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On the degree of approximation by new Durrmeyer type operators [1]

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Abstract

In this paper, we define a new kind of positive linear operators and study basic properties as well as Voronovskaya type results. In the last section of this paper we establish the error estimation for simultaneous approximation in terms of higher order modulus of continuity by using the technique of linear approximating method viz Steklov mean.

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1 Introduction

In the year 1957, Baskakov [1] introduced the following operators

$$B_n(f,x) = \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

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where

$$b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \quad x \in [0, \infty).$$

Important modifications had been studied by Sahai & Prasad [14] and Heilmann [9] on Baskakov operators after these milestone modifications, various researchers have given different type modification of Baskakov operators and studied several good results. Now we are giving another modification of Baskakov operators:

(2) $$V_n(f, x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n+1}\right),$$

where

$$p_{n,k}(x) = \left(\frac{n}{n+1}\right)^{n+1} \begin{pmatrix} n+k \\ k \end{pmatrix} \frac{x^k}{\left(1 - \frac{1}{n+1} + x\right)^{n+k+1}}$$

and Durrmeyer variants of these operators are:

(3) $$D_n(f, x) = (n+1) \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t) dt.$$

Let $C_\gamma[0, \infty) = \{f \in C[0, \infty) : |f(x)| \le Mt^\gamma, \text{ for some } \gamma > 0\}$ we define the norm $\|.\|$ on $C_\gamma[0, \infty)$ by

$$\|f\|_\gamma = \sup_{0 \le t < \infty} |f(t)| t^{-\gamma}$$

We note that the order of approximation by these operators (3) is at best $O(n^{-1})$, howsoever smooth the function may be. Thus to improve the order of approximation, we consider May [13] type linear combination of the operators (3) as described below:

For $d_0, d_1, d_2, ..., d_k$ arbitrary but fixed distinct positive integers, the linear combination $D_n\big(f, \big(d_0,\, d_1, d_2, ..., d_k\big), x\big)$ of $D_{d_j n}(f, x), j = 0, 1, 2, ..., k$ are defined as:

$$D_n(f, (d_0, d_1, d_2, ..., d_k), x) = \sum_{j=0}^{k} C(j, k) D_{d_j n}(f, x),$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i} \ for\ k \neq 0 \quad and \quad C(0,0) = 1.$$

Very recently Deo et al. [4] have studied new Bernstein type operators and established a Voronovskaya type asymptotic formula and an estimate of error in terms of modulus of continuity in simultaneous approximation for the linear combinations. In [5], Deo and Singh have given some theorems on the approximation of the r-th derivative of a function f by the same operators. Deo [3] has studied Voronovskaya type result for Lupaş type operators and he [2] has also given iterative combinations of Baskakov operator.

In the present paper, we study some ordinary approximation results including Voronovskaya type results. At the end of this paper we obtain an estimate of error in terms of higher order modulus of continuity in simultaneous approximation for the linear combination of the operators (3).

2 Properties and Basic Results

In this section we write some basic results to prove our theorem.

Lemma 1 *For $n \geq 1$ one obtains,*

$$V_n(1,x) = 1$$

$$V_n(t,x) = \left(1 + \frac{1}{n}\right) x$$

$$V_n\left(t^2,x\right) = \left(1 + \frac{3}{n} + \frac{2}{n^2}\right) x^2 + \frac{x}{n}$$

Lemma 2 *For $m \in N^0$ (the set of non-negative integers), the m-th order moment of the operator is defined as*

$$U_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \left(\frac{k}{n+1} - x\right)^m.$$

Consequently, $U_{n,0}(x) = 1$ *and* $U_{n,1}(x) = x/n$. *There holds the recurrence relation*

$$nU_{n,m+1}(x) = x\left(1 - \frac{1}{n+1} + x\right)\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right] + xU_{n,m}(x)$$

Proof. It is easily observed that

$$x\left(1 - \frac{1}{n+1} + x\right)p'_{n,k}(x) = \left[\frac{nk}{n+1} - (n+1)x\right]p_{n,k}(x). \tag{4}$$

Hence the result. Thus

(i) $U_{n,m}(x)$ is a polynomial in x of degree $\leq m$;

(ii) For every $x \in [0, \infty)$, $U_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, where $\lfloor \alpha \rfloor$ denotes the integral part of α.

Lemma 3 *Let the m-th order moment be defined by*

$$T_{n,m}(x) = (n+1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)(t-x)^m dt$$

then

$$T_{n,0}(x) = 1, \;\; T_{n,1}(x) = \frac{n(1+2x)+2x}{n^2-1}, \;\; n > 1, \tag{5}$$

$$T_{n,2}(x) = \frac{2\left(n^2 + 4x^2 + 4nx + 3n^2x + 7nx^2 + 2n^2x^2 - n^3x - n^3x^2\right)}{(n+1)(n^2-1)(n-2)} \tag{6}$$

and

(7)

$$(n-m-1)T_{n,m+1}(x) = (m+1)\left(1 - \frac{1}{n+1} + 2x\right)T_{n,m}(x) + x\left(1 - \frac{1}{n+1} + x\right)\left[T'_{n,m}(x) + 2mT_{n,m-1}(x)\right].$$

Further, for all $x \in [0, \infty)$

$$T_{n,m}(x) = O\left(n^{-\left[\frac{(m+1)}{2}\right]}\right). \tag{8}$$

Proof. We can easily obtain (5) and (6) by using the definition of $T_{n,m}(x)$. For the proof of (7), we proceed as follows. First

$$\begin{aligned}
& x\left(1-\frac{1}{n+1}+x\right)T'_{n,m}(x) \\
&= x\left(1-\frac{1}{n+1}+x\right)(n+1)\sum_{k=0}^{\infty}p'_{n,k}(x)\int_0^{\infty}p_{n,k}(t)(t-x)^m dt \\
&\quad - mx\left(1-\frac{1}{n+1}+x\right)T_{n,m-1}(x).
\end{aligned}$$

Now, using inequality (4) two times, then we get

$$\begin{aligned}
& x\left(1-\frac{1}{n+1}+x\right)\left[T'_{n,m}(x)+mT_{n,m-1}(x)\right] \\
&= (n+1)\sum_{k=0}^{\infty}\left[\frac{nk}{n+1}-(n+1)x\right]p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)(t-x)^m dt \\
&= (n+1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}\left[\frac{nk}{n+1}-(n+1)t\right]p_{n,k}(t)(t-x)^m dt+(n+1)T_{n,m+1}(x) \\
&= (n+1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}t\left(1-\frac{1}{n+1}+t\right)p'_{n,k}(t)(t-x)^m dt+(n+1)T_{n,m+1}(x) \\
&= (n+1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}\left[\left(1-\frac{1}{n+1}+2x\right)(t-x)+(t-x)^2+x\left(1-\frac{1}{n+1}+x\right)\right] \\
& .p'_{n,k}(t)(t-x)^m dt+(n+1)T_{n,m+1}(x) \\
&= -(m+1)\left(1-\frac{1}{n+1}+2x\right)T_{n,m}(x)+(n-m-1)T_{n,m+1}(x) \\
& - mx\left(1-\frac{1}{n+1}+x\right)T_{n,m-1}(x).
\end{aligned}$$

This leads to (7). The proof of (8) easily follow from (5) and (7).

Lemma 4 *There exists the polynomials $q_{i,j,r}(x)$ independent of n and k such that*

$$x^r\left(1-\frac{1}{n+1}+x\right)^r\frac{d^r}{dx^r}p_{n,k}(x)=\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+1)^i\{k-(n+1)x\}^j\phi_{i,j,r}(x)p_{n,k}(x).$$

The proof of this lemma proceeds exactly on the lines of that of a results by Lorentz [12, p. 26].

Lemma 5 *Let f be r times differentiable on $[0,\infty)$ such that $f^{(r-1)} = O(t^{\alpha})$, for some $\alpha > 0$ as $t \to \infty$ then for $r = 1, 2, 3, ...$ and $n > \alpha + r$, we have*

$$(9)\quad D_n^{(r)}(f,x) = \frac{(n+1)(n-r)!(n+r)!}{(n!)^2} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} p_{n-r,k+r}(t) f(t) dt$$

Proof. We have by Leibnitz theorem

$$D_n^{(r)}(f,x)$$

$$= (n+1)\left(\frac{n}{n+1}\right)^{n+1} \sum_{i=0}^{r} \sum_{k=i}^{\infty} \begin{pmatrix} r \\ i \end{pmatrix} \frac{(-1)^{r-i}(n+k+r-i)}{n!(k-i)!} \frac{x^{k-i}}{\left(1 - \frac{1}{n+1} + x\right)^{n+k+r+1-i}}$$

$$\cdot \int_0^{\infty} p_{n,k}(t) f(t) dt$$

$$= \frac{(n+1)(n+r)!}{n!} \left(\frac{n+1}{n}\right)^{r} \sum_{k=0}^{\infty} \sum_{i=0}^{r} \begin{pmatrix} r \\ i \end{pmatrix} (-1)^{r-i} p_{n+r,k}(x) \int_0^{\infty} p_{n,k+i}(t) f(t) dt$$

$$= \frac{(n+1)(n+r)!}{n!} \left(\frac{n+1}{n}\right)^{r} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} \sum_{i=0}^{r} (-1)^{r-i} \begin{pmatrix} r \\ i \end{pmatrix} p_{n,k+i}(t) f(t) dt$$

Again applying Leibnitz theorem

$$p_{n-r,k+r}^{(r)}(t) = \sum_{i=0}^{r} \left(\frac{n+1}{n}\right)^{r} \frac{n!}{(n-r)!} (-1)^{i} \begin{pmatrix} r \\ i \end{pmatrix} p_{n,k+i}(t)$$

$$D_n^{(r)}(f,x) = \frac{(n+1)(n-r)!(n+r)!}{(n!)^2} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} (-1)^{r} p_{n-r,k+r}^{(r)}(t) f(t) dt.$$

Further integrating by parts r times, we get the required result.

Lemma 6 *Let* $f \in C_\gamma[0,\infty)$, *if* $f^{(2k+r+2)}$ *exists at a point* $x \in C_\gamma[0,\infty)$, *then*

$$\lim_{n\to\infty} n^{k+1}\left[D_n^{(r)}\left(f\big(d_0, d_1, d_2, ..., d_k\big), x\right) - f^{(r)}(x)\right] = \sum_{i=r}^{2k+r+2} Q(i,k,r,x) f^{(i)}(x),$$

where $Q(i,k,r,x)$ *are certain polynomials in* x.

The proof of the above Lemma follows easily along the lines of [8, 11].

3 Voronovskaya Type Results

Theorem 1 *If a function* f *is such that its first and second order derivatives are bounded in* $[0,\infty)$, *then*

$$\lim_{n\to\infty}(n+1)\left\{D_n(f,x) - f(x)\right\} = f'(x)(1+2x) - x(1+x)f''(x) \tag{10}$$

Proof. Using Taylor's theorem we write that

$$f(t) - f(x) = (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + \frac{(t-x)^2}{2!}\eta(t,x), \tag{11}$$

where $\eta(t,x)$ is a bounded function $\forall t, x$ and $\lim_{t\to x}\eta(t,x) = 0$. Now applying (3) and (11), we get

$$D_n(f,x) - f(x) = f'(x)D_n(t-x,x) + \frac{f''(x)}{2}D_n\left((t-x)^2, x\right) + I_1$$

where

$$I_1 = \frac{1}{2}D_n\left((t-x)^2\eta(t,x), x\right).$$

Using (5) and (6), we get

$$\begin{aligned} &D_n(f,x) - f(x) = f'(x)T_{n,1}(x) + \frac{f''(x)}{2}T_{n,2}(x) + I_1 \\ &= f'(x)\left\{\frac{n(1+2x)+2x}{n^2-1}\right\} \\ &+ f''(x)\left\{\frac{n^2+4x^2+4nx+3n^2x+7nx^2+2n^2x^2-n^3x-n^3x^2}{(n+1)(n^2-1)(n-2)}\right\} + I_1, \end{aligned}$$

therefore

$$(n+1)\left\{D_n(f,x)-f(x)\right\} = f'(x)\left\{\frac{n(1+2x)+2x}{n-1}\right\}$$
$$+ f''(x)\left\{\frac{\left(4x^2+nx(4+7x)+n^2(1+3x+2x^2)-n^3x(1+x)\right)}{(n^2-1)(n-2)}\right\} + (n+1)I_1.$$

Now, we have to show that as $n \to \infty$, the value of $(n+1)I_1 \to 0$. Let $\varepsilon > 0$ be given since $\eta(t,x) \to 0$ as $t \to 0$, then there exists $\delta > 0$ such that when $|t-x| < \delta$ we have $|\eta(t,x)| < \varepsilon$ and when $|t-x| \geq \delta$, we write

$$|\eta(t,x)| \leq M < M\frac{(t-x)^2}{\delta^2}.$$

Thus, for all $t, x \in [0,\infty)$

$$|\eta(t,x)| \leq \varepsilon + M\frac{(t-x)^2}{\delta^2}$$

$$(n+1)I_1 \leq (n+1)D_n\left((t-x)^2\left(\varepsilon+\frac{M(t-x)^2}{\delta^2}\right),x\right)$$
$$\leq \varepsilon(n+1)D_n\left((t-x)^2,x\right) + \frac{M}{\delta^2}(n+1)D_n\left((t-x)^4,x\right)$$

Using (6) and (8), we see that,

$$(n+1)I_1 \to 0 \ \text{ as } \ n \to \infty.$$

This leads to (10).

Corollary 1 *We can also get the following result:*

(12) $$\lim_{n\to\infty}(n-1)\left\{D_n(f,x)-f(x)\right\} = f'(x)(1+2x) - x(1+x)f''(x)$$

Theorem 2 *If $g \in C_B^2[0,\infty)$ then we have*

(13) $$|D_n(g,x)-g(x)| \leq \lambda_n(x)\,\|g\|_{C_B^2}$$

where

$$\lambda_n(x) = \frac{n(1+2x)+2x}{n^2-1}$$

Proof. We write that

$$g(t)g(x)=(t-x)g'(x)+\frac{1}{2}(t-x)^2g''(\xi) \tag{14}$$

where $t\le\xi\le x$. Now applying (3) on (13)

$$\begin{aligned}
&|D_n(g,x)-g(x)|\\
&\le \|g'\|\,|D_n((t-x),x)|+\frac{1}{2}\left|g''\right|\left|D_n((t-x)^2,x)\right|\\
&\le \frac{n(1+2x)+2x}{n^2-1}\|g'\|\\
&\quad+\left\{\frac{n^2+4x^2+4nx+3n^2x+7nx^2+2n^2x^2-n^3x-n^3x^2}{(n+1)(n^2-1)(n-2)}\right\}\|g''\|\\
&\le \lambda_n(x)\left\{\|g'\|+\|g''\|\right\}\le\lambda_n(x)\,\|g\|_{C_B^2}
\end{aligned}$$

Theorem 3 *For $f\in C_B[0,\infty)$, we obtain*

$$|D_n(f,x)-f(x)|\le A\left\{\omega_2\left(f,\frac{\sqrt{\lambda_n(x)}}{2}\right)+\min\left(1,\frac{\lambda_n(x)}{2}\right)\|f\|_{C_B}\right\}, \tag{15}$$

where constant A depends on f and $\lambda_n(x)$.

Proof. for $f\in C_B[0,\infty)$ and $g\in C_B^2[0,\infty)$ we write

$$D_n(f,x)-f(x)=D_n(f,x)-D_n(g,x)+D_n(g,x)-g(x)+g(x)-f(x)$$

by using (13) and Peetre $K-$functions, we get

$$\begin{aligned}
|D_n(f,x)-f(x)|&=|D_n(f,x)-D_n(g,x)|+|D_n(g,x)-g(x)|+|g(x)-f(x)|\\
&\le\|D_nf\|\,\|f-g\|+\lambda_n(x)\,\|g\|_{C_B^2}+\|f-g\|\\
&\le 2\,\|f-g\|+\lambda_n(x)\,\|g\|_{C_B^2}\\
&\le 2\left\{\|f-g\|+\frac{1}{2}\lambda_n(x)\,\|g\|_{C_B^2}\right\}\le 2K\left\{f;\frac{1}{2}\lambda_n(x)\right\}\\
&\le 2A\left\{\omega_2\left(f,\frac{1}{2}\sqrt{\lambda_n(x)}\right)+\min\left(1,\frac{1}{2}\lambda_n(x)\right)\|f\|_{C_B}\right\}.
\end{aligned}$$

This complete the proof.

4 Rate of Convergence

Definition 1 *Let us suppose that* $0 < a < a_1 < b_3 < b_1 < b < \infty$, *for sufficiently small* $\delta > 0$, *the* $(2k+2)$*-th order Steklov mean* $f_{2k+2,\delta}(t)$ *corresponding to* $f(t) \in C_\gamma\,[0,\infty)$ *is defined by*

$$(16)\quad f_{2k+2,\delta}(t) = \delta^{-(2k+2)} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} \cdots \int_{-\delta/2}^{\delta/2} \left\{ f(x) - \Delta_\eta^{2k+2} f(t) \right\} \prod_{i=1}^{2k+2} dt_i,$$

where

$$\eta = \frac{1}{2\mathrm{k}+2} \sum_{i=1}^{2\mathrm{k}+2} t_i \ \text{ and } \ t \in [a,b]\,.$$

It is easily checked (see e.g. [6, 10]) that

(*i*) $f_{2k+2,\delta}$ *has continuous derivatives up to order* $(2k+2)$ *on* $[a,b]$;

(*ii*) $\left\| f_{2k+2,\delta}^{(r)} \right\|_{C[a_1,b_1]} \le M_1 \delta^{-r} \omega_r(f,\delta,a,b),\ r = 1,2,...,(2k+2)$;

(*iii*) $\left\| f - f_{2k+2,\delta} \right\|_{C[a_1,b_1]} \le M_2 \omega_{2k+2}(f,\delta,a,b)$;

(*iv*) $\left\| f_{2k+2,\delta} \right\|_{C[a_1,b_1]} \le M_3 \left\| f \right\|_\gamma$,

where $M_i^{'}s,\ i = 1,2,3,$ *are certain unrelated constants independent of* f *and* δ.

Theorem 4 *For* $f^{(r)} \in C_\gamma\,[0,\infty)$ *and* $0 < a < a_1 < b_1 < b < \infty$. *Then for* n *sufficiently large*

$$\begin{aligned} &\left\| D_n^{(r)}\left(f,(d_0,d_1,d_2,...,d_k),.\right) - f^{(r)} \right\|_{C[a_1,b_1]} \\ &\quad \le \max\left\{ C_1 \omega_{2k+2}\left(f^{(r)}, n^{-1/2}, a, b\right), C_2 n^{-(k+1)} \left\| f \right\|_\gamma \right\}, \end{aligned}$$

where constant $C_1 = C_1(k,r)$ *and* $C_2 = C_2(k,r,f)$.

Proof. By linearity property

$$\begin{aligned}
&\left\| D_n^{(r)}\left(f,\left(\left(d_0,d_1,d_2,...,d_k\right)\right),.\right)\right\|_{C[a_1,b_1]} \\
&\leq \left\| D_n^{(r)}\left(\left(f-f_{2k+2,\delta}\right),\left(d_0,d_1,d_2,...,d_k\right),.\right)\right\|_{C[a_1,b_1]} \\
&\quad + \left\| D_n^{(r)}\left(f_{2k+2,\delta},\left(d_0,d_1,d_2,...,d_k\right),.\right)-f_{2k+2,\delta}^{(r)}\right\|_{C[a_1,b_1]} \\
&\quad + \left\| f^{(r)}-f_{2k+2,\delta}^{(r)}\right\|_{C[a_1,b_1]} \\
&= E_1+E_2+E_3, \text{ say.}
\end{aligned}$$

Since, $f_{2k+2,\delta}^{(r)}(t)=\left(f^{(r)}\right)_{2k+2,\delta}(t)$, by property (iii) of Steklov mean, we have

$$E_3 \leq C_1\omega_{2k+2}\left(f^{(r)},\delta,a,b\right)$$

Next by Lemma 6, we get

$$E_2 \leq C_2 n^{-(k+1)}\sum_{j=r}^{2k+r+2}\left\| f_{2k+2,\delta}^{(j)}\right\|_{C[a,b]}.$$

By applying the interpolation property due to Goldberg and Meir [7] for each $j=r,r+1,...,2k+r+2$, we have

$$\left\| f_{2k+2,\delta}^{(j)}\right\|_{C[a,b]} \leq C_3\left\{\left\| f_{2k+2,\delta}\right\|_{C[a,b]}+\left\| f_{2k+2,\delta}^{(2k+r+2)}\right\|_{C[a,b]}\right\}.$$

Therefore, by applying properties (ii) and (iv) of Steklov mean, we get

$$E_2 \leq C_4 n^{-(k+1)}\left\{\|f\|_\gamma+\delta^{-(2k+2)}\omega_{2k+2}\left(f^{(r)},\delta\right)\right\}.$$

Finally, we shall estimate E_1, choosing a^*,b^* satisfying the condition $0<a<a^*<a_1<b_1<b^*<b<\infty$. Also let $\psi(t)$ denotes the characteristic function of the interval $[a^*,b^*]$, then

$$\begin{aligned}
E_1 &\leq \left\| D_n^{(r)}\left(\psi(t)\left(f(t)-f_{2k+2,\delta}(t)\right)\left(d_0,d_1,d_2,...,d_k\right),.\right)\right\|_{C[a_1,b_1]} \\
&\quad + \left\| D_n^{(r)}\left(\left(1-\psi(t)\right)\left(f(t)-f_{2k+2,\delta}(t)\right)\left(d_0,d_1,d_2,...,d_k\right),.\right)\right\|_{C[a_1,b_1]} \\
&= E_4+E_5, \quad \text{say.}
\end{aligned}$$

We may note here that to estimate E_4 and E_5, it is enough to consider their expressions without the linear combinations. By Lemma 5, we have

$$D_n^{(r)}\left(\psi(t)\left(f(t)-f_{2k+2,\delta}(t)\right),x\right)=\frac{(n+1)(n-r)!(n+r)!}{(n!)^2}\sum_{k=0}^{\infty}p_{n+r,k}(x)$$
$$\cdot\int_0^{\infty}p_{n-r,k+r}(t)\psi(t)\left(f^{(r)}(t)-f^{(r)}_{2k+2,\delta}(t)\right)dt.$$

Hence

$$\left\|D_n^{(r)}\left(\psi(t)\left(f(t)-f_{2k+2,\delta}(t)\right),k,.\right)\right\|_{[a_1,b_1]}\leq C_5\left\|f^{(r)}-f^{(r)}_{2k+2,\delta}\right\|_{[a^*,b^*]}$$

Now for $x\in[a_1,b_1]$ and $t\in[0,\infty)/[a^*,b^*]$, we can choose a $\delta_1>0$ satisfying $|t-x|\geq\delta_1$. Therefore, by Lemma 4 and Schwarz inequality, we have

$$\begin{aligned}
I&\equiv\left|D_n^{(r)}\left((1-\psi(t))\left(f(t)-f_{2k+2,\delta}(t)\right),x\right)\right|\\
&\leq(n+1)\sum_{\substack{2i+j\leq r\\ i,j\geq0}}n^i\frac{|\phi_{i,j,r}(x)|}{x^r\left(1-\frac{1}{n+1}+x\right)^r}\sum_{k=0}^{\infty}p_{n,k}(x)|k-(n+1)x|^j\\
&\quad\cdot\int_0^{\infty}p_{n,k}(t)\left(1-\psi(t)\right)|f(t)-f_{2k+2,\delta}(t)|dt\\
&\leq C_6\|f\|_\gamma(n+1)\sum_{\substack{2i+j\leq r\\ i,j\geq0}}n^i\sum_{k=0}^{\infty}p_{n,k}(x)|k-(n+1)x|^j\int_{|t-x|\geq\delta_1}p_{n,k}(t)dt\\
&\leq C_6\delta_1^{-2s}\|f\|_\gamma(n+1)\sum_{\substack{2i+j\leq r\\ i,j\geq0}}n^i\sum_{k=0}^{\infty}p_{n,k}(x)|k-(n+1)x|^j\\
&\quad\cdot\left(\int_0^{\infty}p_{n,k}(t)dt\right)^{1/2}\left(\int_0^{\infty}p_{n,k}(t)(t-x)^{4s}dt\right)^{1/2}\\
&\leq C_6\delta_1^{-2s}\|f\|_\gamma\sum_{\substack{2i+j\leq r\\ i,j\geq0}}n^i\left\{\sum_{k=0}^{\infty}p_{n,k}(x)\left\{k-(n+1)x\right\}^{2j}\right\}^{1/2}\\
&\quad\cdot\left\{(n+1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)(t-x)^{4s}dt\right\}^{1/2}
\end{aligned}$$

Hence by Lemma 2 and Lemma 3, we have

$$I \leq C_7 \|f\|_\gamma \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} \delta_1^{-2m} O\left(n^{\left(i+\frac{j}{2}-s\right)}\right) \leq C_7 n^{-q} \|f\|_\gamma$$

where $q = (s - r/2)$. Now choose $s > 0$ such that $q \geq k+1$. Now we obtain

$$I \leq C_7 n^{-(k+1)} \|f\|_\gamma .$$

Therefore by property (iii) of Steklov mean, we get

$$\begin{aligned} E_1 &\leq C_8 \left\| f^{(r)} - f^{(r)}_{2k+2,\delta} \right\|_{C[a^*,b^*]} + C_7 n^{-(k+1)} \|f\|_\gamma \\ &\leq C_9 \omega_{2k+2}\left(f^{(r)}, \delta, a, b\right) + C_7 n^{-(k+1)} \|f\|_\gamma . \end{aligned}$$

Choosing $\delta = n^{-1/2}$, the theorem follows.

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