

Proper curvature collineations in non-static plane symmetric space-times ¹

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Abstract

The most general form of non-static plane symmetric space-times is considered to study proper curvature collineations by using the rank of the 6×6 Riemann matrix and direct integration techniques. Studying proper curvature collineations in each non static case of the above space-times it is shown that when the above space-times admit proper curvature collineations, they form an infinite dimensional vector space.

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1 Introduction

This paper is an attempt to investigate the existence of proper curvature collineations (CCS) in the non static plane symmetric space-times. These curvature collineations which preserve the curvature structure of a space-time carry significant information and play an important role in Einstein's theory of general relativity and gravitation. The theory of general relativity, which is actually a field theory of gravitation and is described in terms of geometry, is highly non-linear [2]. Due to this non-linearity it becomes very

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hard to solve the gravitational field equations unless certain symmetry restrictions are imposed on the space-times. These symmetry restrictions may be expressed in terms of Killing vector fields (KVF), homothetic vector fields (HVF), Ricci collineations (RCS) and curvature collineations. Killing vector fields give rise to some conservation laws. Katzin et al. [2, 3] suggests that Riemann curvature tensor may also provide some extra understandings which are not provided by (KVF) and (HVF). It is, therefore, important to study CCS. Here an approach, which is given in [4], is adopted to study proper curvature collineations in non-static plane symmetric space-times by using the rank of the 6×6 Riemann matrix and direct integration techniques. In this paper we are only interested in the non static cases of the above space-times. The cases when the above space-times become static their CCS can be found in [5].

Throughout M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature $(-, +, +, +)$. The curvature tensor associated with g_{ab} , through the Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L , respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. Here, M is assumed non-flat in the sense that the Riemann tensor does not vanish over any non empty open subset of M .

The covariant derivative of any vector field on can be decomposed as

$$(1) \quad X_{a;b} = \frac{1}{2}h_{ab} + F_{ab}$$

where $h_{ab}(= h_{ba}) = L_X g_{ab}$ is a symmetric and $F_{ab} = -F_{ba}$ is a skew symmetric tensor on M . If $h_{ab;c} = 0$, X is said to be affine and further $h_{ab} = 2cg_{ab}$, $c \in R$ then X is said to be homothetic (and Killing if $c = 0$.) The vector field X is said to be proper affine if it is not homothetic vector field and also X is said to be proper homothetic vector field if it is not Killing vector field.

A vector field X on M is called a curvature collineation (CC) if it satisfies [2]

$$(2) \quad L_X R^a{}_{bcd} = 0$$

or equivalently,

$$R^a{}_{bcd;e} X^e + R^a{}_{ecd} X^e{}_{;b} + R^a{}_{bed} X^e{}_{;c} + R^a{}_{bce} X^e{}_{;d} - R^e{}_{bcd} X^a{}_{;e} = 0.$$

The vector field X is said to be proper CC if it is not affine [4] on M . One can expand the above equation in a set of 22 coupled CC equations which can be seen in [6].

2 Classification of the Riemann tensors

In this section we will classify the Riemann tensor in terms of its rank and bivector decomposition.

The rank of the Riemann tensor is the rank of the 6×6 symmetric matrix derived in a well known way [4]. The rank of the Riemann tensor at p is the rank of the linear map β which maps the vector space of all bivectors F at p to itself and is defined by $\beta : F^{ab} \rightarrow R^{ab}_{cd} F^{cd}$. Define the subspace N_p of the tangent space $T_p M$ consisting of those members k of $T_p M$ which satisfy the relation

$$(3) \quad R_{abcd} k^d = 0.$$

Then the Riemann tensor at p satisfies exactly one of the following algebraic conditions [4].

Class B

The rank is 2 and the range of β is spanned by the dual pair of non-null simple bivectors and $\dim N_p = 0$. The Riemann tensor at p takes the form

$$(4) \quad R_{abcd} = \alpha F_{ab} F_{cd} + \eta F_{ab}^* F_{cd}^*,$$

where F and its dual F^* are the unique (up to scaling) simple non-null space-like and timelike bivectors in the range of β , respectively and $\alpha, \eta \in R$.

Class C

The rank is 2 or 3 and there exists a unique (up to scaling) solution say, k of (3) (and so $\dim N_p = 1$). The Riemann tensor at p takes the form

$$(5) \quad R_{abcd} = \sum_{i,j=1}^3 \alpha_{ij} F_{ab}^i F_{cd}^j,$$

where $\alpha_{ij} \in R$ for all i, j and $F_{ab}^i k^b = 0$ for each of the bivectors F^i which span the range of β .

Class D

Here the rank of the curvature matrix is 1. The range of the map β is spanned by a single bivector F , say, which has to be simple because

the symmetry of Riemann tensor $R_{a[bcd]} = 0$ means $F_{a[b}F_{cd]} = 0$. Then it follows from a standard result that F is simple. The curvature tensor admits exactly two independent solutions k, u of equation (3) so that $\dim N_p = 2$. The Riemann tensor at p takes the form

$$(6) \quad R_{abcd} = \alpha F_{ab}F_{cd},$$

where $\alpha \in R$ and F is simple bivector with blade orthogonal to k and u .

Class 0

Here the rank of the curvature matrix is 0 (so that $R_{abcd} = 0$) and $\dim N_p = 4$.

Class A

The Riemann tensor is said to be of class A at p if it is not of class B, C, D or O. Here always $\dim N_p = 0$.

A study of the CCS for the classes A, B, D, C, O and CCS in the two dimensional submanifolds can be found in [4].

3 Main results

Consider a non static plane symmetric space-time in the usual coordinate system (t, x, y, z) (labeled by (x^0, x^1, x^2, x^3) , respectively) with line element [7]

$$(7) \quad ds^2 = -e^{A(t,x)}dt^2 + e^{B(t,x)}dx^2 + e^{C(t,x)}(dy^2 + dz^2).$$

The Ricci tensor Segre type of the above space-time is $\{1, 1(11)\}$ or $\{2(11)\}$ or one of degeneracies. The above space-time admits three linearly independent Killing vector fields which are

$$(8) \quad \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

The non-zero independent components of the Riemann tensor are

$$\begin{aligned} R_{0101} &= \frac{1}{4}[e^{A(t,x)}(A_x^2(t,x) + 2A_{xx}(t,x) - A_x(t,x)B_x(t,x)) - \\ &\quad e^{B(t,x)}(B_t^2(t,x) + 2B_{tt}(t,x) - A_t(t,x)B_t(t,x))] \equiv \alpha_1, \\ R_{0202} = R_{0303} &= -\frac{1}{4}e^{C(t,x)-B(t,x)}[e^{B(t,x)}(C_t^2(t,x) + 2C_{tt}(t,x) - A_t(t,x)C_t(t,x)) \\ &\quad - e^{A(t,x)}A_x(t,x)C_x(t,x)] \equiv \alpha_2, \end{aligned}$$

$$R_{1212} = R_{1313} = -\frac{1}{4}e^{C(t,x)-A(t,x)}[e^{A(t,x)}(C_x^2(t,x)+2C_{xx}(t,x)-B_x(t,x)C_x(t,x)) - e^{B(t,x)}B_t(t,x)C_t(t,x)] \equiv \alpha_3,$$

$$R_{2323} = -\frac{1}{4}e^{A(t,x)+B(t,x)+2C(t,x)}[e^{A(t,x)}(C_x^2(t,x) - e^{B(t,x)}(C_t^2(t,x))] \equiv \alpha_4,$$

$$R_{0212} = R_{0313} = \frac{1}{4}e^{C(t,x)}[C_x(t,x)C_t(t,x) + 2C_{tx}(t,x) - A_x(t,x)C_t(t,x) - B_t(t,x)C_x(t,x)] \equiv \alpha_5,$$

Writing the curvature tensor with components R_{abcd} at p as a 6×6 symmetric matrix

$$(9) \quad R_{abcd} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & \alpha_5 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & \alpha_5 & 0 \\ 0 & \alpha_5 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & \alpha_5 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_4 \end{pmatrix}.$$

It is important to remind the reader that we will consider Riemann tensor components as R^a_{bcd} for calculating CCS. Here, we are only interested in those cases when the rank of the 6×6 Riemann matrix is less than or equal to three. Since we know from theorem [4] that when the rank of the 6×6 Riemann matrix is greater than three there exists no proper CCS. It is also important to remind the reader again that we will only considering the non static cases. The cases when the space-times become static their CCS can be found in [5]. Thus there exist the following possibilities:

- (1) Rank = 3, $A_x(t,x) = 0, B_t(t,x) = 0, C_x(t,x) = 0, A_t(t,x) \neq 0, B_x(t,x) \neq 0, C_t(t,x) \neq 0$ and $C_t^2(t,x) + 2C_{tt}(t,x) \neq 0$.
- (2) Rank = 3, $A_x(t,x) = 0, B_t(t,x) = 0, B_x(t,x) = 0, C_x(t,x) = 0, A_t(t,x) \neq 0, C_t(t,x) \neq 0$ and $C_t^2(t,x) + 2C_{tt}(t,x) \neq 0$.
- (3) Rank = 3, $A_x(t,x) = 0, B_t(t,x) = 0, C_x(t,x) = 0, A_t(t,x) \neq 0, B_x(t,x) \neq 0, C_t(t,x) \neq 0$ and $C_t^2(t,x) + 2C_{tt}(t,x) = 0$.
- (4) Rank = 3, $A_x(t,x) = 0, A_t(t,x) = 0, B_t(t,x) = 0, C_x(t,x) = 0, B_x(t,x) \neq 0, C_t(t,x) \neq 0$ and $C_t^2(t,x) + 2C_{tt}(t,x) \neq 0$.
- (5) Rank = 3, $A_x(t,x) = 0, A_t(t,x) = 0, B_t(t,x) = 0, B_x(t,x) = 0, C_x(t,x) = 0, C_t(t,x) \neq 0$ and $C_t^2(t,x) + 2C_{tt}(t,x) \neq 0$.
- (6) Rank = 3, $A_x(t,x) = 0, A_t(t,x) = 0, C_x(t,x) = 0, B_t(t,x) \neq 0, B_x(t,x) \neq 0, C_t(t,x) \neq 0, C_t^2(t,x) + 2C_{tt}(t,x) = 0$ and $B_t^2(t,x) + 2B_{tt}(t,x) = 0$.

- (7) Rank = 3, $A_x(t, x) = 0, A_t(t, x) = 0, B_x(t, x) = 0, C_x(t, x) = 0, B_t(t, x) \neq 0, C_t(t, x) \neq 0, C_t^2(t, x) + 2C_{tt}(t, x) = 0$ and $B_t^2(t, x) + 2B_{tt}(t, x) = 0$.
- (8) Rank = 3, $A_x(t, x) = 0, A_t(t, x) = 0, B_t(t, x) = 0, B_x(t, x) \neq 0, C_x(t, x) \neq 0, C_t(t, x) \neq 0, C_t^2(t, x) + 2C_{tt}(t, x) = 0, C_t(t, x)C_x(t, x) + 2C_{tx}(t, x) = 0$ and $C_x^2(t, x) + 2C_{xx}(t, x) \neq 0$.
- (9) Rank = 3, $A_x(t, x) = 0, A_t(t, x) = 0, B_t(t, x) = 0, B_x(t, x) \neq 0, C_x(t, x) \neq 0, C_t(t, x) \neq 0, C_t^2(t, x) + 2C_{tt}(t, x) = 0, C_t(t, x)C_x(t, x) + 2C_{tx}(t, x) = 0$ and $C_x^2(t, x) + 2C_{xx}(t, x) = 0$.
- (10) Rank = 3, $A_x(t, x) = 0, A_t(t, x) = 0, B_t(t, x) = 0, B_x(t, x) = 0, C_x(t, x) \neq 0, C_t(t, x) \neq 0, C_t^2(t, x) + 2C_{tt}(t, x) = 0, C_t(t, x)C_x(t, x) + 2C_{tx}(t, x) = 0$ and $C_x^2(t, x) + 2C_{xx}(t, x) \neq 0$.
- (11) Rank = 3, $A_x(t, x) = 0, B_t(t, x) = 0, B_x(t, x) = 0, A_t(t, x) \neq 0, C_x(t, x) \neq 0, C_t(t, x) \neq 0, C_x^2(t, x) + 2C_{xx}(t, x) = 0, C_t(t, x)C_x(t, x) + 2C_{tx}(t, x) = 0$ and $C_t^2(t, x) + 2C_{tt}(t, x) = 0$.
- (12) Rank = 3, $A_x(t, x) = 0, B_t(t, x) = 0, B_x(t, x) = 0, A_t(t, x) \neq 0, C_x(t, x) \neq 0, C_t(t, x) \neq 0, C_t^2(t, x) + 2C_{tt}(t, x) \neq 0, C_t(t, x)C_x(t, x) + 2C_{tx}(t, x) = 0$ and $C_x^2(t, x) + 2C_{xx}(t, x) = 0$.
- (13) Rank = 3, $A_x(t, x) = 0, A_t(t, x) = 0, B_t(t, x) = 0, B_x(t, x) = 0, C_x(t, x) \neq 0, C_t(t, x) \neq 0, C_t^2(t, x) + 2C_{tt}(t, x) \neq 0, C_t(t, x)C_x(t, x) + 2C_{tx}(t, x) = 0$ and $C_x^2(t, x) + 2C_{xx}(t, x) = 0$.
- (14) Rank = 1, $C_t(t, x) = 0, C_x(t, x) = 0, A_x(t, x) \neq 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0, B_x(t, x) \neq 0, A_x^2(t, x) + 2A_{xx}(t, x) - A_x(t, x)B_x(t, x) \neq 0$ and $B_t^2(t, x) + 2B_{tt}(t, x) - A_t(t, x)B_t(t, x) \neq 0$.
- (15) Rank = 1, $A_x(t, x) = 0, C_t(t, x) = 0, C_x(t, x) = 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0, B_x(t, x) \neq 0$ and $B_t^2(t, x) + 2B_{tt}(t, x) - A_t(t, x)B_t(t, x) \neq 0$.
- (16) Rank = 1, $A_x(t, x) = 0, B_x(t, x) = 0, C_t(t, x) = 0, C_x(t, x) = 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0$ and $B_t^2(t, x) + 2B_{tt}(t, x) - A_t(t, x)B_t(t, x) \neq 0$.
- (17) Rank = 1, $C_t(t, x) = 0, C_x(t, x) = 0, A_x(t, x) \neq 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0, B_x(t, x) \neq 0, B_t^2(t, x) + 2B_{tt}(t, x) - A_t(t, x)B_t(t, x) \neq 0$ and $A_x^2(t, x) + 2A_{xx}(t, x) - A_x(t, x)B_x(t, x) = 0$.
- (18) Rank = 1, $C_t(t, x) = 0, C_x(t, x) = 0, A_x(t, x) \neq 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0, B_x(t, x) \neq 0, B_t^2(t, x) + 2B_{tt}(t, x) - A_t(t, x)B_t(t, x) = 0$ and $A_x^2(t, x) + 2A_{xx}(t, x) - A_x(t, x)B_x(t, x) \neq 0$.
- (19) Rank = 1, $C_t(t, x) = 0, C_x(t, x) = 0, A_x(t, x) \neq 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0, B_x(t, x) \neq 0, B_t^2(t, x) + 2B_{tt}(t, x) - A_t(t, x)B_t(t, x) \neq 0$ and $A_x^2(t, x) + 2A_{xx}(t, x) = 0$.

(20) Rank = 1, $C_t(t, x) = 0, C_x(t, x) = 0, A_x(t, x) \neq 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0, B_x(t, x) \neq 0, B_t^2(t, x) + 2B_{tt}(t, x) = 0$ and $A_x^2(t, x) + 2A_{xx}(t, x) - A_x(t, x)B_x(t, x) \neq 0$.

(21) Rank = 1, $A_t(t, x) = 0, B_x(t, x) = 0, C_t(t, x) = 0, C_x(t, x) = 0, A_x(t, x) \neq 0, B_t(t, x) \neq 0, B_t^2(t, x) + 2B_{tt}(t, x) \neq 0$ and $A_x^2(t, x) + 2A_{xx}(t, x) \neq 0$.

(22) Rank = 1, $A_t(t, x) = 0, B_x(t, x) = 0, C_t(t, x) = 0, C_x(t, x) = 0, A_x(t, x) \neq 0, B_t(t, x) \neq 0, B_t^2(t, x) + 2B_{tt}(t, x) \neq 0$ and $A_x^2(t, x) + 2A_{xx}(t, x) = 0$.

(23) Rank = 1, $A_x(t, x) = 0, A_t(t, x) = 0, B_x(t, x) = 0, C_t(t, x) = 0, C_x(t, x) = 0, B_t(t, x) \neq 0$ and $B_t^2(t, x) + 2B_{tt}(t, x) \neq 0$.

(24) Rank = 1, $A_x(t, x) = 0, B_x(t, x) = 0, C_t(t, x) = 0, C_x(t, x) = 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0$ and $B_t^2(t, x) + 2B_{tt}(t, x) = 0$.

(25) Rank = 1, $A_x(t, x) = 0, C_t(t, x) = 0, C_x(t, x) = 0, B_x(t, x) \neq 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0$ and $B_t^2(t, x) + 2B_{tt}(t, x) = 0$.

(26) Rank = 1, $C_t(t, x) = 0, C_x(t, x) = 0, A_x(t, x) \neq 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0, B_x(t, x) \neq 0, A_x^2(t, x) + 2A_{xx}(t, x) = 0$ and $B_t^2(t, x) + 2B_{tt}(t, x) = 0$.

(27) Rank = 1, $A_x(t, x) = 0, A_t(t, x) = 0, B_t(t, x) = 0, B_x(t, x) = 0, C_x(t, x) = 0, C_t(t, x) \neq 0$ and $C_t^2(t, x) + 2C_{tt}(t, x) = 0$.

(28) Rank = 1, $A_x(t, x) = 0, A_t(t, x) = 0, B_t(t, x) = 0, B_x(t, x) = 0, C_x(t, x) \neq 0, C_t(t, x) \neq 0, C_x^2(t, x) + 2C_{xx}(t, x) = 0$ and $C_t(t, x)C_x(t, x) + 2C_{tx}(t, x) = 0$.

We will consider each case in turn.

Case 1

In this case we have $A_x(t, x) = 0, B_t(t, x) = 0, C_x(t, x) = 0, A_t(t, x) \neq 0, B_x(t, x) \neq 0, C_t(t, x) \neq 0, C_t^2(t, x) + 2C_{tt}(t, x) \neq 0$ and the rank of the 6×6 Riemann matrix is three. Here, there exists a unique (up to a multiple) non zero spacelike vector field $x_a = x_{,a}$ satisfying $x_{a;b} = 0$. From the Ricci identity $R^a_{bcd}x_a = 0$. From the above constraints we have $A_x(t, x) = 0, B_t(t, x) = 0$, and $C_x(t, x) = 0 \Rightarrow A(t, x) = \alpha(t), B(t, x) = \beta(t)$, and $C(t, x) = \eta(t)$. Substituting the information of $A(t, x), B(t, x)$, and $C(t, x)$ in (7) and after a rescaling of x , the line element can be written in the form

$$(10) \quad ds^2 = -e^{\alpha(t)} dt^2 + dx^2 + e^{\eta(t)} (dy^2 + dz^2),$$

The above space-time (10) is clearly 1+3 decomposable and belongs to curvature class C. CCS in this case [4] are

$$(11) \quad X = f(x) \frac{\partial}{\partial x} + X',$$

where $f(x)$ is an arbitrary function of x only and X' is a homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant x . The completion of case 1 needs finding a homothetic vector field in the induced geometry of the submanifolds of constant x . The induced metric $g_{\lambda w}$ (where $\lambda, w = 0, 2, 3$) with nonzero components is given by

$$(12) \quad g_{00} = -e^{\alpha(t)}, \quad g_{22} = g_{33} = e^{\eta(t)}.$$

A vector field X' is a homothetic vector field if it satisfies

$$(13) \quad L_{X'} g_{\lambda w} = 2c g_{\lambda w}, \quad c \in R,$$

where $\lambda, w = 0, 2, 3$. One can expand equation (13) using (12) to get

$$(14) \quad \alpha X^0 + 2X^0_{,0} = c$$

$$(15) \quad -e^{\alpha} X^0_{,2} + e^{\eta} X^2_{,0} = 0$$

$$(16) \quad -e^{\alpha} X^0_{,3} + e^{\eta} X^3_{,0} = 0$$

$$(17) \quad \eta X^0 + 2X^2_{,2} = c$$

$$(18) \quad X^2_{,3} + X^3_{,2} = 0$$

$$(19) \quad \eta X^0 + 2X^3_{,3} = c$$

Equation (14), (15) and (16) give

$$X^0 = \frac{1}{2} c e^{-\frac{\alpha}{2}} \int e^{\frac{\alpha}{2}} dt + e^{-\frac{\alpha}{2}} N^1(y, z), \quad X^2 = N^1_y(y, z) \int e^{\frac{\alpha}{2} - \eta} dt + N^2(y, z),$$

$$X^3 = N^1_z(y, z) \int e^{\frac{\alpha}{2} - \eta} dt + N^3(y, z),$$

where $N^1(y, z)$, $N^2(y, z)$ and $N^3(y, z)$ are functions of integration. If one proceeds further, after a straightforward calculation one can find that the proper homothetic vector fields exist if and only if $\eta(t) = \ln(at+b)^2$ and $\alpha(t) = d$, where $a, b, d \in R$. Substituting this information into (9), one finds that the rank of the Riemann matrix reduces to one, thus giving a contradiction. So the only homothetic vector fields in the induced geometry are the Killing vector fields which are

$$(20) \quad X^0 = 0, \quad X^2 = z c_1 + c_2, \quad X^3 = -y c_1 + c_3,$$

where $c_1, c_2, c_3 \in R$. CCS in this case are given by use of equation (11) and (20)

$$(21) \quad X^0 = 0, X^1 = f(x), X^2 = zc_1 + c_2, X^3 = -yc_1 + c_3,$$

where $f(x)$ is an arbitrary function of x . One can write the above equation (21) after subtracting Killing vector fields as

$$(22) \quad X = (0, f(x), 0, 0).$$

CCS clearly form an infinite dimensional vector space. Cases 2 to 5 are precisely the same.

Case 6

In this case we have $A_x(t, x) = 0, A_t(t, x) = 0, C_x(t, x) = 0, B_t(t, x) \neq 0, B_x(t, x) \neq 0, C_t(t, x) \neq 0, C_t^2(t, x) + 2C_{tt}(t, x) = 0, B_t^2(t, x) + 2B_{tt}(t, x) = 0$ and rank of the 6×6 Riemann matrix is three. Here, there exists a unique (up to a multiple) no where zero timelike vector field $t_a = t_{,a}$ which is the solution of equation (3). The vector field t_a is not covariantly constant. From the above constraints we have $A_x(t, x) = 0, A_t(t, x) = 0$ and $C_x(t, x) = 0 \Rightarrow A(t, x) = b$ and $C(t, x) = D(t)$, where $b \in R$ and $D(t)$ is a function of integration. Equations $D_t^2(t) + 2D_{tt}(t) = 0$ and $B_t^2(t, x) + 2B_{tt}(t, x) = 0 \Rightarrow D(t) = \ln(at + d)^2$ and $B(t, x) = \ln(P(x)t + Q(x))^2$, where $P(x)$ is no where zero functions of integration and $Q(x)$ is function of integration and $a, d \in R(a \neq 0)$. The line element in this case can, after a rescaling of t , be written in the form

$$(23) \quad ds^2 = -dt^2 + (P(x)t + Q(x))^2 dx^2 + (at + d)^2 (dy^2 + dz^2).$$

The above space-time (23) belongs to curvature class C. Substituting the above information into the CC equations in [6] and after some calculation one finds CCS in this case are

$$(24) \quad X^0 = 0, X^1 = 0, X^2 = zc_1 + c_2, X^3 = -yc_1 + c_3,$$

where $c_1, c_2, c_3 \in R$. CCS in this case are Killing vector fields. Cases 7 to 13 are precisely the same.

Case 14

In this case one has $A_x(t, x) \neq 0, A_t(t, x) \neq 0, B_t(t, x) \neq 0, B_x(t, x) \neq 0, A_x^2(t, x) + 2A_{xx}(t, x) - A_x(t, x)B_x(t, x) \neq 0, B_t^2(t, x) + 2B_{tt}(t, x) - A_t(t, x)B_t(t, x) \neq 0, C_t(t, x) = 0, C_x(t, x) = 0$ and the rank of the 6×6 Riemann matrix is one. From the above constraints we have $A = A(t, x)$,

$B = B(t, x)$ and $C(t, x) = e$, where $e \in R$. Here there exist two linear independent solutions $y_a = y_{,a}$ and $z_a = z_{,a}$ of equation (3) and satisfying $y_{a;b} = 0$ and $z_{a;b} = 0$. The line element can, after rescaling of y and z , be written in the form

$$(25) \quad ds^2 = -e^{A(t,x)} dt^2 + e^{B(t,x)} dx^2 + (dy^2 + dz^2).$$

The above space-time (25) is 1+1+2 decomposable and belongs to curvature class D. CCS in this case are [4]

$$(26) \quad X = f(y, z) \frac{\partial}{\partial y} + g(y, z) \frac{\partial}{\partial z} + X',$$

where $f(y, z)$ and $g(y, z)$ are arbitrary functions of y and z and X' is a CC on each of two dimensional submanifolds of constant y and z . The next step is to find the CCS in the induced geometry of the submanifolds of constant y and z . A method for finding CCS in 2-dimensional submanifolds is given in [4]. If one proceeds further the non zero components of the induced metric on each of the two dimensional submanifolds of constant y and z are given by

$$(27) \quad g_{00} = -e^{A(t,x)}, g_{11} = -e^{B(t,x)}.$$

The nonzero components of the Ricci tensor are

$$(28) \quad \begin{aligned} R_{00} &= \frac{1}{4} e^{-B(t,x)} [(A_x^2 + 2A_{xx} - A_x B_x) e^{A(t,x)} - e^{B(t,x)} (B_t^2 + 2B_{tt} - A_t B_t)], \\ R_{11} &= -\frac{1}{4} e^{-A(t,x)} [(A_x^2 + 2A_{xx} - A_x B_x) e^{A(t,x)} - e^{B(t,x)} (B_t^2 + 2B_{tt} - A_t B_t)]. \end{aligned}$$

The Ricci scalar is

$$R = -\frac{1}{2} [(A_x^2 + 2A_{xx} - A_x B_x) e^{-B(t,x)} - e^{-A(t,x)} (B_t^2 + 2B_{tt} - A_t B_t)].$$

It follows from [4] that CCS in the two dimensional submanifolds of constant y and z are the solution of the equation $L_{X'} G_{\alpha\beta} = 0$, where $G_{\alpha\beta} \equiv \frac{R}{2} g_{\alpha\beta}$ and $\alpha, \beta = 0, 1$. The non-zero components of $G_{\alpha\beta}$ are

$$(29) \quad \begin{aligned} G_{00} &= \frac{1}{4} e^{-B(t,x)} [(A_x^2 + 2A_{xx} - A_x B_x) e^{A(t,x)} - e^{B(t,x)} (B_t^2 + 2B_{tt} - A_t B_t)], \\ G_{11} &= -\frac{1}{4} e^{-A(t,x)} [(A_x^2 + 2A_{xx} - A_x B_x) e^{A(t,x)} - e^{B(t,x)} (B_t^2 + 2B_{tt} - A_t B_t)]. \end{aligned}$$

Expanding the equation $L_{X'} G_{\alpha\beta} = 0$ and using equation (29) we get

$$(30) \quad \dot{G}_{00} X^0 + G'_{00} X^1 + G_{00} X^0_{,0} = 0$$

$$(31) \quad e^{B(t,x)} X^1_{,0} - e^{A(t,x)} X^0_{,1} = 0$$

$$(32) \quad \dot{G}_{11} X^0 + G'_{11} X^1 + G_{11} X^0_{,0} = 0$$

Here, the above system of equations give trivial solution which is $X^0 = X^1 = 0$. Proper CCS in this case can be written as

$$(33) \quad X = (0, 0, f(y, z), g(y, z)).$$

Clearly CCS in this case form an infinite dimensional vector space. Cases 15 to 26 are precisely the same.

Case 27

In this case we have $A_x(t, x) = 0, A_t(t, x) = 0, B_t(t, x) = 0, B_x(t, x) = 0, C_x(t, x) = 0, C_t(t, x) \neq 0, C_t^2(t, x) + 2C_{tt}(t, x) = 0$ and the rank of the 6×6 Riemann matrix is one. From the above constraints we have $A(t, x) = a, B(t, x) = b$ and $C(t, x) = \ln(et + d)^2$, where $a, b, e, d \in R(e \neq 0)$. Here there exist two linear independent solutions $t_a = t_{,a}$ and $x_a = x_{,a}$ of equation (3). The vector field t_a is not a covariantly constant whereas x_a is covariantly constant. Substituting the information of $A(t, x), B(t, x)$ and $C(t, x)$ in (7) and after rescaling of t and x , the line element can be written in the form

$$(34) \quad ds^2 = -dt^2 + dx^2 + (et + d)^2(dy^2 + dz^2).$$

The above space-time (34) is 1+3 decomposable and belongs to curvature class D. Substituting the above information into CC equations in [6] one finds

$$(35) \quad X^0_{,2} = X^0_{,3} = X^1_{,2} = X^1_{,3} = X^2_{,0} = X^2_{,1} = X^2_{,2} = X^3_{,0} = X^3_{,1} = X^3_{,3} = 0,$$

$$(36) \quad X^2_{,3} + X^3_{,2} = 0$$

Equation (35) gives $X^0 = f(t, x), X^1 = g(t, x), X^2 = M(z)$ and $X^3 = M(y)$, where $f(t, x)$ and $g(t, x)$ are arbitrary functions and $M(z)$ and $M(y)$ are functions of integration. Substituting the above information about X^2 and X^3 in (36) gives $M_z(z) + N_y(y) = 0$, upon differentiating with respect to y gives $N_{yy}(y) = 0 \Rightarrow N(y) = c_1 y + c_2$, substituting back in the same equation we get $M(z) = -c_1 z + c_3$, where $c_1, c_2, c_3 \in R$. CCS in this case

$$(37) \quad X^0 = f(t, x), X^1 = g(t, x), X^2 = -c_1 z + c_3, X^3 = c_1 y + c_2.$$

One can write the above equation (37) after subtracting Killing vector fields as

$$(38) \quad X = (f(t, x), g(t, x), 0, 0).$$

Clearly CCS form an infinite dimensional vector space.

Case 28

In this case we have $A_x(t, x) = 0, A_t(t, x) = 0, B_t(t, x) = 0, B_x(t, x) = 0, C_x(t, x) \neq 0, C_t(t, x) \neq 0, C_x^2(t, x) + 2C_{xx}(t, x) = 0, C_t(t, x)C_x(t, x) + 2C_{tx}(t, x) = 0$ and the rank of the 6×6 Riemann matrix is one. From the above constraints we have $A(t, x) = a, B(t, x) = b$ and $C(t, x) = \ln(ct+ex+d)^2$, where $a, b, e, d \in R(c, e \neq 0)$. Here there exist two linear independent solutions $t_a = t_{,a}$ and $x_a = x_{,a}$ of equation (3). The vector field t_a and x_a are not a covariantly constant. The line element after rescaling of t and x , can be written in the form

$$(39) \quad ds^2 = -dt^2 + dx^2 + (ct + ex + d)^2(dy^2 + dz^2).$$

The above space-time (39) belongs to curvature class D. Substituting the above information into the CC equations in [6] and after some calculation one find CCS in this case are Killing vector fields which are given in equation (24).

Summary

In this paper a study of non-static plane symmetric space-times according to their proper CCS is given. An approach is adopted to study the above space-times by using the rank of the 6×6 Riemann matrix and also using the theorem given in [4], which suggested where proper curvature collineations exist. From the above study we obtain the following results:

- (i) We get the space-time (10) that admits proper curvature collineations (see case 1) when the rank of the 6×6 Riemann matrix is three and there exists a nowhere zero independent covariantly constant spacelike vector field, which is the solution of equation (3).
- (ii) We obtain the space-time (23) that admits curvature collineations when the rank of the 6×6 Riemann matrix is three and there exists a unique nowhere zero independent timelike vector field, which is the solution of equation (3) and is not covariantly constant. In this case the curvature collineations are Killing vector fields (for details see case 6).
- (iii) The space-time (25) is obtained, which admits proper curvature collineations (see case 14) when the rank of the 6×6 Riemann matrix is one and there exist two independent covariantly constant spacelike vector fields being the solutions of equation (3).

(iv) The space-time (34) is obtained, which admits proper curvature collineations (see case 27) when the rank of the 6×6 Riemann matrix is one and there exist two nowhere zero independent solutions of equation (3) of which only one is covariantly constant vector field.

(v) The space-time (39) is achieved, which admits curvature collineations (see case 28) when the rank of the 6×6 Riemann matrix is one and there exist two nowhere zero independent timelike and spacelike vector fields being the solutions of equation (3) and are not covariantly constant. In this case curvature collineations are Killing vector fields.

References

- [1] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, Freeman, San Francisco, 1973.
- [2] G. H. Katzin, J. Levine, W. R. Davis, *Curvature collineations: a fundamental symmetry property of the space-times of general relativity defined by the vanishing Lie derivative of the Riemann curvature tensor*, J. Math. Physics, 10, 1969, 617-629.
- [3] G. H. Katzin, J. Levine, *Application of Lie derivatives to symmetries, geodesic mapping, and first integrals in Riemannian spaces*, Colloq. Mathematicum, 26, 1972, 21-38.
- [4] G. S. Hall, J. da. Costa, *Curvature collineations in general relativity II*, J. Math. Physics, 32, 1991, 2854-2862.
- [5] G. Shabbir, *Proper curvature collineations in plane symmetric static space-times*, Applied and Computational Mathematics An International Journal, 2, 2003, 65-76.
- [6] G. Shabbir, A. H. Bokhari, A. R. Kashif, *Proper curvature collineations in cylindrically symmetric static space-times*, Nuovo Cimento B, 118, 2003, 873-886.
- [7] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselears, E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 2003.

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