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On certain subclass of meromorphic p-valent functions with negative coefficients [1]

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Abstract

A certain subclass $B_n^*(p, \alpha, \lambda, A, B)$ consisting of meromorphioc p-valent functions with negative coefficients in $U^* = \{z : z \in C$ and $0 < |z| < 1\}$ is introduced. In this paper we obtain coefficient inequalities, distortion theorem, closure theorems and class preserving integral operators for functions in the class $B_n^*(p, \alpha, \lambda, A, B)$.

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1 Introduction

Let Σ_p denote the class of functions of the form :

$$(1) \qquad f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} a_k z^k \quad (a_{-p} \neq 0; p \in N = \{1, 2, ...\})$$

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which are regular in the punctured disc $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U\backslash\{0\}$. For a function $f(z) \in \Sigma_p$, we define the following differential operator:

$$D^0_{\lambda,p}f(z) = f(z), \tag{2}$$

$$\begin{aligned} D^1_{\lambda,p}f(z) &= (1-\lambda)f(z) + \frac{\lambda}{z^p}(z^{p+1}f(z))' \quad (\lambda \geq 0; p \in N) \\ &= \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}[1+\lambda(k+p)]a_k z^k, \end{aligned} \tag{3}$$

$$\begin{aligned} D^2_{\lambda,p}f(z) &= D(D^1_{\lambda,p}f(z)) \\ &= \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}[1+\lambda(k+p)]^2 a_k z^k, \end{aligned} \tag{4}$$

and for $n = 1, 2, ...,$

$$\begin{aligned} D^n_{\lambda,p}f(z) &= D(D^{n-1}_{\lambda,p}f(z)) \\ &= \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}[1+\lambda(k+p)]^n a_k z^k. \end{aligned} \tag{5}$$

Also we can write $D^n_{\lambda,p}f(z)$ as follows :

$$\begin{aligned} D^n_{\lambda,p}f(z) &= (f * \{\frac{1}{z^p} + \sum_{k=1}^{\infty}[1+\lambda(k+p)]^n z^k\})(z) \\ &= (f * \phi^p_{n,\lambda})(z), \end{aligned} \tag{6}$$

where $\phi^p_{n,\lambda}(z) = \frac{1}{z^p} + \sum\limits_{k=1}^{\infty}[1+\lambda(k+p)]^n z^k$ and $(\star)$ denotes convolution.

In [1] Aouf and Hossen obtained new criteria for meromorphic p-valent starlike functions of order $\alpha(0 \leq \alpha < p)$ via the basic inclusion relationship $B_{n+1}(\alpha) \subset B_n(\alpha)(0 \leq \alpha < p, n \in N_0 = N \cup \{0\}, p \in N)$, where $B_n(\alpha)$ is the class consisting of functions in Σ_p satisfying

$$Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - (p+1)\right\} < -\alpha \ (z \in U^*; 0 \leq \alpha < p; p \in N; n \in N_0). \tag{7}$$

We note that $B_0(\alpha) = \Sigma_p^*(\alpha)$ (the class of meromorphic p-valent starlike functions of order α).

Let σ_p denote the subclass of Σ_p consisting of functions of the form :

$$f(z) = \frac{a_{-p}}{z^p} - \sum_{k=1}^{\infty} a_k z^k \quad (a_{-p} > 0; a_k \geq 0; p \in N). \tag{8}$$

With the aid of the differential operator $D^n_{\lambda,p}f(z)$ we define the class $B_n(p, \alpha, \lambda, A, B)$ as follows :

A function $f(z) \in \Sigma_p$ is said to be in the class $B_n(p, \alpha, \lambda, A, B)$ if it satisfies the condition

$$\left| \frac{\frac{D^{n+1}_{\lambda,p}f(z)}{D^n_{\lambda,p}f(z)} - 1}{B\left[\frac{D^{n+1}_{\lambda,p}f(z)}{D^n_{\lambda,p}f(z)} - (p+1)\right] + [pB + (A - B)(p - \alpha)]} \right| < 1 \, (z \in U^*) \tag{9}$$

for some $0 \leq \alpha < p; \lambda \geq 0; p \in N, n \in N_0, -1 \leq A < B \leq 1$ and $0 < B \leq 1$.

Let us write :

$$B_n^*(p, \alpha, \lambda, A, B) = B_n(p, \alpha, \lambda, A, B) \cap \sigma_p \ .$$

We note that :

(i) $B_n^*(p, \alpha, 0, -1, 1) = \sigma_p(n, \alpha)$ (Darwish [2]);

(ii) $B_0(p, \alpha, -1, 1) = \Sigma_p^*(\alpha)(0 \leq \alpha < p)$;

(iii) $B_0(p, \alpha, 1, -\beta, \beta) = \Sigma_p^*(\alpha, \beta)(0 \leq \alpha < p; 0 < \beta \leq 1)$ is the class of meromorphically p-valent starlike functions of order α and type β;

(iv) $B_0(\alpha, 1, -\beta, \beta) = \Sigma^*(\alpha, \beta)(0 \leq \alpha < 1; 0 < \beta \leq 1)$ is the class of meromorphically starlike functions of order α and type β (Mogra et al. [3]).

Also we note that:

(i) $B_n(p, \alpha, \lambda, -\beta, \beta) = B_n(p, \alpha, \lambda, \beta)$

$$= \left\{ f(z) \in \Sigma_p : \left| \frac{\frac{D^{n+1}_{\lambda,p}f(z)}{D^n_{\lambda,p}f(z)} - 1}{\frac{D^{n+1}_{\lambda,p}f(z)}{D^n_{\lambda,p}f(z)} + 2\alpha - (2p+1)} \right| < \beta \right. \ ,$$

$$(z \in U^*, 0 \le \alpha < p; 0 < \beta \le 1; \lambda \ge 0; n \in N_0; p \in N)\} . \tag{10}$$

In this paper coefficient inequalities, distortion theorem and closure theorems for the class $B_n^*(p,\alpha,\lambda,A,B)$ are obtained. Finally, the class preserving integral operators of the form

$$F_{c+p-1}(z) = c \int_0^1 u^{c+p-1} f(uz)du \ \ (c > 0)$$

for the class $B_n^*(p,\alpha,\lambda,A,B)$ is considered. We employ techniques similar to these used earlier by Silverman [4] (see also Srivastava and Owa [5]).

2 Coefficient Inequalities

Theorem 1 *Let the function $f(z)$ be defined by (1). If*

$$\sum_{k=1}^{\infty} [1+\lambda(k+p)]^n \{\lambda(k+p)(1+B) + (A-B)(p-\alpha)\} |a_k| \le (B-A)(p-\alpha) |a_{-p}| . \tag{11}$$

Then $f(z) \in B_n(p,\alpha,\lambda,A,B)$.

Proof. It suffices to show that

$$\left| \frac{\frac{D_{\lambda,p}^{n+1} f(z)}{D_{\lambda,p}^{n} f(z)} - 1}{B\left[\frac{D_{\lambda,p}^{n+1} f(z)}{D_{\lambda,p}^{n} f(z)} - (p+1)\right] + [pB + (A-B)(p-\alpha)]} \right| < 1 \quad (|z| < 1). \tag{12}$$

We have

$$\left|\frac{\frac{D^{n+1}_{\lambda,p}f(z)}{D^{n}_{\lambda,p}f(z)}-1}{B\frac{D^{n+1}_{\lambda,p}f(z)}{D^{n}_{\lambda,p}f(z)}+[pB+(A-B)(p-\alpha)]}\right|$$

$$=\left|\frac{\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\lambda(k+p)a_kz^{k+p}}{(A-B)(p-\alpha)a_{-p}+\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n[\lambda B(k+p)+(A-B)(p-\alpha)]a_kz^{k+p}}\right|$$

$$\leq\frac{\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\lambda(k+p)|a_k|}{(B-A)(p-\alpha)|a_{-p}|+\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n[\lambda(k+p)B+(A-B)(p-\alpha)]|a_k|}.$$

The last expression is bounded by 1 if

$$\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\lambda(k+p)|a_k|\leq(B-A)(p-\alpha)|a_{-p}|-$$

$$\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\left\{\lambda(k+p)B+(A-B)(p-\alpha)\right\}|a_k|$$

which reduces to

$$\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n[\lambda(k+p)(1+B)+(A-B)(p-\alpha)]|a_k| \leq(B-A)(p-\alpha)|a_{-p}|. \tag{13}$$

This completes the proof of Theorem 1.

Theorem 2 *Let the function $f(z)$ be defined by (8). Then $f(z)\in B^*_n(p,\alpha,\lambda,A,B)$ if and only if*

$$\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\left\{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\right\}a_k\leq(B-A)(p-\alpha)a_{-p}\ .$$

Proof. In view of Theorem 1, it is sufficient to prove the "only if" part. Let us assume that $f(z)$ defined by (8) is in $B_n^*(p,\alpha,\lambda,A,B)$. We have

$$\left|\frac{\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}-1}{B\left[\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}-(p+1)\right]+[pB+(A-B)(p-\alpha)]}\right|$$

$$=\left|\frac{\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\lambda(k+p)a_kz^{k+p}}{(B-A)(p-\alpha)a_{-p}-\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\{\lambda(k+p)B+(A-B)(p-\alpha)\}a_kz^{k+p}}\right|$$

$$<1 \quad (z\in U^*)\ .$$

Since $|Re(z)|\le|z|$ for all z, we have

(14)
$$Re\left\{\frac{\sum_{k=0}^{\infty}[1+\lambda(k+p)]^n\lambda(k+p)a_kz^{k+p}}{(B-A)(p-\alpha)a_{-p}-\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\{\lambda(k+p)B+(A-B)(p-\alpha)\}a_k}\right\}<1.$$

Choose values of z on the real axis so that $\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}$ is real. Upon clearing the denominator in (14) and letting $z\to 1^-$ through real values, we obtain
$$\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\lambda(k+p)a_k\le(B-A)(p-\alpha)a_{-p}-$$

$$\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\{\lambda(k+p)B+(A-B)(p-\alpha)\}a_k\,.$$

Thus

$$\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n\{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}a_k\le(B-A)(p-\alpha)a_{-p}\,.$$

Hence the result follows.

Corollary 1 *Let the function* $f(z)$ *be defined by (8) be in the class* $B_n^*(p,\alpha,\lambda,A,B)$. *Then*

(15) $$a_k\le\frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(k+p)]^n\{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}}\qquad(k\in N).$$

The result is sharp for the functions of the form

(16)
$$f_k(z) = \frac{a_{-p}}{z^p} - \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(k+p)]^n\{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}} z^k \; (k \in N).$$

3 Distortion Theorem

Theorem 3 *Let the function $f(z)$ be defined by (8) be in the class $B_n^*(p, \alpha, \lambda, A, B)$. Then for $0 < |z| = r < 1$,*

$$\frac{a_{-p}}{r^p} - \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}} r \leq |f(z)| \leq$$

(17)
$$\frac{a_{-p}}{r^p} + \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}} r$$

with equality holds for the function

(18)
$$f_1(z) = \frac{a_{-p}}{z^p} - \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}} z \quad (z = ir, r),$$

and

$$\frac{pa_{-p}}{r^{p+1}} - \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}} \leq |f'(z)| \leq$$

(19)
$$\frac{pa_{-p}}{r^{p+1}} + \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}},$$

where equality holds for the function $f_1(z)$ given by (18) at $z = \pm ir, \pm r$.

Proof. In view of Theorem 2, we have

(20)
$$\sum_{k=1}^{\infty} a_k \leq \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}}.$$

Thus, for $0 < |z| = r < 1$,

$$(21) \qquad |f(z)| \leq \frac{a_{-p}}{r^p} + r\sum_{k=1}^{\infty} a_k \leq \frac{a_{-p}}{r^p} + \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}} r$$

and

$$(22) \qquad |f(z)| \geq \frac{a_{-p}}{r^p} - r\sum_{k=1}^{\infty} a_k \geq \frac{a_{-p}}{r^p} - \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}} r,$$

which, together, yield (17). Furthermore, it follows from Theorem 2 that

$$[1+\lambda(1+p)]^n \{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\} \sum_{k=1}^{\infty} k a_k \leq$$

$$\sum_{k=1}^{\infty} [1+\lambda(1+p)]^n \{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\} a_k \leq (B-A)(p-\alpha)a_{-p} \ ,$$

that is, that

$$(23) \qquad \sum_{k=1}^{\infty} k a_k \leq \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}}.$$

Hence

$$(24) \qquad |f'(z)| \leq \frac{pa_{-p}}{r^{p+1}} + \sum_{k=1}^{\infty} k a_k r^{k-1} \leq \frac{pa_{-p}}{r^{p+1}} + \sum_{k=1}^{\infty} k a_k \leq \frac{pa_{-p}}{r^{p+1}} + \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}}$$

and

$$(25) \qquad |f'(z)| \geq \frac{pa_{-p}}{r^{p+1}} - \sum_{k=1}^{\infty} k a_k r^{k-1} \geq \frac{pa_{-p}}{r^{p+1}} - \sum_{k=1}^{\infty} k a_k \geq \frac{pa_{-p}}{r^{p+1}} - \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n\{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}},$$

which, together, yield (19). It can easily be seen that the function $f_1(z)$ defined by (18) is extremal for Theorem 3.

4 Closure Theorems

Let the functions $f_j(z)$ be defined for $j = 1, 2, ..., m$, by

(26) $$f_j(z) = \frac{a_{-p,j}}{z^p} - \sum_{k=1}^{\infty} a_{k,j} z^k \quad (a_{-p,j} > 0; a_{k,j} \geq 0; p \in N)$$

for $z \in U^*$.

Theorem 4 *Let the functions $f_j(z)$ be defined by (26) be in the class $B_n^*(p, \alpha, \lambda, A, B)$ for every $j = 1, 2, ..., m$. Then the function $F(z)$ defined by*

(27) $$F(z) = \frac{b_{-p}}{z^p} - \sum_{k=1}^{\infty} b_k z^k \quad (b_{-p} > 0; b_k \geq 0; p \in N)$$

is a member of the class $B_n^*(p, \alpha, \lambda, A, B)$, where

(28) $$b_{-p} = \frac{1}{m} \sum_{j=1}^{m} a_{-p,j} \quad \textit{and} \quad b_k = \frac{1}{m} \sum_{j=1}^{m} a_{k,j} \ (k \in N).$$

Proof. Since $f_j(z) \in B_n^*(p, \alpha, \lambda, A, B)$, it follows from Theorem 2 that

(29) $$\sum_{k=1}^{\infty} [1 + \lambda(k+p)]^n \{\lambda(k+p)(1+B) + (A-B)(p-\alpha)\} a_{k,j} \leq (B-A)(p-\alpha) a_{-p,j},$$

for every $j = 1, 2, ..., m$. Hence,

$$\begin{aligned} & \sum_{k=1}^{\infty} [1 + \lambda(k+p)]^n \{\lambda(k+p)(1+B) + (A-B)(p-\alpha)\} b_k \\ = & \sum_{k=1}^{\infty} [1 + \lambda(k+p)]^n \{\lambda(k+p)(1+B) + (A-B)(p-\alpha)\} \left\{ \frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right\} \\ = & \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} [1 + \lambda(k+p)]^n \{\lambda(k+p)(1+B) + (A-B)(p-\alpha)\} a_{k,j} \right) \\ \leq & (B-A)(p-\alpha) \left(\frac{1}{m} \sum_{j=1}^{m} a_{-p,j} \right) = (B-A)(p-\alpha) b_{-p} \ , \end{aligned}$$

which (in view of Theorem 2) implies that $F(z) \in B_n^*(p, \alpha, \lambda, A, B)$.

Theorem 5 *The class $B_n^*(p,\alpha,\lambda,A,B)$ is closed under convex linear combination.*

Proof. Let the functions $f_j(z)(j = 1,2)$ defined by (26) be in the class $B_n^*(p,\alpha,\lambda,A,B)$, it is sufficient to prove that the function

$$H(z) = tf_1(z) + (1-t)f_2(z) \quad (0 \le t \le 1) \tag{30}$$

is also in the class $B_n^*(p,\alpha,\lambda,A,B)$. Since, for $0 \le t \le 1$,

$$H(z) = \frac{ta_{-p,1} + (1-t)a_{-p,2}}{z^p} + \sum_{k=1}^{\infty} \{ta_{k,1} + (1-t)a_{k,2}\} z^k, \tag{31}$$

we observe that

$$\begin{aligned} &\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B) \\ &+ (A-B)(p-\alpha)\} \{ta_{k,1} + (1-t)a_{k,2}\} \\ &= t\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B) + (A-B)(p-\alpha)\} a_{k,1} + \\ &(1-t)\sum_{k=1}^{\infty}[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B) + (A-B)(p-\alpha)\} a_{k,2} \\ &\le (B-A)(p-\alpha) \{ta_{-p,1} + (1-t)a_{-p,2}\} \end{aligned} \tag{32}$$

with the aid of Theorem 2. Hence $H(z) \in B_n^*(p,\alpha,\lambda,A,B)$. This completes the proof of Theorem 5.

Theorem 6 *Let*

$$f_0(z) = \frac{a_{-p}}{z^p} \tag{33}$$

and

$$f_k(z) = \frac{a_{-p}}{z^p} - \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B) + (A-B)(p-\alpha)\}} z^k \; (k \in N). \tag{34}$$

Then $f(z) \in B_n^*(p, \alpha, \lambda, A, B)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z), \tag{35}$$

where $\mu_k \geq 0 (k \geq 0)$ *and* $\sum_{k=0}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z)\,,$$

where $\mu_k \geq 0 (k \geq 0)$ and $\sum_{k=0}^{\infty} \mu_k = 1$. Then

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \mu_k f_k(z) = \mu_0 f_0(z) + \sum_{k=1}^{\infty} \mu_k f_k(z) \\ &= \frac{a_{-p}}{z^{-p}} - \sum_{k=1}^{\infty} \mu_k \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}} z^k, \\ &(k \in N). \end{aligned} \tag{36}$$

Since

$$\begin{aligned} &\sum_{k=1}^{\infty} [1+\lambda(k+p)]^n \{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}\,. \\ &\cdot \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}} \mu_k \\ &(B-A)(p-\alpha)a_{-p} \sum_{k=1}^{\infty} \mu_k = (B-A)(p-\alpha)a_{-p}(1-\mu_0) \\ &\leq (B-A)(p-\alpha)a_{-p}, \end{aligned} \tag{37}$$

we have $f(z) \in B_n^*(p, \alpha, \lambda, A, B)$, by Theorem 2. Conversely, suppose that the function $f(z)$ defined by (8) belongs to the class $B_n^*(p, \alpha, \lambda, A, B)$. Since

$$a_k \leq \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}} \quad (k \in N) \tag{38}$$

by Corollary 1, setting

$$\mu_k = \frac{[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}}{(B-A)(p-\alpha)a_{-p}} a_k \quad (k \in N) \tag{39}$$

and

$$\mu_0 = 1 - \sum_{k=1}^{\infty} \mu_k, \tag{40}$$

it follows that

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z) \ .$$

This completes the proof of Theorem 6.

5 Integral Operators

In this section we consider integral transforms of functions in the class $B_n^*(p,\alpha,\lambda,A,B)$.

Theorem 7 *Let the function $f(z)$ defined by (8) be in the class $B_n^*(p,\alpha,\lambda,A,B)$. Then the integral transforms*

$$F_{c+p-1}(z) = c\int_0^1 u^{c+p-1} f(uz)du \quad (c>0) \tag{41}$$

are in the class $B_n^*(p,\gamma,\lambda,A,B)$, where

$$\gamma = p - \frac{c\lambda(1+p)(1+B)(p-\alpha)}{(c+p+1)[\lambda(1+p)(1+B)+(A-B)(p-\alpha)] - c(A-B)(p-\alpha)}$$

for $F_{c+p-1}(z) \neq 0 (z \in U^*)$. The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{a_{-p}}{z^p} - \frac{(B-A)(p-\alpha)a_{-p}}{[1+\lambda(1+p)]^n \{\lambda(1+p)(1+B)+(A-B)(p-\alpha)\}} z\,. \tag{42}$$

Proof. Suppose $f(z) = \frac{a_{-p}}{z^p} - \sum_{k=1}^{\infty} a_k z^k (a_{-p} > 0; a_k \geq 0; p \in N) \in B_n^*(p,\alpha,\lambda,A,B)$. Then we have

$$F_{c+p-1}(z) = c\int_0^1 u^{c+p-1} f(uz)du = \frac{a_{-p}}{z^p} - \sum_{k=1}^{\infty} \frac{c}{c+p+k} a_k z^k. \tag{43}$$

Since $f(z) \in B_n^*(p,\alpha,\lambda,A,B)$, we have

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}}{(B-A)(p-\alpha)a_{-p}} a_k \leq 1\,. \tag{44}$$

In view of Theorem 2, we shall find the largest γ for which

(45)

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k+p)]^n \{\lambda(k+p)(1+B)+(A-B)(p-\gamma)\}}{(B-A)(p-\gamma)a_{-p}} \left(\frac{c}{c+p+k}\right) a_k \leq 1.$$

It suffices to find the range of values of γ for which

$$\frac{c\{\lambda(k+p)(1+B)+(A-B)(p-\gamma)\}}{(c+p+k)(p-\gamma)} \leq \frac{\{\lambda(k+p)(1+B)+(A-B)(p-\alpha)\}}{(p-\alpha)}$$

for each $k \in N$. Form the above inequality, we obtain

$$\gamma \leq p - \frac{c\lambda(k+p)(1+B)(p-\alpha)}{(c+p+k)[\lambda(k+p)(1+B)+(A-B)(p-\alpha)] - c(A-B)(p-\alpha)}$$

for each α, p, λ, A and for c fixed, let

$$F(k) = p - \frac{c\lambda(k+p)(1+B)(p-\alpha)}{(c+p+k)[\lambda(k+p)(1+B)+(A-B)(p-\alpha)] - c(A-B)(p-\alpha)}\,.$$

Then

$$F(k+1) - F(k) = \frac{c\lambda^2(1+B)^2(p-\alpha)(k+p)(k+p+1)}{A_1 B_1} > 0$$

for each $k \in N$, where

$$A_1 = (c+p+k)[\lambda(k+p)(1+B)+(A-B)(p-\alpha)] - c(A-B)(p-\alpha)$$

and

$$B_1 = (c+p+k+1)\,[\lambda(k+p+1)(1+B) + (A-B)(p-\alpha)] - c(A-B)(p-\alpha)\ .$$

Hence $F(k)$ is an increasing function of k. Since

$$F(1) = p - \frac{c\lambda(1+p)(1+B)(p-\alpha)}{(c+p+1)\,[\lambda(1+p)(1+B) + (A-B)(p-\alpha)] - c(A-B)(p-\alpha)}\ ,$$

the assertion of Theorem 7 follows.

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