sign changes *of a function* $g \in \mathbb{R}^A$, $A \subseteq \mathbb{R}$.

Considering Remark 2, the following features of the *rational* Schoenberg operator, also pointed out by Gonska et al. [4, Proposition 1, Example 1], are immediate consequences of its definition.

Remark 3.

(a) $\mathcal{R}_{d,\mathbf{t},\mathbf{w}}$ *is* discretely defined, linear, *and* positive.

(b) $\mathcal{R}_{d,\mathbf{t},\mathbf{w}} = \mathcal{S}_{d,\mathbf{t}}$ *if all weights coincide.*

(c) $\mathcal{R}_{d,\mathbf{t},\mathbf{w}}C = C$ *for all* constant *polynomials* $C \in \Pi_0[t_0,t_m]$.

In [11, Conjecture 2.6], Tachev presumes that $\mathcal{R}_{d,\mathbf{t},\mathbf{w}}$ reproduces linear polynomials generally if and only if all weights coincide and provides proofs for $d \in \{1,2\}$ exploiting specific characteristics of linear and quadratic B-splines, respectively. Based on an observation regarding the local linear independence of certain products of cubic B-splines, Piţul [6] successfully verifies Tachev's conjecture for $d = 3$.

Unfortunately, it seems as if either technique cannot be easily extended to higher degrees [11, Remark 2.5], [6, Remark 2.8].

2 A Proof of Tachev's Conjecture

The aforementioned proofs for $d \leq 3$ employ the subsequent statement, essentially given by Tachev [11, Theorem 2.1]. Since the original source contains a misprint, we include a complete, but concise verification.

Lemma 1. *Let* $e_1 : [t_0, t_m] \ni x \mapsto x \in \mathbb{R}$. *Then it holds*

$$\mathcal{R}_{d,\mathbf{t},\mathbf{w}}e_1 - e_1 = \frac{1}{\mathcal{W}_{d,\mathbf{t},\mathbf{w}}e_0} \sum_{j=-d}^{m-1} \sum_{i=j+1}^{m-1} (w_i - w_j)(\xi_{i,d,\mathbf{t}} - \xi_{j,d,\mathbf{t}}) N_{i,d,\mathbf{t}} N_{j,d,\mathbf{t}}. \tag{6}$$

Proof. Omitting insignificant subscripts, we obtain

$$\begin{aligned}
\mathcal{W}e_0(\mathcal{R}e_1 - e_1) &= \mathcal{W}e_1 \mathcal{S}e_0 - \mathcal{W}e_0 \mathcal{S}e_1 \\
&= \sum_{i=-d}^{m-1} w_i \xi_i N_i \sum_{j=-d}^{m-1} N_j - \sum_{i=-d}^{m-1} w_i N_i \sum_{j=-d}^{m-1} \xi_j N_j \\
&= \sum_{i=-d}^{m-1} \sum_{j=-d}^{m-1} w_i(\xi_i - \xi_j) N_i N_j \\
&= \sum_{i=-d}^{m-1} \sum_{j=-d}^{i-1} w_i(\xi_i - \xi_j) N_i N_j + \sum_{i=-d}^{m-1} \sum_{j=i+1}^{m-1} w_i(\xi_i - \xi_j) N_i N_j.
\end{aligned}$$

Changing the order of summation in the first, and relabeling indices in the second term yields

$$\begin{aligned}
\mathcal{W}e_0(\mathcal{R}e_1 - e_1) &= \sum_{j=-d}^{m-1} \sum_{i=j+1}^{m-1} w_i(\xi_i - \xi_j) N_i N_j - \sum_{j=-d}^{m-1} \sum_{i=j+1}^{m-1} w_j(\xi_i - \xi_j) N_i N_j \\
&= \sum_{j=-d}^{m-1} \sum_{i=j+1}^{m-1} (w_i - w_j)(\xi_i - \xi_j) N_i N_j.
\end{aligned}$$

The main contribution of this work comprises

Proposition 1.

$$\mathcal{R}_{d,\mathbf{t},\mathbf{w}}e_1 = e_1 \Rightarrow w_i = w_j, \quad -d \leq i, j \leq 0. \tag{7}$$

Proof. For the sake of clarity, we refrain from writing subscripts which do not affect our reasoning. Let $\mathcal{R}_d e_1 = e_1$. Then we have

$$\mathcal{W}_d e_1 = e_1 \mathcal{W}_d e_0. \tag{8}$$

Since $\operatorname{supp} N_{i,d} = [t_i, t_{i+d+1}]$ (cf. [3, p. 88]), for $x \in [t_0, t_1)$ we obtain the *polynomial* identity

$$\sum_{i=-d}^{0} w_i \xi_{i,d} N_{i,d}(x) = x \sum_{i=-d}^{0} w_i N_{i,d}(x). \tag{9}$$

Following de Boor [3, p. 117], differentiating (9) r times, $1 \le r \le d$, with respect to x yields

$$\begin{aligned} \sum_{i=-d+r}^{0} a_i^{[r]} N_{i,d-r}(x) &= \sum_{j=0}^{r} \binom{r}{j} e_1^{(j)}(x) \sum_{i=-d+r-j}^{0} w_i^{[r-j]} N_{i,d-r+j}(x) \\ &= x \sum_{i=-d+r}^{0} w_i^{[r]} N_{i,d-r}(x) + r \sum_{i=-d+r-1}^{0} w_i^{[r-1]} N_{i,d-r+1}(x), \end{aligned} \tag{10}$$

where

$$a_i^{[r]} = \begin{cases} w_i \xi_{i,d} & , r = 0, \\ \frac{a_i^{[r-1]} - a_{i-1}^{[r-1]}}{\xi_{i,d-r+1} - \xi_{i-1,d-r+1}} & , 1 \le r \le d, \end{cases} \tag{11}$$

$$w_i^{[r]} = \begin{cases} w_i & , r = 0, \\ \frac{w_i^{[r-1]} - w_{i-1}^{[r-1]}}{\xi_{i,d-r+1} - \xi_{i-1,d-r+1}} & , 1 \le r \le d. \end{cases} \tag{12}$$

Since $t_i < t_{i+d-r+1}$, $-d + r \le i \le 0$, $1 \le r \le d$, we necessarily have $\xi_{i-1,d-r+1} < \xi_{i,d-r+1}$. Putting $x = t_0 = 0$, equation (10) simplifies to

$$a_{-d+r}^{[r]} = r w_{-d+r-1}^{[r-1]}, \quad 1 \le r \le d. \tag{13}$$

For $r = 1$ we obtain the base case

$$w := w_{-d} = w_{-d}^{[0]} = a_{-d+1}^{[1]} = \frac{a_{-d+1}^{[0]} - a_{-d}^{[0]}}{\xi_{-d+1,d} - \xi_{-d,d}} = \frac{w_{-d+1} \xi_{-d+1,d}}{\xi_{-d+1,d}} = w_{-d+1}. \tag{14}$$

For $2 \le r \le d$ we shall show

$$a^{[r-l]}_{-d+r} = \begin{cases} 0 & , 0 \le l \le r-2, \\ w_{-d+r} = w & , l = r-1. \end{cases} \tag{15}$$

Taking into account that (15) reduces to (14) for $r = 1$, we assume validity of (15) up to $r - 1$. Then $w = w_{-d} = \cdots = w_{-d+r-1}$, and we have

$$0 = rw^{[r-1]}_{-d+r-1} = a^{[r]}_{-d+r}. \tag{16}$$

Thus (15) is satisfied for r and $l = 0$. For $1 \le l \le r-1$, assuming (15) for $l - 1$, it holds

$$0 = a^{[r-l+1]}_{-d+r} = a^{[r-l]}_{-d+r} - a^{[(r-1)-(l-1)]}_{-d+r-1} = \begin{cases} a^{[r-l]}_{-d+r} & , 1 \le l \le r-2, \\ a^{[1]}_{-d+r} - w & , l = r-1. \end{cases} \tag{17}$$

Finally,

$$\begin{aligned} 0 &= a^{[1]}_{-d+r} - w \\ &= \frac{a^{[0]}_{-d+r} - a^{[0]}_{-d+r-1}}{\xi_{-d+r,d} - \xi_{-d+r-1,d}} - w \\ &= \frac{w_{-d+r}\xi_{-d+r,d} - w_{-d+r-1}\xi_{-d+r-1,d}}{\xi_{-d+r,d} - \xi_{-d+r-1,d}} - w \\ &= \frac{\xi_{-d+r,d}}{\xi_{-d+r,d} - \xi_{-d+r-1,d}}(w_{-d+r} - w) \end{aligned}$$

implies $w_{-d+r} = w$.

Remark 4.

(a) *The last proof points up an alternative way to show [4, Proposition 3] concerning the linear precision of the rational Bernstein operator.*

(b) Since rational Schoenberg splines are invariant under affine transformations in the domain, Proposition 1 also holds for basic intervals $[t_0, t_m] \neq [0, 1]$.

We enhance the last result utilizing a technique introduced by Piţul [6].

Corollary 1.

$$\mathcal{R}_{d,\mathbf{t},\mathbf{w}} e_1 = e_1 \Rightarrow w_i = w_j, \quad -d \leq i, j \leq m-1. \tag{18}$$

Proof. Let $\mathcal{R}_{d,\mathbf{t},\mathbf{w}} e_1 = e_1$. We show

$$w := w_{-d} = w_{-d+r}, \quad 0 \leq r \leq d + l,\, 0 \leq l \leq m-1. \tag{19}$$

For $l = 0$ this is Proposition 1. For $1 \leq l \leq m-1$, we assume (19) for $l-1$. Let $x \in (t_l, t_{l+1})$. Then, restricting ourselves to writing significant subscripts as before, by Lemma 1 and the B-splines' local support property we have

$$\begin{aligned} 0 &= \sum_{j=l-d}^{l-1} \sum_{i=j+1}^{l} \underbrace{(w_i - w_j)}_{=0,\ i \leq l-1} (\xi_i - \xi_j) N_i(x) N_j(x) \\ &= \sum_{j=l-d}^{l-1} (w_l - w)(\xi_l - \xi_j) N_l(x) N_j(x) \\ &= (w_l - w) \underbrace{N_l(x)}_{>0} \sum_{j=l-d}^{l-1} \underbrace{(\xi_l - \xi_j)}_{>0} \underbrace{N_j(x)}_{>0} \\ &= w_l - w. \end{aligned}$$

Combining Corollary 1 with Remarks 2 and 3, we arrive at

Theorem 1 (Tachev [11, Conjecture 2.6])**.**

$$\mathcal{R}_{d,\mathbf{t},\mathbf{w}} L = L \text{ for all } L \in \Pi_1[t_0, t_m] \Leftrightarrow w_i = w_j, \quad -d \leq i, j \leq m-1. \tag{20}$$

As an immediate consequence we obtain

Corollary 2. *The rational Schoenberg operator* $\mathcal{R}_{d,\mathbf{t},\mathbf{w}}$ *possesses the convex hull property (in the strong or classical sense) and is geometrically variation-diminishing, respectively, if and only if all weights coincide.*

3 Conclusion

Historically, Schoenberg introduced his piecewise polynomial approximation scheme $\mathcal{S}_{d,\mathbf{t}}$ as a generalization of Bernstein's operator [2] preserving its variation-diminishing properties. It is effective in the sense that it enables an improved rate of convergence at the expense of reduced smoothness.

Visualizing the success of NURBS in CAGD, it is a natural question to ask for rational generalizations of Schoenberg's classical method. Following Schoenberg, those schemes seem to be most valuable which maintain vital shape characteristics while reducing the error of approximation.

The rational Schoenberg-type operator $\mathcal{R}_{d,\mathbf{t},\mathbf{w}}$ suggested by Gonska et al. [4] does not belong to this class. Our analysis confirms that its inability to reproduce linear polynomials in the truly rational case is an intrinsic feature.

Further aspects of this method and alternative approaches are investigated in [7–9].

Acknowledgment. Financial support by all sponsors of the *Romanian-German Symposium on Mathematics and its Applications 2009*, notably by DFG, the Romanian Academy, and the University of Duisburg-Essen, as well as its organizers' huge effort and warm hospitality are gratefully acknowledged.

Furthermore, the author expresses his gratitude to all members, former members and guests of the research group *Mathematical Informatics*, particularly to Heiner Gonska, Paula Pabst, Maria Rusu, and Gancho Tachev, for their friendly support, helpful advice, and valuable suggestions.

References

[1] Robert E. Barnhill and Richard F. Riesenfeld, editors. *Computer Aided Geometric Design: Proceedings of a Conference held at the University of Utah, Salt Lake City, UT, USA, March 18–21, 1974*, New York, NY, USA, 1974. Academic Press. ISBN 978-0-12-079050-0.

[2] Sergei N. Bernstein. Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités. *Communications de la Société Mathématique de Kharkov*, 13:1–2, 1912.

[3] Carl R. de Boor. *A Practical Guide to Splines*. Springer, New York, NY, USA, revised edition, 2001. ISBN 978-0-387-95366-3.

[4] Heinz H. Gonska, Daniela P. Kacsó, and Paula A. Piţul. On rational B-spline functions. In Borislav D. Bojanov, editor, *Constructive Theory of Functions: Varna 2005*, pages 145–157, Sofia, Bulgaria, 2006. Marin Drinov Academic Publishing House. ISBN 978-954-322-144-8.

[5] Les A. Piegl and Wayne Tiller. *The NURBS Book*. Springer, Berlin, Germany, 2nd edition, 1997. ISBN 978-3-540-61545-3.

[6] Paula A. Piţul. *Evaluation of the Approximation Order by Positive Linear Operators*. PhD thesis, Faculty of Mathematics and Informatics, Babeş-Bolyai University, Cluj-Napoca, Romania, 2007.

[7] Paula A. Piţul and Paul Sablonnière. A family of univariate rational Bernstein operators. *Journal of Approximation Theory*, 160(1–2):39–55, 2009. ISSN 0021-9045.

[8] Maria D. Rusu. True NURBS in Approximation Theory: A Survey. Master's thesis, Faculty of Mathematics, University of Duisburg-Essen, Duisburg, Germany, 2009.

[9] Paul Sablonnière. Rational Bernstein and spline approximation. A new approach. *Jaén Journal on Approximation*, 1(1):37–53, 2009. ISSN 1889-3066.

[10] Isaac J. Schoenberg. On spline functions. In Oved Shisha, editor, *Inequalities: Proceedings of a Symposium held at Wright-Patterson Air Force Base, OH, USA, August 19–27, 1965*, pages 255–291, New York, NY, USA, 1967. Academic Press. ISBN 978-0-12-640303-9. With a supplement by Thomas N. E. Greville.

[11] Gancho T. Tachev. Approximation by rational spline functions. *Calcolo*, 43(4):279–286, 2006. ISSN 0008-0624.

[12] Kenneth J. Versprille. *Computer-Aided Design Applications of the Rational B-Spline Approximation Form.* Dissertation, Graduate School, Syracuse University, Syracuse, NY, USA, 1975.

Michael Wozniczka
Faculty of Mathematics
University of Duisburg-Essen
47048 Duisburg
Germany
E-mail: michael.wozniczka@uni-due.de