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Convolution of the subclass of Salagean-type harmonic univalent functions with negative coefficients [1]

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Abstract

A recent result of Sibel Yalcin et al. [4] appeared in "Journal of Inequalities in Pure and Applied Mathematics"(2007) concerning the convolution of two harmonic univalent functions in the class $\overline{RS}_H(k,\gamma)$ is improved.

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1 Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $\left|h'(z)\right| > \left|g'(z)\right|, z \in D$. See Clunie and Sheil-Small [1].

Denote by S_H the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) =$

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$f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, |b_1| < 1. \tag{1}$$

For $f = h + \overline{g}$ given by (1), Jahangiri et al. [2] defined the modified Salagean operator of f as

$$D^k f(z) = D^k h(z) + (-1)^k \overline{D^k g(z)} \tag{2}$$

where $D^k h(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n$ and $D^k g(z) = \sum_{n=1}^{\infty} n^k b_n z^n$,
where D^k stands for the differential operator introduced by Salagean [3].

We let $RS_H(k, \gamma)$ denote the family of harmonic functions f of the form (1) such that

$$Re\left\{\left(1 + e^{i\alpha}\right) \frac{D^{k+1} f(z)}{D^k f(z)} - e^{i\alpha}\right\} \geq \gamma, 0 \leq \gamma < 1, \alpha \in R \quad \text{and} \quad k \in N_0 \tag{3}$$

where $D^k f$ is defined by (2).

Also, we let the subclass $\overline{RS}_H(k, \gamma)$ consist of harmonic functions $f_k = h + \overline{g_k}$ in $RS_H(k, \gamma)$ so that h and g_k are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, g_k(z) = (-1)^k \sum_{n=1}^{\infty} |b_n| z^n. \tag{4}$$

Let us define the convolution of two harmonic functions of the form

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \bar{z}^n$$

and

$$F_k(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |B_n| \bar{z}^n$$

as

$$(f_k * F_k)(z) = f_k(z) * F_k(z) = z - \sum_{n=2}^{\infty} |a_n| |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| |B_n| \bar{z}^n. \tag{5}$$

Recently, Yalcin et al. [4, Theorem 2.6] has obtained the following result for the convolution of two harmonic univalent functions in class $\overline{RS}_H(k,\gamma)$.

Theorem A. For $0 \leq \beta \leq \gamma < 1$, let $f_k \in \overline{RS}_H(k,\gamma)$ and $F_k \in \overline{RS}_H(k,\beta)$. Then the convolution $f_k * F_k \in \overline{RS}_H(k,\gamma) \subseteq \overline{RS}_H(k,\beta)$.

In the present paper we prove the following theorem and then we critically observe that it improves the above stated theorem of Yalcin et al. [4].

Theorem 1 *Let the functions*

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \bar{z}^n$$

and

$$F_k(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |B_n| \bar{z}^n$$

belong to the classes $\overline{RS}_H(k,\gamma)$ *and* $\overline{RS}_H(k,\beta)$ *respectively. Then*
$(f_k * F_k)(z) \in \overline{RS}_H(2k+1,\gamma)$ *(If k is an odd integer),*
$(f_k * F_k)(z) \in \overline{RS}_H(2k,\gamma)$ *(If k is an even integer) where* $0 \leq \beta \leq \gamma < 1$.

To prove this theorem, we require the following lemmas. Lemma1 and 2 are due to Yalcin et al.[4].

Lemma 1 *[4, Theorem2.2] Let* $f_k = h + \overline{g_k}$ *be given by* (4). *Then* $f_k \in \overline{RS}_H(k,\gamma)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^k(2n+\gamma+1)}{1-\gamma} |b_n| \leq 1, \tag{6}$$

where $0 \leq \gamma < 1$, $k \in N_0$.

Lemma 2 $\overline{RS}_H(k,\gamma) \subseteq \overline{RS}_H(k,\beta)$ *if* $0 \leq \beta \leq \gamma < 1$.

Lemma 3 **(i).**$\overline{RS}_H(2k+1,\gamma) \subseteq \overline{RS}_H(k,\gamma)$ *(if k is an odd integer)*
(ii) $\overline{RS}_H(2k,\gamma) \subseteq \overline{RS}_H(k,\gamma)$ *(if k is an even integer)*

Proof. (i). Let $f_{2k+1}(z) \in \overline{RS}_H(2k+1,\gamma)$ then by Lemma1 we have

$$\sum_{n=2}^{\infty} \frac{n^{2k+1}(2n-\gamma-1)}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^{2k+1}(2n+\gamma+1)}{1-\gamma} |b_n| \leq 1. \tag{7}$$

Now

$$\sum_{n=2}^{\infty} \frac{n^k (2n-\gamma-1)}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^k (2n+\gamma+1)}{1-\gamma} |b_n|$$

$$\leq \sum_{n=2}^{\infty} \frac{n^{2k+1} (2n-\gamma-1)}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^{2k+1} (2n+\gamma+1)}{1-\gamma} |b_n|$$

≤ 1. (Using (7))

Thus $f_{2k+1}(z) \in \overline{RS}_H(2k+1, \gamma)$.

The proof of Lemma 3 (i) is established.

(ii). The proof of Lemma 3 (ii) is similar to that of Lemma 3 (i), hence it is omitted.

2 Proof of the Theorem 1

Here we only prove the Theorem 1 for the case when k is an odd integer. For the case when k is an even integer one can prove the theorem in similar way. Therefore it is omitted.

Since $f_k(z) \in \overline{RS}_H(k, \gamma)$, then by Lemma1 we have

$$\sum_{n=2}^{\infty} \frac{n^k (2n-\gamma-1)}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^k (2n+\gamma+1)}{1-\gamma} |b_n| \leq 1. \tag{8}$$

Similarly $F_k(z) \in \overline{RS}_H(k, \beta)$ we have

$$\sum_{n=2}^{\infty} \frac{n^k (2n-\beta-1)}{1-\beta} |A_n| + \sum_{n=1}^{\infty} \frac{n^k (2n+\beta+1)}{1-\beta} |B_n| \leq 1.$$

Therefore $\frac{n^k(2n-\beta-1)}{1-\beta} |A_n| \leq 1 \ \forall n = 2, 3, ...$ and $\frac{n^k(2n+\beta+1)}{1-\beta} |B_n| \leq 1 \ \forall n = 1, 2, 3, ...$

Now for the convolution function $f_k * F_k$ we obtain

$$\sum_{n=2}^{\infty} \frac{n^{2k+1}(2n-\gamma-1)}{1-\gamma} |a_n| |A_n| + \sum_{n=1}^{\infty} \frac{n^{2k+1}(2n+\gamma+1)}{1-\gamma} |b_n| |B_n|$$
$$= \sum_{n=2}^{\infty} \frac{n^{k}(2n-\gamma-1)}{1-\gamma} |a_n| n^{k+1} |A_n| + \sum_{n=1}^{\infty} \frac{n^{k}(2n+\gamma+1)}{1-\gamma} |b_n| n^{k+1} |B_n|$$
$$\leq \sum_{n=2}^{\infty} \frac{n^{k}(2n-\gamma-1)}{1-\gamma} |a_n| \frac{n^{k}(2n-\beta-1)}{1-\beta} |A_n|$$
$$+ \sum_{n=1}^{\infty} \frac{n^{k}(2n+\gamma+1)}{1-\gamma} |b_n| \frac{n^{k}(2n+\beta+1)}{1-\beta} |B_n|$$
$$\leq \sum_{n=2}^{\infty} \frac{n^{k}(2n-\gamma-1)}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^{k}(2n+\gamma+1)}{1-\gamma} |b_n| \leq 1 \qquad \text{(using (8))}.$$

Therefore we have
$(f_k * F_k)(z) \in \overline{RS}_H(2k+1,\gamma)$ (if k is an odd integer)
Similarly
$(f_k * F_k)(z) \in \overline{RS}_H(2k,\gamma)$ (if k is an even integer)

3 Improvement on the result of Theorem A

In this section we consider the following two cases and, in each case, we observe that our result improves the result of Yalcin et al.[4,Theorem2.6].

Case(i) When k is an odd integer

Case(ii) When k is an even integer

Here we discuss these cases one by one.

Case(i) When k is an odd integer our Theorem states that $f_k * F_k \in \overline{RS}_H(2k+1,\gamma)$, whereas result of Yalcin et al. gives $f_k * F_k \in \overline{RS}_H(k,\gamma)$. But by Lemma 2 and 3(i) we have $\overline{RS}_H(2k+1,\gamma) \subseteq \overline{RS}_H(k,\gamma) \subseteq \overline{RS}_H(k,\beta)$. Therefore our result provides smaller class in comparison to the class given by Yalcin et al. to which $(f_k * F_k)(z)$ belongs.

Case (ii) When k is an even integer we use our result $(f_k * F_k)(z) \in \overline{RS}_H(2k,\gamma)$

. Since $\overline{RS}_H\,(2k,\gamma) \subseteq \overline{RS}_H\,(k,\gamma) \subseteq \overline{RS}_H\,(k,\beta)$ (by Lemma2 and 3(ii)). Our result provides better estimate in this case also.

Hence we conclude that for all values of $k \in N_0 = \{0, 1, 2, 3.....\}$ our result improves the result of Yalcin et al.[4,Theorem2.6].

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