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# Width-Integrals of Mixed Projection Bodies and Mixed Affine Surface Area [1]

Zhao Chang-jian, Mihály Bencze

**Abstract**

The main purposes of this paper are to establish some new Brunn-Minkowski inequalities for width-integrals of mixed projection bodies and affine surface area of mixed bodies, and get their inverse forms.

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## 1 Introduction

In recent years some authors including Ball[1], Bourgain[2], Gardner[3], Schneider[4] and Lutwak[5-10] et al have given considerable attention to the Brunn-Minkowski theory and Brunn-Minkowski-Firey theory and their various generalizations. In particular, Lutwak[7] had generalized the Brunn-Minkowski inequality (1) to mixed projection body and get inequality (2):

**The Brunn-Minkowski inequality** *If* $K, L \in \mathcal{K}^n$, *then*

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{1}$$

*with equality if and only if* $K$ *and* $L$ *are homothetic.*

**The Brunn-Minkowski inequality for mixed projection bodies** *If* $K, L \in \mathcal{K}^n$, *then*

$$V(\Pi(K+L))^{1/n(n-1)} \geq V(\Pi K)^{1/n(n-1)} + V(\Pi L)^{1/n(n-1)}, \tag{2}$$

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*with equality if and only if $K$ and $L$ are homothetic.*

On the other hand, width-integral of convex bodies and affine surface areas play an important role in the Brunn-Minkowski theory. Width-integrals were first considered by Blaschke[11] and later by Hadwiger[12]. In addition, Lutwak had established the following results for the width-integrals of convex bodies and affine surface areas.

**The Brunn-Minkowski inequality for width-integrals of convex bodies**[10]

*If $K, L \in \mathcal{K}^n$, $i < n-1$*

$$B_i(K+L)^{1/(n-i)} \leq B_i(K)^{1/(n-i)} + B_i(L)^{1/(n-i)} \tag{3}$$

*with equality if and only if $K$ and $L$ have similar width.*

**The Brunn-Minkowski inequality for affine surface area** [9]

*If $K, L \in \kappa^n$, and $i \in \mathbb{R}$, then for $i < -1$*

$$\Omega_i(K\tilde{+}L)^{(n+1)/(n-i)} \leq \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)} \tag{4}$$

*with equality if and only if $K$ and $L$ are homothetic , while for $i > -1$*

$$\Omega_i(K\tilde{+}L)^{(n+1)/(n-i)} \geq \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)} \tag{5}$$

*with equality if and only if $K$ and $L$ are homothetic.*

In this paper, there are two purposes:

Firstly, we generalize inequality (3) to mixed projection bodies and get its inverse version.

**Result A** *If $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$,* let $C = (K_3, \ldots, K_n)$, *then for $i < n-1$*

$$B_i(\Pi(C, K_1+K_2))^{1/(n-i)} \leq B_i(\Pi(C,K_1))^{1/(n-i)} + B_i(\Pi(C,K_2)^{1/(n-i)}, \tag{6}$$

*with equality if and only if $\Pi(C,K_1)$ and $\Pi(C,K_2)$ are homothetic.*

*While for $i > n$ or $n > i > n-1$,*

$$B_i(\Pi(C, K_1+K_2))^{1/(n-i)} \geq B_i(\Pi(C,K_1))^{1/(n-i)} + B_i(\Pi(C,K_2)^{1/(n-i)}, \tag{7}$$

*with equality if and only if $\Pi(C,K_1)$ and $\Pi(C,K_2)$ are homothetic.*

Secondly, we prove that analogs of inequalities (4)-(5) for affine surface area of mixed bodies.

**Result B** *If $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ and all of mixed bodies of $K_1, K_2, \ldots, K_n$ have positive continuous curvature functions, respectively, then for $i < -1$*

$$\Omega_i([K_1+K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)}$$

(8) $\leq \Omega_i([K_1, K_3, K_4 \ldots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)}$.

*with equality if and only if* $[K_1, K_3, K_4, \ldots, K_n]$ *and* $[K_2, K_3 \ldots, K_n]$ *are homothetic.*

*While for* $i > -1$

$$\Omega_i([K_1 + K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)}$$

(9) $\geq \Omega_i([K_1, K_3, K_4, \ldots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)}$

*with equality if and only if* $[K_1, K_3, K_4, \ldots, K_n]$ *and* $[K_2, K_3 \ldots, K_n]$ *are homothetic.*

Please see the next section for above interrelated notations, definitions and their background materials.

## 2 Notations and Preliminary works

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathbb{C}^n$ denote the set of non-empty convex figures(compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$, and if $p \in \mathcal{K}^n$, let $\mathcal{K}^n_p$ denote the subset of $\mathcal{K}^n$ that contains the centered (centrally symmetric with respect to $p$) bodies. We reserve the letter $u$ for unit vectors, and the letter $B$ is reserved for the unit ball centered at the origin. The surface of $B$ is $S^{n-1}$. For $u \in S^{n-1}$, let $E_u$ denote the hyperplane, through the origin, that is orthogonal to $u$. We will use $K^u$ to denote the image of $K$ under an orthogonal projection onto the hyperplane $E_u$.

*2.1 Mixed volumes*

We use $V(K)$ for the $n$-dimensional volume of convex body $K$. Let $h(K, \cdot) : S^{n-1} \to \mathbb{R}$, denote the support function of $K \in \mathcal{K}^n$; i.e.

(10) $$h(K, u) = Max\{u \cdot x : x \in K\}, u \in S^{n-1},$$

where $u \cdot x$ denotes the usual inner product $u$ and $x$ in $\mathbb{R}^n$.

Let $\delta$ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$,

$$\delta(K, L) = |h_K - h_L|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

For a convex body $K$ and a nonnegative scalar $\lambda, \lambda K$, is used to denote $\{\lambda x : x \in K\}$. For $K_i \in \mathcal{K}^n, \lambda_i \geq 0, (i = 1, 2, \ldots, r)$ ,the Minkowski linear combination $\sum_{i=1}^{r} \lambda_i K_i \in \mathcal{K}^n$ is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r = \{\lambda_1 x_1 + \cdots + \lambda_r x_r \in K^n : x_i \in K_i\}. \tag{11}$$

It is trivial to verify that

$$h(\lambda_1 K_1 + \cdots + \lambda_r K_r, \cdot) = \lambda_1 h(K_1, \cdot) + \cdots + \lambda_r h(K_r, \cdot). \tag{12}$$

If $K_i \in \mathcal{K}^n (i = 1, 2, \ldots, r)$ and $\lambda_i (i = 1, 2, \ldots, r)$are nonnegative real numbers, then of fundamental impotence is the fact that the volume of $\sum_{i=1}^{r} \lambda_i K_i$ is a homogeneous polynomial in $\lambda_i$ given by [4]

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{13}$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding $r$. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (13), it is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_{i_1} = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1 \ldots K_n)$ is usually written $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the $i$th projection measure(Quermassintegral) of $K$ and is written as $W_i(K)$. With this notation, $W_0 = V(K)$, while $nW_1(K)$ is the surface area of $K$, $S(K)$.

*2.2 Width-integrals of convex bodies*

For $u \in S^{n-1}$, $b(K, u)$ is defined to be half the width of $K$ in the direction $u$. Two convex bodies $K$ and $L$ are said to have similar width if there exists a constant $\lambda > 0$ such that $b(K, u) = \lambda b(L, u)$ for all $u \in S^{n-1}$. For $K \in \mathcal{K}^n$ and $p \in intK$, we use $K^p$ to denote the polar reciprocal of $K$ with respect to the unit sphere centered at $p$. The width-integral of index $i$ is defined by Lutwak[10]: For $K \in \mathcal{K}^n, i \in \mathbb{R}$

$$B_i(K) = \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i} dS(u), \tag{14}$$

where $dS$ is the $(n-1)$-dimensional volume element on $S^{n-1}$.

The width-integral of index $i$ is a map

$$B_i : \mathcal{K}^n \to \mathbb{R}.$$

It is positive, continuous, homogeneous of degree $n-i$ and invariant under motion. In addition, for $i \leq n$ it is also bounded and monotone under set inclusion.

The following results[10] will be used later

$$b(K+L,u) = b(K,u) + b(L,u), \tag{15}$$

$$B_{2n}(K) \leq V(K^p), \tag{16}$$

with equality if and only if $K$ is symmetric with respect to $p$.

*2.3 The radial function and the Blaschke linear combination*

The radial function of convex body $K$, $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K,\cdot) = Max\{\lambda \geq 0 : \lambda\mu \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, $K$ will be call a star body. Let $\varphi^n$ denote the set of star bodies in $\mathbb{R}^n$.

A convex body $K$ is said to have a positive continuous curvature function[5],

$$f(K,\cdot) : S^{n-1} \to [0,\infty),$$

if for each $L \in \varphi^n$, the mixed volume $V_1(K,L)$ has the integral representation

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} f(K,u)h(L,u)dS(u).$$

The subset of $\mathcal{K}^n$ consisting of bodies which have a positive continuous curvature function will be denoted by $\kappa^n$. Let $\kappa^n_c$ denote the set of centrally symmetric member of $\kappa^n$.

The following result is true[6], for $K \in \kappa^n$

$$\int_{S^{n-1}} uf(K,u)dS(u) = 0.$$

Suppose $K, L \in \kappa^n$ and $\lambda, \mu \geq 0$(not both zero). From above it follows that the function $\lambda f(K,\cdot) + \mu f(L,\cdot)$ satisfies the hypothesis of Minkowski's existence theorem(see [13]). The solution of the Minkowski problem for this function is denoted by $\lambda \cdot K \tilde{+} \mu \cdot L$ that is

$$f(\lambda \cdot K \tilde{+} \mu \cdot L, \cdot) = \lambda f(K,\cdot) + \mu f(L,\cdot), \tag{17}$$

where the linear combination $\lambda \cdot K \tilde{+} \mu \cdot L$ is called a Blaschke linea combination.

The relationship between Blaschke and Minkowski scalar multiplication is given by

$$\lambda \cdot K = \lambda^{1/(n-1)} K. \tag{18}$$

*2.4 Mixed affine area and mixed bodies*

The affine surface area of $K \in \kappa^n$, $\Omega(K)$, is defined by

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{n/(n+1)} dS(u). \tag{19}$$

It is well known that this functional is invariant under unimodular affine transformations. For $K, L \in \kappa^n$, and $i \in \mathbb{R}$, the $i$th mixed affine surface area of $K$ and $L$, $\Omega_i(K,L)$, was defined in[5] by

$$\Omega_i(K,L) = \int_{S^{n-1}} f(K,u)^{(n-i)/(n+1)} f(L,u)^{i/(n+1)} dS(u). \tag{20}$$

Now, we define the $i$th affine area of $K \in \kappa^n, \Omega_i(K)$, to be $\Omega_i(K,B)$, since $f(B,\cdot) = 1$ one has

$$\Omega_i(K) = \int_{S^{n-1}} f(K,u)^{(n-i)/(n+1)} dS(u), \quad i \in \mathbb{R}. \tag{21}$$

Lutwak[8] defined mixed bodies of convex bodies $K_1, \ldots, K_{n-1}$ as $[K_1, \ldots, K_{n-1}]$. The following property will be used later:

$$[K_1 + K_2, K_3, \ldots, K_n] = [K_1, K_3, \ldots, K_n] \tilde{+} [K_2, K_3, \ldots, K_n] \tag{22}$$

*2.5 Mixed projection bodies and their polars*

If $K$ is a convex that contains the origin in its interior, we define the polar body of $K$, $K^*$ ,by

$$K^* := \{x \in \mathbb{R}^n | x \cdot y \le 1, y \in K\}. \tag{23}$$

If $K_i(i = 1, 2, \ldots, n-1) \in K^n$, then the mixed projection body of $K_i(i = 1, 2, \ldots, n-1)$ is denoted by $\Pi(K_1, \ldots, K_{n-1})$, and whose support function is given, for $u \in S^{n-1}$, by[7]

$$h\left(\Pi(K_1, \ldots, K_{n-1}), u\right) = v(K_1^u, \ldots, K_{n-1}^u). \tag{24}$$

It is easy to see, $\Pi(K_1, \ldots, K_{n-1})$ is centered.

We use $\Pi^*(K_1,\ldots,K_{n-1})$ to denote the polar body of $\Pi(K_1,\ldots,K_{n-1})$, and is called polar of mixed projection body of $K_i (i=1,2,\ldots,n-1)$. If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\Pi(K_1,\ldots,K_{n-1})$ will be written as $\Pi_i(K,L)$. If $L = B$, then $\Pi_i(K,B)$ is called the $i$th projection body of $K$ and is denoted $\Pi_i K$. We write $\Pi_0 K$ as $\Pi K$. We will simply write $\Pi_i^* K$ and $\Pi^* K$ rather than $(\Pi_i K)^*$ and $(\Pi K)^*$ ,respectively.

The following property will be used:

$$\Pi(K_3,\ldots,K_n,K_1+K_2) = \Pi(K_3,\ldots,K_n,K_1) + \Pi(K_3,\ldots,K_n,K_2) \tag{25}$$

# 3 Main results and their proofs

Our main results are The following Theorems which were stated in the introduction.

**Theorem 1** *If* $K_1,K_2,\ldots,K_n \in \mathcal{K}^n$, let $C = (K_3,\ldots,K_n)$, *then for* $i < n-1$

$$B_i(\Pi(C,K_1+K_2))^{1/(n-i)} \le B_i(\Pi(C,K_1))^{1/(n-i)} + B_i(\Pi(C,K_2)^{1/(n-i)}, \tag{26}$$

*with equality if and only if* $\Pi(C,K_1)$ *and* $\Pi(C,K_2)$ *are homothetic.*

While for $i > n$,

$$B_i(\Pi(C,K_1+K_2))^{1/(n-i)} \ge B_i(\Pi(C,K_1))^{1/(n-i)} + B_i(\Pi(C,K_2)^{1/(n-i)}, \tag{27}$$

*with equality if and only if* $\Pi(C,K_1)$ *and* $\Pi(C,K_2)$ *are homothetic.*

*Proof* Here, we only give the proof of (27).

From (12), (14),(15),(25) and notice for $i > n$ to use inverse the Minkowski inequality for integral[14,P.147], we obtain that

$$B_i(\Pi(C,K_1+K_2))^{1/(n-i)} = \left(\frac{1}{n}\int_{S^{n-1}} b(\Pi(C,K_1+K_2),u)^{n-i} dS(u)\right)^{1/(n-i)}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} b(\Pi(C,K_1)+\Pi(C,K_2),u)^{n-i} dS(u)\right)^{1/(n-i)}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} (b(\Pi(C,K_1),u)+b(\Pi(C,K_2),u))^{n-i} dS(u)\right)^{1/(n-i)}$$

$$\ge \left(\frac{1}{n}\int_{S^{n-1}} b(\Pi(C,K_1),u)^{n-i} dS(u)\right)^{1/(n-i)} +$$

$$+ \left(\frac{1}{n}\int_{S^{n-1}} b(\Pi(C,K_1),u)^{n-i} dS(u)\right)^{1/(n-i)}$$

$$= B_i(\Pi(C, K_1))^{1/(n-i)} + B_i(\Pi(C, K_2))^{1/(n-i)},$$

with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ have similar width, in view of $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are centered (centrally symmetric with respect to origin),then with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic.

The proof of inequality (27) is complete.

Taking $i = 0$ to (26), inequality (26) changes to the following result

**Corollary 1** *If* $K_1, K_2, \dots, K_n \in \mathcal{K}^n$, *let* $C = (K_3, \dots, K_n)$, *then*

(28) $$B(\Pi(C, K_1 + K_2))^{1/n} \leq B(\Pi(C, K_1))^{1/n} + B(\Pi(C, K_2)^{1/n},$$

*with equality if and only if* $\Pi(C, K_1)$ *and* $\Pi(C, K_2)$ *are homothetic.*

Taking $i = 2n$ to (27), inequality (27) changes to the following result

**Corollary 2** *If* $K_1, K_2, \dots, K_n \in \mathcal{K}^n$, *let* $C = (K_3, \dots, K_n)$, *then*

(29) $$B_{2n}(\Pi(C, K_1 + K_2))^{-1/n} \geq B_{2n}(\Pi(C, K_1))^{-1/n} + B_{2n}(\Pi(C, K_2)^{-1/n},$$

*with equality if and only if* $\Pi(C, K_1)$ *and* $\Pi(C, K_2)$ *are homothetic.*

From (16),(29) and notice that projection body is centered(centrally symmetric with respect to origin), we get

**Corollary 3** *If* $K_1, K_2, \dots, K_n \in \mathcal{K}^n$, let $C = (K_3, \dots, K_n)$, *then*

(30) $$V(\Pi^*(C, K_1 + K_2))^{-1/n} \geq V(\Pi^*(C, K_1)^{-1/n} + V(\Pi^*(C, K_2))^{-1/n}$$

*with equality if and only if* $\Pi(C, K_1)$ *and* $\Pi(C, K_2)$ *are homothetic.*

This is just Brunn-Minkowski inequality of polars of mixed projection bodies. This result first is given in here.

**Theorem 2** *If* $K_1, K_2, \dots, K_n \in \mathcal{K}^n$ *and all of mixed bodies of* $K_1, K_2, \dots, K_n$ *have positive continuous curvature functions*, *then for* $i < -1$

$$\Omega_i([K_1 + K_2, K_3, \dots, K_n])^{(n+1)/(n-i)}$$

(31) $$\leq \Omega_i([K_1, K_3, K_4 \dots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \dots, K_n])^{(n+1)/(n-i)}$$

*with equality if and only if* $[K_1, K_3, K_4, \dots, K_n]$ *and* $[K_2, K_3 \dots, K_n]$ *are homothetic.*

*While for* $i > -1$

$$\Omega_i([K_1 + K_2, K_3, \dots, K_n])^{(n+1)/(n-i)}$$

(32) $$\geq \Omega_i([K_1, K_3, K_4, \dots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \dots, K_n])^{(n+1)/(n-i)}$$

*with equality if and only if* $[K_1, K_3, K_4, \ldots, K_n]$ *and* $[K_2, K_3 \ldots, K_n]$ *are homothetic.*

*Proof* Firstly, we give the proof of (31).

From (17), (21),(22) and in view of the Minkowski inequality for integral[14,P.147], we obtain that

$$
\begin{aligned}
&\Omega_i([K_1+K_2, K_3, K_4, \ldots, K_n])^{(n+1)/(n-i)} \\
&= \left(\int_{S^{n-1}} f([K_1+K_2, K_3, K_4, \ldots, K_n], u)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\
&= \left(\int_{S^{n-1}} f([K_1, K_3, K_4, \ldots, K_n]\tilde{+}[K_2, K_3, \ldots, K_n], u)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\
&= \left(\int_{S^{n-1}} (f([K_1, K_3, K_4, \ldots, K_n], u) + f([K_2, K_3, \ldots, K_n], u))^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\
&\le \left(\int_{S^{n-1}} f([K_1, K_3, K_4, \ldots, K_n], u)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\
&\quad + \left(\int_{S^{n-1}} f([K_2, K_3, \ldots, K_n], u)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\
&= \Omega_i([K_1, K_3, K_4, \ldots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)},
\end{aligned}
$$

with equality if and only if $[K_1, K_3, K_4, \ldots, K_n]$ and $[K_2, K_3, \ldots, K_n]$ are homothetic.

Similarly, from (17),(21),(22) and in view of inverse Minkowski inequality[14,P.147], we can also prove (32).

The proof of Theorem 2 is complete.

Taking $i = 0$ to (32), we have

**Corollary 4** *If* $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ *and all of mixed bodies of* $K_1, K_2, \ldots, K_n$ *have positive continuous curvature functions, then*

$$
\begin{aligned}
&\Omega([K_1+K_2, K_3, \ldots, K_n])^{(n+1)/n} \\
&\ge \Omega([K_1, K_3, K_4, \ldots, K_n])^{(n+1)/n} + \Omega([K_2, K_3, \ldots, K_n])^{(n+1)/n} \qquad (33)
\end{aligned}
$$

*with equality if and only if* $[K_1, K_3, K_4, \ldots, K_n]$ *and* $[K_2, K_3 \ldots, K_n]$ *are homothetic.*

Taking $i = 2n$ to (32), inequality (32) changes to the following result

**Corollary 5** *If* $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ *and all of mixed bodies of* $K_1, K_2, \ldots, K_n$ *have positive continuous curvature functions, then*

$$
\begin{aligned}
&\Omega_{2n}([K_1+K_2, K_3, \ldots, K_n])^{-(n+1)/n} \\
&\ge \Omega_{2n}([K_1, K_3, K_4 \ldots, K_n])^{-(n+1)/n} + \Omega_{2n}([K_2, K_3, \ldots, K_n])^{-(n+1)/n}, \qquad (34)
\end{aligned}
$$

*with equality if and only if* $[K_1, K_3, K_4, \ldots, K_n]$ *and* $[K_2, K_3 \ldots, K_n]$ *are homothetic.*

Taking $i = -n$ to (31), we have

**Corollary 6** *If* $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ *and all of mixed bodies of* $K_1, K_2, \ldots, K_n$ *have positive continuous curvature functions, then*

$$\Omega_{-n}([K_1 + K_2, K_3, \ldots, K_n])^{(n+1)/2n}$$

$$\leq \Omega_{-n}([K_1, K_3, K_4 \ldots, K_n])^{(n+1)/2n} + \Omega_{-n}([K_2, K_3, \ldots, K_n])^{(n+1)/2n}, \tag{35}$$

*with equality if and only if* $[K_1, K_3, K_4, \ldots, K_n]$ *and* $[K_2, K_3 \ldots, K_n]$ *are homothetic.*

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Zhao Chang-jian
Department of Information and Mathematics Sciences
China Jiliang University
Hangzhou 310018, P.R.China
e-mail: chjzhao@yahoo.com.cn chjzhao@163.com

Mihály Bencze
Str. Hărmanului 6
505600 Săcele-Négyfalu
Jud. Braşov, Romania
e-mail: benczemihaly@yahoo.com aedit@metanet.ro