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On a class of p-valent non-Bazilevic functions [1]

Khalida Inayat Noor, Ali Muhammad, Muhammad Arif

Abstract

In this paper, we introduce a class $N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$. We investigate a number of inclusion relationships, distortion theorems for the class $N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$, the lower and upper bounds of $Re\left(\frac{z^p}{I_p^\lambda(a,c)f(z)}\right)^\mu$ for $f(z)\in N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$ and some other interesting properties of p-valent functions which are defined here by means of a certain linear integral operator $I_p^\lambda(a,c)f(z)$.

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1 Introduction

Let $A(p)$ denote the class of functions $f(z)$ normalized by

$$(1)\qquad f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p},\ (p\in\mathbb{N}=\{1,2,\ldots\}),$$

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which are analytic and p-valent in the open unit disc $E = \{z : |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in E, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \text{ in } E \text{ or } f(z) \prec g(z),\ z \in E,$$

if there exists a Schwarz function $w(z)$, which is analytic in E with

$$|w(0)| = 0 \text{ and } |w(z)| < 1,\ z \in E,$$

such that

$$f(z) = g(w(z)),\ z \in E.$$

Indeed it is known that

$$f(z) \prec g(z)\ \ (z \in E) \Rightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

Furthermore, if the function $g(z)$ is univalent in E, then we have the following equivalence, see [6, 7],

$$f(z) \prec g(z)\ \ (z \in E) \Leftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

For functions $f_j(z) \in A(p)$, given by

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,j} z^{k+p} \ \ (j = 1, 2)\,, \tag{2}$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 \star f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p} = (f_2 \star f_1)(z) \ \ (z \in E)\,. \tag{3}$$

In our present investigation we shall make use of the Gauss hypergeometric functions defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k \ \ (z \in E)\,, \tag{4}$$

where $a, b, c \in \mathbb{C}$, $c \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$ and $(k)_n$ denote the Pochhammer symbol (or the shifted factorial) given, in terms of the Gamma function Γ, by

$$(k)_n = \frac{\Gamma(k+n)}{\Gamma(k)} = \begin{cases} k(k+1)(k+2)\ldots(k+n-1), \, n \in \mathbb{N} \\ 1, \, n = 0. \end{cases}$$

We note that the series defined by (4) converges absolutely for $z \in E$ and hence ${}_2F_1(a, b; c; z)$ represents an analytic function in E, see [13].

We define a function $\Phi_p(a, c; z)$ by

$$\Phi_p(a, c; z) = z^p + \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \, z^{k+p} \quad \left(a \in \mathbb{R}; \; c \in \mathbb{R} \backslash \mathbb{Z}_0^- = \{0, -1, \ldots\}\right).$$

With the aid of the function $\Phi_p(a, c; z)$, we consider a function $\Phi_p^{\dagger}(a, c; z)$ defined by

$$\Phi_p(a, c; z) \star \Phi_p^{\dagger}(a, c; z) = \frac{z^p}{(1-z)^{\lambda+p}}, \, z \in E,$$

where $\lambda > -p$. This function yields the following family of linear operators

$$I_p^{\lambda}(a, c) f(z) = \Phi_p^{\dagger}(a, c; z) \star f(z), \, z \in E, \tag{5}$$

where $a, c \in \mathbb{R} \backslash \mathbb{Z}_0^-$. For a function $f(z) \in A(p)$, given by (1), it follows from (5) that for $\lambda > -p$ and $a, c \in \mathbb{R} \backslash \mathbb{Z}_0^-$

$$\begin{aligned} I_p^{\lambda}(a, c) f(z) &= z^p + \sum_{k=0}^{\infty} \frac{(c)_k \, (\lambda+p)_k}{(a)_k \, (1)_k} \, a_{p+k} z^{p+k} \\ &= z^p {}_2F_1(c, \lambda+p; a; z) \star f(z), \, z \in E. \end{aligned} \tag{6}$$

From equation (6) we deduce that

$$z\left(I_p^{\lambda}(a, c) f(z)\right)' = (\lambda+p) I_p^{\lambda+1}(a, c) f(z) - \lambda I_p^{\lambda}(a, c) f(z), \tag{7}$$

and

$$z\left(I_p^{\lambda}(a+1, c) f(z)\right)' = a I_p^{\lambda}(a, c) f(z) - (a-p) I_p^{\lambda}(a+1, c) f(z). \tag{8}$$

We also note that

$$
\begin{aligned}
I_p^0(a+1,1)f(z) &= p\int_0^z \frac{f(t)}{t}dt, \\
I_p^0(p,1)f(z) &= I_p^1(p+1,1)f(z) = f(z), \\
I_p^1(p,1)f(z) &= \frac{zf'(z)}{p}, \\
I_p^2(p,1)f(z) &= \frac{2zf'(z)+z^2f''(z)}{p(p+1)}, \\
I_p^2(p+1,1)f(z) &= \frac{f(z)+zf'(z)}{p(p+1)}, \\
I_p^n(a,a)f(z) &= D^{n+p-1}f(z),\ n\in\mathbb{N},\ n>-p,
\end{aligned}
$$

where $D^{n+p-1}f(z)$ is the Ruscheweyh derivative of $(n+p-1)$th order, see [4].

The operator $I_p^\lambda(a,c)$ $\left(\lambda>-p,\ a;c\in\mathbb{R}\backslash\mathbb{Z}_0^-\right)$ was recently introduced by Cho et al [1], who investigated (among other things) some inclusion relationships and argument properties of various subclasses of multivalent functions in $A(p)$, which were defined by means of the operator $I_p^\lambda(a,c)$.

For $\lambda=c=1$ and $a=n+p$, the Cho-Kown-Srivastava operator yields

$$
I_p^1(n+p,1)f(z) = I_{n,p}\ \ (n>-p),
$$

where $I_{n,p}$ denotes an integral operator of the $(n+p-1)$th order, which was studied by Liu and Noor [5], see also [9, 10]. The linear operator $I_1^\lambda(\mu+2,1)$ $(\lambda>-1,\ \mu>-2)$ was also recently introduced and studied by Choi et al [2]. For relevant details about further special cases of the Choi-Saigo-Srivastava operator $I_1(\lambda+2,1)$, the interested reader may refer to the works by Cho et al [2] and Choi et al [1], see also [3].

Using the Cho-Kown-Srivastava operator $I_p^\lambda(a,c)$, we now define a subclass of $A(p)$ as follows:

Definition 1 *Assume that* $0 < \mu < 1$, $\alpha \in \mathbb{C}$, $-1 \leq B \leq 1$, $A \neq B$, $A \in \mathbb{R}$, *we say that a function* $f(z) \in A(p)$ *is in the class* $N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$ *if it satisfies:*

$$\left\{(1-\alpha)\left(\frac{z^p}{I_p^{\lambda}(a,c)\,f(z)}\right)^{\mu}-\alpha\left(\frac{I_p^{\lambda+1}(a,c)}{I_p^{\lambda}(a,c)}\right)\left(\frac{z^p}{I_p^{\lambda}(a,c)\,f(z)}\right)^{\mu}\right\}\prec\frac{1+Az}{1+Bz},\ z\in E,$$

where the powers are understood as a principal values.

In particular, we let $N_{p,\alpha}^{\lambda,\mu}(a,c,1-2\rho,-1) = N_{p,\alpha}^{\lambda,\mu}(a,c,\rho)$ denote the subclass $N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$ for $A = 1-2\rho$, $B = -1$ and $0 \leq \rho < p$. It is obvious that $f(z) \in N_{p,\alpha}^{\lambda,\mu}(a,c,\rho)$ if and only if $f(z) \in A(p)$ and it satisfies

$$Re\left\{(1-\alpha)\left(\frac{z^p}{I_p^{\lambda}(a,c)\,f(z)}\right)^{\mu}-\alpha\left(\frac{I_p^{\lambda+1}(a,c)}{I_p^{\lambda}(a,c)}\right)\left(\frac{z^p}{I_p^{\lambda}(a,c)\,f(z)}\right)^{\mu}\right\}>\rho,\ z\in E.$$

Special Cases

(i) When $a = c = p = 1$, $\lambda = 0$, then $N_{1,\alpha}^{0,\mu}(1,1,A,B)$ is the class studied by Z. Wang et al [14].

(ii) The subclass $N_{1,-1}^{0,\mu}(1,1,1,-1) = N(\mu)$ has been studied by Obradovic [11].

(iii) If $a = c = p = 1$, $\lambda = 0$, $\alpha = B = -1$ and $A = 1-2\rho$, then the class $N_{1,-1}^{0,\mu}(1,1,1-2\rho,-1)$ reduces to the class of non-Bazilevic functions of order $\rho(0 \leq \rho < 1)$. The Fekete-Szegö problem of the class $N_{1,-1}^{0,\mu}(1,1,1-2\rho,-1)$ were considered by N. Tuneski and M .Darus [12].

2 Preliminary Results

In this section we recall some known results.

Lemma 1 *Let the function $h(z)$ be analytic and convex (univalent) in E with $h(0) = 1$. Suppose also that the function $\Phi(z)$ given by*

$$\Phi(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

is analytic in E. If

$$\Phi(z) + \frac{z\,\Phi'(z)}{\gamma} \prec h(z) \ \ (z \in E;\ Re\gamma \geq 0;\ \gamma \neq 0)\,, \tag{9}$$

then

$$\Phi(z) \prec \Psi(z) = \frac{\gamma}{z^{\gamma}} \int_0^z t^{\gamma-1} h(t)dt \prec h(z) \ \ (z \in E)\,,$$

and $\Psi(z)$ is the best dominant of (9).

3 Main Result

Theorem 1 *Let $Re\alpha > 0$ and $f(z) \in N_{p,\alpha}^{\lambda;\mu}(a, c, A, B)$. Then*

$$\left(\frac{z^p}{I_p^{\lambda}(a,c)\, f(z)}\right)^{\mu} \prec \frac{(\lambda+p)\,\mu}{\alpha} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{(\lambda+p)\mu}{\alpha}-1} du \prec \frac{1+Az}{1+Bz}. \tag{10}$$

Proof. Let

$$\Phi(z) = \left(\frac{z^p}{I_p^{\lambda}(a,c)\, f(z)}\right)^{\mu}. \tag{11}$$

Then $\Phi(z)$ is analytic in E with $\Phi(0) = 1$. Taking logarithmic differentiation of (11) both sides and using the identity (7) in the resulting equation, we deduce that

$$\left\{(1-\alpha)\left(\frac{z^p}{I_p^{\lambda}(a,c)\, f(z)}\right)^{\mu} - \alpha\left(\frac{I_p^{\lambda+1}(a,c)}{I_p^{\lambda}(a,c)}\right)\left(\frac{z^p}{I_p^{\lambda}(a,c)\, f(z)}\right)^{\mu}\right\}$$

$$= \Phi(z) + \frac{\alpha z\ \Phi'(z)}{(\lambda+p)\,\mu} \prec \frac{1+Az}{1+Bz}.$$

Now, by Lemma 1 for $\gamma = \frac{(\lambda+p)\mu}{\alpha}$, we deduce that

$$\left(\frac{z^p}{I_p^\lambda(a,c)\,f(z)}\right)^\mu \prec q(z) = \frac{(\lambda+p)\,\mu}{\alpha} z^{-\frac{(\lambda+p)\mu}{\alpha}} \int_0^z t^{\frac{(\lambda+p)\mu}{\alpha}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$= \frac{(\lambda+p)\,\mu}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{(\lambda+p)\mu}{\alpha}-1} du \prec \frac{1+Az}{1+Bz},$$

and the proof is complete.

Theorem 2 *Let* $0 \le \alpha_2 \le \alpha_1$. *Then*

$$N_{p,\alpha_1}^{\lambda,\mu}(a,c,A,B) \subset N_{p,\alpha_2}^{\lambda,\mu}(a,c,A,B).$$

Proof. Let $f(z) \in N_{p,\alpha_1}^{\lambda,\mu}(a,c,A,B)$. Then by Theorem 3.1 we have

$$f(z) \in N_{p,0}^{\lambda,\mu}(a,c,A,B).$$

Since

$$\left\{(1+\alpha_2)\left(\frac{z^p}{I_p^\lambda(a,c)\,f(z)}\right)^\mu - \alpha_2\left(\frac{I_p^{\lambda+1}(a,c)}{I_p^\lambda(a,c)}\right)\left(\frac{z^p}{I_p^\lambda(a,c)\,f(z)}\right)^\mu\right\}$$

$$= \left(1+\frac{\alpha_2}{\alpha_1}\right)\left(\frac{z^p}{I_p^\lambda(a,c)\,f(z)}\right)^\mu - \frac{\alpha_2}{\alpha_1}\left\{(1+\alpha_1)\left(\frac{z^p}{I_p^\lambda(a,c)\,f(z)}\right)^\mu\right.$$

$$\left. -\alpha_1\left(\frac{I_p^{\lambda+1}(a,c)}{I_p^\lambda(a,c)}\right)\left(\frac{z^p}{I_p^\lambda(a,c)\,f(z)}\right)^\mu\right\} \prec \frac{1+Az}{1+Bz}.$$

Wee see that $f(z) \in N_{p,\alpha_2}^{\lambda,\mu}(a,c,A,B)$.

Theorem 3 *Let* $Re\alpha > 0$, $0 < \mu < 1$, $-1 \leq B < A \leq 1$ *and* $f(z) \in N_{p,\alpha}^{\lambda,\mu}(a, c, A, B)$. *Then*

$$\frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du < Re\left(\frac{z^p}{I_p^{\lambda}(a,c)f(z)}\right)^{\mu}$$

$$< \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du, \tag{12}$$

and the inequality (12) *is sharp, with the extremal function defined by*

$$I_p^{\lambda}(a,c)F_{\alpha,\mu,A,B}(z) = z^p\left\{\frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\right\}^{\frac{-1}{\mu}}. \tag{13}$$

Proof. Since $f(z) \in N_{p,\alpha}^{\lambda,\mu}(a, c, A, B)$, according to Theorem 1, we have

$$\left(\frac{z^p}{I_p^{\lambda}(a,c)f(z)}\right)^{\mu} \prec \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du,$$

Therefore it follows from the definition of subordination and $A > B$ that

$$\begin{aligned} Re\left(\frac{z^p}{I_p^{\lambda}(a,c)f(z)}\right)^{\mu} &< \sup_{z\in E} Re\left\{\frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\right\} \\ &\leq \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \sup_{z\in E} Re\left\{\frac{1+Azu}{1+Bzu}\right\}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du \\ &< \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du. \end{aligned}$$

Also

$$
\begin{aligned}
Re\left(\frac{z^{p}}{I_{p}^{\lambda}(a,c)f(z)}\right)^{\mu} &> \inf_{z\in E} Re\left\{\frac{(\lambda+p)\mu}{\alpha}\int_{0}^{1}\frac{1+Azu}{1+Bzu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\right\} \\
&\geq \frac{(\lambda+p)\mu}{\alpha}\int_{0}^{1}\inf_{z\in E} Re\left\{\frac{1+Azu}{1+Bzu}\right\}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du \\
&> \frac{(\lambda+p)\mu}{\alpha}\int_{0}^{1}\frac{1-Au}{1-Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du.
\end{aligned}
$$

Note that the function $I_{p}^{\lambda}(a,c)\,F_{\alpha,\mu,A,B}(z)$ defined by (13) belongs to the class $N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$ and hence we obtain that the inequality (12) is sharp. By applying the similar techniques that we used in proving Theorem 12, we have the following result.

Theorem 4 *Let* $Re\alpha > 0$, $0 < \mu < 1$, $-1 \leq A < B \leq 1$ *and* $f(z) \in N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$. *Then*

$$
\begin{aligned}
\frac{(\lambda+p)\mu}{\alpha}\int_{0}^{1}\frac{1+Au}{1+Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du &< Re\left(\frac{z^{p}}{I_{p}^{\lambda}(a,c)f(z)}\right)^{\mu} \\
&< \frac{(\lambda+p)\mu}{\alpha}\int_{0}^{1}\frac{1-Au}{1-Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du, \qquad (14)
\end{aligned}
$$

and the inequality (14) *is sharp, with the extremal function defined by* (13).

Theorem 5 *Let* $0 < \mu < 1$, $Re\alpha \geq 0$, $-1 \leq B < A \leq 1$ *and* $f(z) \in N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$. *Then*

$$
\begin{aligned}
\left(\frac{(\lambda+p)\mu}{\alpha}\int_{0}^{1}\frac{1-Au}{1-Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\right)^{\frac{1}{2}} &< Re\left(\frac{z^{p}}{I_{p}^{\lambda}(a,c)f(z)}\right)^{\frac{\mu}{2}} \qquad (15) \\
&< \left(\frac{(\lambda+p)\mu}{\alpha}\int_{0}^{1}\frac{1+Au}{1+Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\right)^{\frac{1}{2}},
\end{aligned}
$$

and inequality (15) *is sharp, with the extremal function defined by* (13).

Proof. According to Theorem 1, we have

$$\left(\frac{z^p}{I_p^\lambda(a,c)f(z)}\right)^\mu \prec \frac{1+Az}{1+Bz}.$$

Since $-1 \le B < A \le 1$, we have

$$0 < \frac{1-A}{1-B} < Re\left(\frac{z^p}{I_p^\lambda(a,c)f(z)}\right)^\mu < \frac{1+A}{1+B}.$$

Hence the result follows by Theorem 3.

Note that the function defined by (13) belongs to $N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$, we obtain that the inequality (15) is sharp. By applying the similar arguments as in Theorem 5, we have the following Theorem.

Theorem 6 *Let* $0 < \mu < 1$, $Re\alpha \ge 0$, $-1 \le A < B \le 1$ *and* $f(z) \in N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$. *Then*

$$\left(\frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\right)^{\frac{1}{2}} < Re\left(\frac{z^p}{I_p^\lambda(a,c)f(z)}\right)^{\frac{\mu}{2}} \tag{16}$$

$$< \left(\frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\right)^{\frac{1}{2}},$$

and inequality (16) *is sharp, with the extremal function defined by* (13).

Theorem 7 *Let* $0 < \mu < 1$, $Re\alpha \ge 0$, $-1 \le B < A \le 1$ *and* $f(z) \in N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$. *Then*

(i)If $\alpha = 0$, *the for* $|z| = r < 1$, *we have*

$$r^p\left(\frac{1+Br}{1+Ar}\right)^{\frac{1}{\mu}} \le \left|I_p^\lambda(a,c)f(z)\right| \le r^p\left(\frac{1-Br}{1-Ar}\right)^{\frac{1}{\mu}} \tag{17}$$

and inequality (17) *is sharp , with the extremal function defined by*

$$I_p^{\lambda}(a,c)\,f(z) = z^p\left(\frac{1+Bz}{1+Az}\right)^{\frac{1}{\mu}}. \tag{18}$$

(ii) If $\alpha \neq 0$, *the for* $|z| = r < 1$, *we have*

$$r^p\left(\frac{(\lambda+p)\,\mu}{\alpha}\int_0^1 \frac{1+Aru}{1+Bru}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\right)^{-\frac{1}{\mu}} \leq \left|I_p^{\lambda}(a,c)\,f(z)\right| \tag{19}$$

$$\leq r^p\left(\frac{(\lambda+p)\,\mu}{\alpha}\int_0^1 \frac{1-Aru}{1-Bru}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\right)^{-\frac{1}{\mu}},$$

and inequality (19) *is sharp with the extremal function defined by* (13).

Proof. (i) If $\alpha = 0$. Since $f(z) \in N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$, $-1 \leq B < A \leq 1$, we obtain from the definition of $N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$ that

$$\left(\frac{z^p}{I_p^{\lambda}(a,c)\,f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}.$$

Therefore it follows from the definition of the subordination that

$$\left(\frac{z^p}{I_p^{\lambda}(a,c)\,f(z)}\right)^{\mu} = \frac{1+Aw(z)}{1+Bw(z)},$$

where $w(z) = c_1 z + c_2 z^2 + \ldots$ is analytic E and $|w(z)| \leq |z|$, so when $|z| = r < 1$, we have

$$\left|\left(\frac{z^p}{I_p^{\lambda}(a,c)\,f(z)}\right)\right|^{\mu} = \left|\frac{1+Aw(z)}{1+Bw(z)}\right| \leq \frac{1+A\,|w(z)|}{1+B\,|w(z)|} \leq \frac{1+Ar}{1+Br},$$

and

$$\left|\left(\frac{z^p}{I_p^{\lambda}(a,c)\,f(z)}\right)\right|^{\mu} \geq Re\left(\frac{z^p}{I_p^{\lambda}(a,c)\,f(z)}\right)^{\mu} \geq \frac{1-Ar}{1-Br}.$$

It is obvious that (17) is sharp, with the extremal function defined by (18).

(ii) If $\alpha \neq 0$. according to Theorem 1 we have

$$\left(\frac{z^p}{I_p^\lambda(a,c)f(z)}\right)^\mu \prec \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du.$$

Therefore it follows from the definition of the subordination

$$\left(\frac{z^p}{I_p^\lambda(a,c)f(z)}\right)^\mu = \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Aw(z)u}{1+Bw(z)u}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du,$$

where $w(z)=c_1z+c_2z^2+\ldots$ is analytic E and $|w(z)|\leq|z|$, so when $|z|=r<1$, we have

$$\begin{aligned}\left|\left(\frac{z^p}{I_p^\lambda(a,c)f(z)}\right)\right|^\mu &\leq \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \left|\frac{1+Aw(z)u}{1+Bw(z)u}\right|u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\\ &\leq \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Au|w(z)|}{1+Bu|w(z)|}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du\\ &\leq \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du,\end{aligned}$$

and

$$\left|\left(\frac{z^p}{I_p^\lambda(a,c)f(z)}\right)\right|^\mu \geq Re\left(\frac{z^p}{I_p^\lambda(a,c)f(z)}\right)^\mu \geq \frac{(\lambda+p)\mu}{\alpha}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{(\lambda+p)\mu}{\alpha}-1}du.$$

Note that the function defined by (13) belongs to the class $N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$, we obtain that the inequality (19) is sharp. By applying the similar method as in Theorem 5 we have

Theorem 8 *Let* $0<\mu<1$, $Re\alpha\geq 0$, $-1\leq A<B\leq 1$ *and* $f(z)\in N_{p,\alpha}^{\lambda,\mu}(a,c,A,B)$. *Then*

(i) If $\alpha = 0$, the for $|z| = r < 1$, we have

$$r^p \left(\frac{1 - Br}{1 - Ar} \right)^{\frac{1}{\mu}} \leq \left| I_p^{\lambda}(a, c) f(z) \right| \leq r^p \left(\frac{1 + Br}{1 + Ar} \right)^{\frac{1}{\mu}} \tag{20}$$

and inequality (20) *is sharp, with the extremal function defined by* (18).

(ii) If $\alpha \neq 0$, the for $|z| = r < 1$, we have

$$r^p \left(\frac{(\lambda + p)\mu}{\alpha} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{(\lambda+p)\mu}{\alpha} - 1} du \right)^{-\frac{1}{\mu}} \leq \left| I_p^{\lambda}(a, c) f(z) \right| \tag{21}$$

$$\leq r^p \left(\frac{(\lambda + p)\mu}{\alpha} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{(\lambda+p)\mu}{\alpha} - 1} du \right)^{-\frac{1}{\mu}},$$

and inequality (21) *is sharp with the extremal function defined by* (13).

Theorem 9 *Let $Re\alpha \geq 0$ and $f(z) \in N_{p,0}^{\lambda\mu}(a, c, A, B)$. Then $f(z) \in N_{p,\alpha}^{\lambda,\mu}(a, c, A, B)$ for $|z| < R(\lambda, \alpha, \mu, p)$, where*

$$R(\lambda, \alpha, \mu, p) = \frac{(\lambda + p)\mu}{\alpha + \sqrt{\alpha^2 + (\lambda + p)^2 \mu^2}}. \tag{22}$$

Proof. Set

$$\left(\frac{z^p}{I_p^{\lambda}(a, c) f(z)} \right)^{\mu} = \rho + (p - \rho) h(z). \tag{23}$$

Then clearly, $h(z)$ is analytic in E and $h(0) = 1$. Taking logarithmic differentiation of (23) both sides and using identity (7) in the resulting equation, we observe that

$$Re \left\{ (1 - \alpha) \left(\frac{z^p}{I_p^{\lambda}(a, c) f(z)} \right)^{\mu} - \alpha \left(\frac{I_p^{\lambda+1}(a, c)}{I_p^{\lambda}(a, c)} \right) \left(\frac{z^p}{I_p^{\lambda}(a, c) f(z)} \right)^{\mu} - \rho \right\} \tag{24}$$

$$= (p - \rho) Re \left\{ h(z) + \frac{\alpha z h'(z)}{(\lambda + p)\mu} \right\} \geq (p - \rho) Re \left\{ h(z) - \frac{\alpha |z h'(z)|}{(\lambda + p)\mu} \right\}.$$

Now by using the following well known estimate, see [8],

$$\left|zh'(z)\right| \leq \frac{2rReh(z)}{1-r^2} \quad (|z| = r < 1),$$

in (24), we have

$$Re\left\{(1-\alpha)\left(\frac{z^p}{I_p^{\lambda}(a,c)f(z)}\right)^{\mu} - \alpha\left(\frac{I_p^{\lambda+1}(a,c)}{I_p^{\lambda}(a,c)}\right)\left(\frac{z^p}{I_p^{\lambda}(a,c)f(z)}\right)^{\mu} - \rho\right\}$$

$$= (p-\rho)\, Reh(z)\left\{1 - \frac{2\alpha r}{(\lambda+p)\,\mu\,(1-r^2)}\right\}. \tag{25}$$

The right hand side of (25) is positive if $r < R(\lambda, \alpha, \mu, p)$ where $R(\lambda, \alpha, \mu, p)$ is given by (22).

Sharpness of this result follows by taking

$$\left(\frac{z^p}{I_p^{\lambda}(a,c)f(z)}\right)^{\mu} = \rho + (p-\rho)\frac{1+z}{1-z}.$$

where $0 \leq \rho < p$, $\lambda > -p$ and $z \in E$.

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Khalida Inayat Noor
Department of Mathematics
COMSATS Institute of Information Technology, Islamabad Pakistan
email: khalidanoor@hotmail.com

Ali Muhammad
Department of Basic Sciences
University of Engineering and Technology, Peshawar Pakistan
email: ali7887@gmail.com

Muhammad Arif
Department of Mathematics
Abdul Wali Khan University Mardan Pakistan
email: marifmaths@yahoo.com