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Inclusion and neighborhood properties of some analytic p-valent functions [1]

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Abstract

By means of a certain extended derivative operator of Salagean type, the authors introduce and investigate two new subclasses of p-valently analytic function of complex ordor. The various results obtained here for each of these function classes include coefficient inqualities and the consequent inclusion relationships involving the neighborhoods of the p-valently analytic functions.

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1 Introduction

Let $T(j, p)$ denote the class of functions of the form :

(1) $$f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \;\; (a_k \geq 0; p, j \in N = \{1, 2, ...\}),$$

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which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. For a function $f(z)$ in $T(j,p)$, we define

$$\begin{aligned} D^0_{\lambda,p}f(z) &= f(z), \\ D^1_{\lambda,p}f(z) &= D_{\lambda,p}(D^0_{\lambda,p}f(z)) = (1-\lambda)f(z) + \frac{\lambda}{p}zf'(z) \qquad (\lambda \geq 0) \\ &= z^p - \sum_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]a_k z^k, \\ D^2_{\lambda,p}f(z) &= D_{\lambda,p}(D^1_{\lambda,p}f(z)) \\ &= z^p - \sum_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^2 a_k z^k \end{aligned}$$

and

$$D^n_{\lambda,p}f(z) = D_{\lambda,p}(D^{n-1}_{\lambda,p}f(z)) \qquad (n \in N).$$

It can be easily seen that

$$D^n_{\lambda,p}f(z) = z^p - \sum_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^n a_k z^k \quad (p,j \in N; n \in N_0 = N \cup \{0\}). \tag{2}$$

We note that :

(i) By taking $j = p = \lambda = 1$, the differential operator $D^n_{1,1} = D^n$ was introduced by Salagean[11];

(ii) By taking $j = p = 1$, the differential operator $D^n_{\lambda,1} = D^n_{\lambda}$ was introduced by Al-Oboudi[1].

Now, making use of the differential operator $D^n_{\lambda,p}f(z)$ given by (2), we introduce a new subclass $H_j(n,p,\lambda,b,\beta)$ of the p-valent analytic function class $T(j,p)$ which consist of function $f(z) \in T(j,p)$ satisfying the following inequality :

$$\left| \frac{1}{b}(\frac{z(D^n_{\lambda,p}f(z))'}{D^n_{\lambda,p}f(z)} - p) \right| < \beta \quad , \tag{3}$$

$$(z \in U; p,j \in N; n \in N_0; \lambda \geq 0; b \in C\backslash\{0\}; 0 < \beta \leq 1).$$

We note that :

(i) $H_j(n,p,1,b,\beta) = H_j(n,p,b,\beta) = \{f(z) \in T(j,p) : \left|\frac{1}{b}(\frac{z(D_p^n f(z))'}{D_p^n f(z)} - p)\right| < \beta$

(4) $$(z \in U; p,j \in N; n \in N_0; b \in C\backslash\{0\}; 0 < \beta \leq 1)\};$$

(ii) $H_j(0,p,0,b,\beta) = S_j(p,b,\beta) = \{f(z) \in T(j,p) : \left|\frac{1}{b}(\frac{zf'(z)}{f(z)} - p)\right| < \beta$

(5) $$(z \in U; p,j \in N; b \in C\backslash\{0\}; 0 < \beta \leq 1)\};$$

(iii) $H_j(1,p,\lambda,b,\beta) = C_j(p,\lambda,b,\beta) = \{f(z) \in T(j,p) : \left|\frac{1}{b}(\frac{zF'_{\lambda,p}(z)}{F_{\lambda,p}(z)} - p)\right| < \beta$

(6)

$(z \in U; p,j \in N; \lambda \geq 0; b \in C\backslash\{0\}; 0 < \beta \leq 1; F_{\lambda,p}(z) = (1-\lambda)f(z) + \frac{\lambda}{p}zf^{'}(z))\}.$

Now following the earlier investigations by Goodman [7], Ruscheweyh [10], and others including Altintas and Owa [3], Altintas et al.([4] and [5]), Murugusundaramoorthy and Srivastava [8], Raina and Srivastava [9], Aouf [6] and Srivastava and Orhan [13] (see also [2], [12] and [14]), we define the $(j,\delta)-$ neighborhood of a function $f(z) \in T\ (j,p)$ by (see, for example, [5, p. 1668])

(7) $$N_{j,\delta}(f) = \{g : g \in T(j,p), g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k \text{ and } \sum_{k=j+p}^{\infty} k\,|a_k - b_k| \leq \delta\}.$$

In particular, if

(8) $$h(z) = z^p \qquad (p \in N),$$

we immediately have

(9) $$N_{j,\delta}(h) = \{g : g \in T(j,p), g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k \text{ and } \sum_{k=j+p}^{\infty} k\,|b_k| \leq \delta\}.$$

Also, let $L_j(n,p,\lambda,b,\beta,\mu)$ denote the subclass of $T(j,p)$ consisting of function $f(z) \in T(j,p)$ which satisfy the inequality :

$$\left|\frac{1}{b}\{[(1-\mu)\frac{D_{\lambda,p}^n f(z)}{z^p} + \mu\frac{D_{\lambda,p}^n f^{'}(z)}{pz^{p-1}}] - 1\}\right| < \beta$$

$$(z \in U; p, j \in N; n \in N_0; \lambda \geq 0; b \in C\backslash\{0\}; 0 < \beta \leq 1; \mu \geq 0)\,. \tag{10}$$

We note that :

(i) $L_j(0, p, 0, b, \beta, \mu) = L_j(p, b, \beta, \mu)$

$$= \left\{ f(z) \in T(j,p) : \left| \frac{1}{b} \left\{ \left[(1-\mu)\frac{f(z)}{z^p} + \mu \frac{f^{'}(z)}{pz^{p-1}} \right] - 1 \right\} \right| < \beta \right.$$

$$(z \in U; p, j \in N; b \in C\backslash\{0\}; 0 < \beta \leq 1; \mu \geq 0\}. \tag{11}$$

2 Neighborhoods for the classes $H_j(n, p, \lambda, b, \beta)$ and $L_j(n, p, \lambda, b, \beta, \mu)$

In our investigation of the inclution relations involving $N_{j,\delta}(h)$, we shall require Lemmas 1 and 2 below.

Lemma 1 *Let the function* $f(z) \in T(j,p)$ *be defined by* (1). *Then* $f(z) \in H_j(n, p, \lambda, b, \beta)$ *if and only if*

$$\sum_{k=j+p}^{\infty} [1 + \lambda(\frac{k-p}{p})]^n (k + \beta\,|b| - p) a_k \leq \beta\,|b|\,. \tag{12}$$

Proof. Let a function $f(z)$ of the form (1) belong to the class $H_j(n, p, \lambda, b, \beta)$. Then, in view of (2) and (3), we obtain the following inequality :

$$Re\{\frac{z(D^n_{\lambda,p}f(z))'}{D^n_{\lambda,p}f(z)} - p\} > -\beta\,|b| \qquad (z \in U), \tag{13}$$

or, equivalently,

$$Re\{\frac{-\sum\limits_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^n (k-p) a_k z^{k-p}}{1 - \sum\limits_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^n a_k z^{k-p}}\} > -\beta\,|b| \qquad (z \in U). \tag{14}$$

Setting $z = r$ $(0 \le r < 1)$ in (14), we observe that the expression in the denominator of the left-hand side of (14) is positive for $r = 0$ and also for all r $(0 < r < 1)$. Thus, by letting $r \to 1^-$ through real values, (14) leads us to the desired assertion (12) of Lemma 1.

Conversaly, by applying the hypothesis (12) and letting $|z| = 1$, we find from (3) that

$$\begin{aligned}\left|\frac{z(D^n_{\lambda,p}f(z))'}{D^n_{\lambda,p}f(z)} - p\right| &= \left|\frac{\sum\limits_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^n(k-p)a_kz^{k-p}}{1-\sum\limits_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^na_kz^{k-p}}\right| \\ &\le \frac{\sum\limits_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^n(k-p)a_k}{1-\sum\limits_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^na_k} \\ &\le \frac{\beta\,|b|\,\{1-\sum\limits_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^na_k\}}{1-\sum\limits_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^na_k} = \beta\,|b|\,.\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z) \in H_j(n, p, \lambda, b, \beta)$, which evidently completes the proof of Lemma 1.

Similarly, we can prove the following lemma.

Lemma 2 *Let the function* $f(z) \in T(j,p)$ *be defined by* (1). *Then* $f(z) \in L_j(n, p, \lambda, b, \beta, \mu)$ *if and only if*

$$\sum_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^n[p+\mu(k-p)]a_k \le p\beta\,|b|\,. \tag{15}$$

Our first inclusion relation involving $N_{j,\delta}(h)$ is given in the following theorem.

Theorem 1 *Let*

$$\delta = \frac{(j+p)\beta\,|b|}{(1+\frac{\lambda j}{p})^n(j+\beta\,|b|)} \qquad (p > |b|), \tag{16}$$

then

$$H_j(n,p,\lambda,b,\beta) \subset N_{j,\delta}(h). \tag{17}$$

Proof. Let $f(z) \in H_j(n,p,\lambda,b,\beta)$. Then, in view of the assertion (12) of Lemma 1, we have

$$(1+\frac{\lambda j}{p})^n(j+\beta\,|b|)\sum_{k=j+p}^{\infty} a_k \leq \sum_{k=j+p}^{\infty}[1+\lambda(\frac{k-p}{p})]^n(k+\beta|b|-p)a_k \leq \beta\,|b|\,, \tag{18}$$

which readily yeilds

$$\sum_{k=j+p}^{\infty} a_k \leq \frac{\beta\,|b|}{(1+\frac{\lambda j}{p})^n(j+\beta\,|b|)}. \tag{19}$$

Making use of (12) again, in conjunction with (19), we get

$$\begin{aligned}(1+\frac{\lambda j}{p})^n\sum_{k=j+p}^{\infty} ka_k &\leq \beta\,|b| + (p-\beta\,|b|)\,(1+\frac{\lambda j}{p})^n\sum_{k=j+p}^{\infty} a_k \\ &\leq \beta\,|b| + \frac{\beta\,|b|\,(p-\beta\,|b|)}{(j+\beta\,|b|)} = \frac{(j+p)\beta\,|b|}{(j+\beta\,|b|)}.\end{aligned}$$

Hence

$$\sum_{k=j+p}^{\infty} ka_k \leq \frac{(j+p)\beta\,|b|}{(1+\frac{\lambda j}{p})^n(j+\beta\,|b|)} = \delta \qquad (p > |b|), \tag{20}$$

which, by means of the definition (9), establishes the inclusion (17) asserted by Theorem 1.

Putting (i) $n=\lambda=0$ and (ii) $n=1$ in Theorem 1, we obtain the following results.

Corollary 1 *Let*

$$\delta = \frac{(j+p)\beta\,|b|}{(j+\beta\,|b|)} \quad (p > |b|), \tag{21}$$

then

$$S_j(p,b,\beta) \subset N_{j,\delta}(h). \tag{22}$$

Corollary 2 *Let*

$$\delta = \frac{(j+p)p\beta\,|b|}{(p+\lambda j)(j+\beta\,|b|)} \qquad (p > |b|), \tag{23}$$

then

$$C_j(p, \lambda, b, \beta) \subset N_{j,\delta}(h). \tag{24}$$

In a similar manner, by applying the assertion (15) of Lemma 2 instead of the assertion (12) of Lemma 1 to functions in the class $L_j(n, p, \lambda, b, \beta, \mu)$, we can prove the following inclusion relationship.

Theorem 2 *If*

$$\delta = \frac{(j+p)p\beta\,|b|}{(1+\frac{\lambda j}{p})^n(p+\mu j)} \qquad (\mu > 1), \tag{25}$$

then

$$L_j(n, p, \lambda, b, \beta, \mu) \subset N_{j,\delta}(h). \tag{26}$$

Putting $n = \lambda = 0$ in Theorem 2, we obtain the following result.

Corollary 3 *If*

$$\delta = \frac{(j+p)p\beta\,|b|}{(p+\mu j)}, \tag{27}$$

then

$$L_j(p, b, \beta, \mu) \subset N_{j,\delta}(h). \tag{28}$$

3 Neighborhoods for the classes $H_j^{(\alpha)}(n,p,\lambda,b,\beta)$ and $L_j^{(\alpha)}(n,p,\lambda,b,\beta,\mu)$

In this section, we determine the neighborhood for the each classes $H_j^{(\alpha)}(n,p,\lambda,b,\beta)$ and $L_j^{(\alpha)}(n,p,\lambda,b,\beta,\mu)$, which we define as follows. A function $f(z) \in T(j,p)$ is said to be in the class $H_j^{(\alpha)}(n,p,\lambda,b,\beta)$ if there exists a function $g(z) \in H_j(n,p,\lambda,b,\beta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha \qquad (z \in U; 0 \leq \alpha < p). \tag{29}$$

Analogously, a function $f(z) \in T(j,p)$ is said to be in the class $L_j^{(\alpha)}(n,p,\lambda,b,\beta,\mu)$ if there exists a function $g(z) \in L_j(n,p,\lambda,b,\beta,\mu)$ such that the inequality (29) holds true.

Theorem 3 *If* $g(z) \in H_j(n,p,\lambda,b,\beta)$ *and*

$$\alpha = p - \frac{\delta(1+\frac{\lambda j}{p})^n(j+\beta\,|b|)}{(j+p)[(1+\frac{\lambda j}{p})^n(j+\beta\,|b|)-\beta\,|b|]}, \tag{30}$$

then

$$N_{j,\delta}(g) \subset H_j^{(\alpha)}(n,p,\lambda,b,\beta), \tag{31}$$

where

$$\delta \leq p(j+p)\{1-\beta\,|b|\,[(1+\frac{\lambda j}{p})^n(j+\beta\,|b|)]^{-1}\}. \tag{32}$$

Proof. Suppose that $f(z) \in N_{j,\delta}(g)$. We find from (7) that

$$\sum_{k=j+p}^{\infty} k\,|a_k - b_k| \leq \delta, \tag{33}$$

which readily implies that

$$\sum_{k=j+p}^{\infty} |a_k - b_k| \leq \frac{\delta}{j+p} \qquad (p, j \in N). \tag{34}$$

Next, since $g(z) \in H_j(n, p, \lambda, b, \beta)$, we have [cf. equation (19)]

$$\sum_{k=j+p}^{\infty} b_k \leq \frac{\beta\,|b|}{(1+\dfrac{\lambda j}{p})^n (j+\beta\,|b|)}, \tag{35}$$

so that

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - 1\right| &\leq \frac{\sum\limits_{k=j+p}^{\infty} |a_k - b_k|}{1 - \sum\limits_{k=j+p}^{\infty} b_k} \leq \frac{\delta}{j+p} . \frac{(1+\dfrac{\lambda j}{p})^n (j+\beta\,|b|)}{[(1+\dfrac{\lambda j}{p})^n (j+\beta\,|b|) - \beta\,|b|]} \\ &= p - \alpha, \end{aligned} \tag{36}$$

provided that α is given by (30). Thus, by the above definition, $f(z) \in H_j^{(\alpha)}(n, p, \lambda, b, \beta)$ for α given by (30). This evidently proves Theorem 3.

Putting (i) $n = \lambda = 0$ and (ii) $n = 1$ in Theorem 3, we obtain the following results.

Corollary 4 *If $g(z) \in S_j(p, b, \beta)$ and*

$$\alpha = p - \frac{\delta(j+\beta\,|b|)}{(j+p)[(j+\beta\,|b|) - \beta\,|b|]}, \tag{37}$$

then

$$N_{j,\delta}(g) \subset S_j^{(\alpha)}(p, b, \beta), \tag{38}$$

where

$$\delta \leq p(j+p)\{1 - \beta\,|b|\,(j+\beta\,|b|)^{-1}\}. \tag{39}$$

Corollary 5 *If* $g(z) \in C_j(p, \lambda, b, \beta)$ *and*

$$\alpha = p - \frac{\delta(p + \lambda j)(j + \beta |b|)}{(j + p)[(p + \lambda j)(j + \beta |b|) - p\beta |b|]}, \tag{40}$$

then

$$N_{j,\delta}(g) \subset C_j^{(\alpha)}(p, \lambda, b, \beta),$$

where

$$\delta \leq p(j + p)\{1 - p\beta |b| [(p + \lambda j)(j + \beta |b|)]^{-1}\}. \tag{41}$$

The proof of Theorem 4 below is similar to that the proof of Theorem 3 above. We, therefore, skip its proof.

Theorem 4 *If* $g(z) \in L_j(n, p, \lambda, b, \beta, \mu)$ *and*

$$\alpha = p - \frac{\delta(1 + \frac{\lambda j}{p})^n (p + \mu j)}{(j + p)[(1 + \frac{\lambda j}{p})^n (p + \mu j) - p\beta |b|]}, \tag{42}$$

then

$$N_{j,\delta}(g) \subset L_j^{(\alpha)}(n, p, \lambda, b, \beta, \mu), \tag{43}$$

where

$$\delta \leq p(j + p)\{1 - p\beta |b| [(1 + \frac{\lambda j}{p})^n (p + \mu j)]^{-1}\}. \tag{44}$$

Putting $n = \lambda = 0$ in Theorem 4, we obtain the following result.

Corollary 6 *If* $g(z) \in L_j(p, b, \beta, \mu)$ *and*

$$\alpha = p - \frac{\delta(p + \mu j)}{(j + p)[(p + \mu j) - p\beta |b|]}, \tag{45}$$

then

$$N_{j,\delta}(g) \subset L_j^{(\alpha)}(p, b, \beta, \mu), \tag{46}$$

where

$$\delta \leq p(j + p)\{1 - p\beta |b| (p + \mu j)^{-1}\}. \tag{47}$$

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