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SINGULAR INTEGRAL OPERATORS ON MANIFOLDS WITH A BOUNDARY

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Abstract. This paper deals with singular integral operators that are bounded, completely continuous, and Noetherian on manifolds with a boundary in weighted Hölder spaces.

We shall investigate the matrix singular operator

$$A(u)(x)=a(x)u(x)+\int_D f\big(x,\frac{x-y}{|x-y|}\big)|x-y|^{-m}u(y)dy,$$
$$x\in D,\quad D\subset\mathbb{R}^m,$$

in weighted Hölder spaces and develop the results obtained in [1] for one-dimensional singular operators and in [2–8] for multidimensional singular operators in Lebesgue spaces.

This paper consists of two sections. In Section I we shall prove the theorems of integral operators that are bounded and completely continuous in Hölder spaces with weight. Section II will contain the proof of the theorem of factorization of matrix-functions and present the theorem of singular operators that are Noetherian in weighted spaces.

1. Let $\mathbb{R}^m$ $(m\geq 2)$ be an m-dimensional Euclidean space, $x=(x_1,...,x_m)$, $y=(y_1,...,y_m)$ be points of the space $\mathbb{R}^m$,

$$|x|=\big(\sum_{i=1}^m x_i^2\big)^{\frac12},\quad \Gamma=\{x:\ x\in\mathbb{R}^m,\ x_m=0\},$$
$$\mathbb{R}^m_+=\{x:\ x\in\mathbb{R}^m,\ x_m>0\},\quad x'=(x_1,\dots,x_{m-1}),$$
$$B(x,a)=\{y:\ y\in\mathbb{R}^m,\ |y-x|<a\},$$
$$S(x,a)=\{y:\ y\in\mathbb{R}^m,\ |y-x|=a\}.$$

Definition 1. A function u defined on $\mathbb{R}^m\backslash\Gamma$ belongs to the space $H^\nu_{\alpha,\beta}(\mathbb{R}^m\backslash\Gamma)$ $(0<\nu,\ \alpha<1,\ \beta\geq 0,\ \alpha+\beta<m)$ if

$$\text{(i)}\quad \forall x\in\mathbb{R}^m\backslash\Gamma\quad |u(x)|\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta},$$

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(ii) $\forall x \in \mathbb{R}^m \backslash \Gamma, \quad \forall y \in B(x, \dfrac{1}{2}|x_m|)$

$$|u(x) - u(y)| \leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-y|^\nu.$$

The norm in the space $H^\nu_{\alpha,\beta}(\mathbb{R}^m \backslash \Gamma)$ is defined by the equality

$$\|u\| = \sup_{x \in \mathbb{R}^m \backslash \Gamma} |x|^\alpha (1+|x|)^\beta |u(x)| +$$

$$+ \sup_{\substack{x \in \mathbb{R}^m \backslash \Gamma \\ y \in B(x, \frac{1}{2}|x_m|)}} |x_m|^{\alpha+\nu}(1+|x|)^\beta \frac{|u(x)-u(y)|}{|x-y|^\nu}.$$

The space $H^\nu_{\alpha,\beta}(\mathbb{R}^m_+)$ is defined similarly.

Note that if $y \in B(x, \frac{1}{2}|x_m|)$, then

$$|y| \leq \frac{3}{2}|x|, \quad |y| \geq \frac{1}{2}|x|; \quad |y_m| \leq \frac{3}{2}|x_m|, \quad |y_m| \geq \frac{1}{2}x_m|. \tag{1}$$

Thus for $y \in B(x, \frac{1}{2}x_m)$ we have $|x| \sim |y|$, $|x_m| \sim |y_m|$.

Let $x, y \in \mathbb{R}^m \backslash \Gamma$ and $|x-y| \geq \frac{1}{2}\min(|x_m|, |y_m|)$. Then the condition (i) implies

$$|u(x)-u(y)| \leq c(\min(|x_m|, |y_m|))^{-\alpha}(\min(1+|x|, 1+|y|))^{-\beta} \leq$$
$$\leq c|x-y|^\nu (\min(|x_m|, |y_m|)^{-\alpha-\nu}(\min(1+|x|, 1+|y|)^{-\beta};$$

and therefore the condition (ii) can be replaced by the condition

$$\forall x, y \in \mathbb{R}^m \backslash \Gamma$$
$$|u(x)-u(y)| \leq c|x-y|^\nu (\min(|x_m|, |y_m|)^{-\alpha-\nu}(\min(1+|x|, 1+|y|)^{-\beta}.$$

One can easily prove that the space $H^\nu_{\alpha,\beta}(\mathbb{R}^m \backslash \Gamma)[H^\nu_{\alpha,\beta}(\mathbb{R}^m_+)]$ is the Banach one.

Consider the singular integral operator

$$v(x) = A(u)(x) = \int_{\mathbb{R}^m} K(x, x-y)u(y)dy,$$

where $K(x,z) = f\big(x, \frac{z}{|z|}\big)|z|^{-m}$.

Theorem 1. *Let the characteristic f defined on $(\mathbb{R}^m \backslash \Gamma) \times (S(0,1) \backslash \Gamma)$ satisfy the conditions*

(a) $\forall x \in \mathbb{R}^m \backslash \Gamma \quad \displaystyle\int_{S(0,1)} f(x,z) d_z S = 0;$

(b) $\forall x \in \mathbb{R}^m \backslash \Gamma, \ \forall z \in S(0,1) \backslash \Gamma \quad |f(x,z)| \leq c|z_m|^{-\sigma} \quad (0 \leq \sigma \leq \alpha);$

(c) $\forall x, y \in \mathbb{R}^m \backslash \Gamma, \ \forall z, \theta, \omega \in S(0,1) \backslash \Gamma$

$$|f(x,z)-f(y,z)| \le c|x-y|^{\nu}(\min(|x_m|,|y_m|))^{-\nu}|z_m|^{-\sigma}$$
$$|f(x,\theta)-f(x,\omega)| \le c|\theta-\omega|^{\nu_1}(\min(|\theta_m|,|\omega_m|))^{-\nu_1-\sigma}$$
$$\nu_1>\nu, \quad \nu_1+\sigma<1.$$

Then the operator A is bounded in the space $H_{\alpha,\beta}(\mathbb{R}^m\backslash\Gamma)$.

Proof. In the first place note that the second inequality of the condition (c) yields the inequality

$$\forall x,y,z \in \mathbb{R}^m\backslash\Gamma$$
$$\Big|f\big(x,\frac{y}{|y|}\big)-f\big(x,\frac{z}{|z|}\big)\Big| \le c|y-z|^{\nu_1}\Big(\frac{|y|^{\sigma}}{|y_m|^{\nu_1+\sigma}}+\frac{|z|^{\sigma}}{|z_m|^{\nu_1+\sigma}}\Big). \tag{2}$$

Set

$$\begin{gathered} D_1=B(x,\tfrac12|x_m|), \quad D_2=B(x,\tfrac12(1+|x|))\backslash D_1, \\ D_3=B(0,\tfrac12(1+|x|))\backslash(D_1\cup D_2), \quad D_4=\mathbb{R}^m\backslash(D_1\cup D_2\cup D_3), \\ D=\{y:\ (y',y_m)\in\mathbb{R}^m,\ |y_m|\le 2(|y'|+1)\}. \end{gathered} \tag{3}$$

We have

$$v(x)=\int_{D_1}K(x,x-y)[u(y)-u(x)]dy+\sum_{i=2}^{4}\int_{D_i}K(x,x-y)u(y)dy \equiv$$
$$\equiv \sum_{i=1}^{4} I_i(x).$$

By virtue of the condition (ii) and the inequalities (1)

$$|I_1(x)| \le c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}\int_{D_1}\Big|f\big(\frac{x-y}{|x-y|}\big)\Big||x-y|^{\nu-m}dy\|u\| \le$$
$$\le c|x_m|^{-\alpha}(1+|x|)^{-\beta}\int_{S(0,1)}|f(z)|d_zS\|u\|. \tag{4}$$

If $y\notin D_1$, then $|x-y|\ge\frac16(|x'-y'|+|x_m|+|y_m|)$ and if $y\in D_2$, then $1+|y|\ge 1+|x|-|x-y|\ge\frac{1+|x|}{2}$. Therefore due to the conditions (i) and (b)

$$|I_2(x)| \le c(1+|x|)^{-\beta}\int_{D_2}\frac{|y_m|^{-\alpha}|y_m-x_m|^{-\sigma}}{(|x'-y'|+|x_m|+|y_m|)^{m-\sigma}}dy\|u\|.$$

After performing the spherical transformation of $y'-x'$, we obtain

$$|I_2(x)| \le c(1+|x|)^{-\beta}\int_0^{\infty}r^{m-2}dr\int_{-\infty}^{\infty}\frac{|y_m|^{-\alpha}|x_m-y_m|^{-\sigma}}{(r+|x_m|+|y_m|)^{m-\sigma}}dy_m\|u\|.$$

The transformation of $r = |x_m|\widetilde{r}$, $y_m = |x_m|\widetilde{y}_m$ leads to

$$|I_2(x)| \le c|x_m|^{-\alpha}(1+|x|)^{-\beta}\int_{-\infty}^{\infty}|y_m|^{-\alpha}|y_m - \operatorname{sign} x_m|^{-\sigma}dy_m \times$$
$$\times\int_0^{\infty}(r+|y_m|+1)^{\sigma-2}dr\,\|u\| \le c|x_m|^{-\alpha}(1+|x|)^{-\beta}\times$$
$$\times\int_{-\infty}^{\infty}|y_m|^{-\alpha}|y_m - \operatorname{sign} x_m|^{-\sigma}(1+|y_m|)^{\sigma-1}dy_m\|u\|. \tag{5}$$

The term $I_4(x)$ is evaluated in the same manner, since in that case, too, $1+|y| \ge \frac{1}{2}(|x|+1)$. Represent $I_3(x)$ in the form

$$I_3(x) = \int_{D_3\cap B(0,\frac{1}{2}|x_m|)}K(x,x-y)u(y)dy +$$
$$+\int_{D_3\setminus B(0,\frac{1}{2}|x_m|)}K(x,x-y)u(y)dy \equiv J_1(x)+J_2(x).$$

If $y \in D_3$, then $|x-y| \ge \frac{1}{2}(1+|x|) \ge |y|$; if $y \notin B(0,\frac{1}{2}|x_m|)$, then $|y| \ge \frac{|x_m|}{2}$ and hence $|y| \ge \frac{1}{4}(|y'|+|x_m|+|y_m|)$. Therefore, in evaluating $J_2(x)$, we shall have

$$|J_2(x)| \le c(1+|x|)^{-\beta}\times$$
$$\times\int_{D_3\setminus B(0,\frac{1}{2}|x_m|)}|y_m|^{-\alpha}|x_m-y_m|^{-\sigma}(|y'|+|x_m|+|y_m|)^{\sigma-m}dy\|u\|.$$

After performing the spherical transformation of y', we obtain, as in the case of evaluating $I_2(x)$,

$$|J_2(x)| \le c|x_m|^{-\alpha}(1+|x|)^{-\beta}\|u\|. \tag{6}$$

Write $J_1(x)$ in the form

$$J_1(x) = \int_{D_3\cap B(0,\frac{1}{2}|x_m|)\cap D}K(x,x-y)u(y)dy +$$
$$+\int_{D_3\cap B(0,\frac{1}{2}|x_m|)\cap(\mathbb{R}^m\setminus D)}K(x,x-y)u(y)dy \equiv J_1'(x)+J_1''(x).$$

If $y \in B(0, \frac{1}{2}|x_m|)$, then $|x_m - y_m| \geq |x_m| - |y_m| \geq \frac{1}{2}|x_m|$. We have

$$|J_1'(x)| \leq c(1+|x|)^{-m+\sigma}|x_m|^{-\sigma} \int_{|y'|\leq\frac{1}{2}(1+|x|)} (1+|y'|)^{-\beta} dy' \times$$
$$\times \int_{|y_m|\leq 2(|y'|+1)} |y_m|^{-\alpha} dy_m \|u\| \leq$$
$$\leq c|x_m|^{-\sigma}(1+|x|)^{\sigma-m} \int_{|y'|\leq\frac{1}{2}(1+|x|)} (1+|y'|)^{1-\beta-\alpha} dy' \|u\| \leq$$
$$\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}|x_m|^{\alpha-\sigma}(1+|x|)^{\beta+\sigma-m}\big(c+(1+|x|)^{m-(\beta+\alpha)}\big)\|u\| \leq$$
$$\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\|u\|, \tag{7}$$

since $\sigma \leq \alpha$, $\alpha+\beta < m$.

If $y \in \mathbb{R}^m \backslash D$, then $1+|y| < 1+|y'|+|y_m| < \frac{3}{2}|y_m|$. Therefore

$$|J_1''(x)| \leq c|x_m|^{-\sigma}(1+|x|)^{\sigma-m} \int_{|y|\leq\frac{1}{2}(1+|x|)} (1+|y|)^{-\beta-\alpha} dy \|u\| \leq$$
$$\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\|u\|. \tag{8}$$

From the estimates (4)–(8) we obtain

$$|v(x)| \leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\|u\|. \tag{9}$$

Let us evaluate the difference $v(x) - v(z)$. It is assumed that $|x-z| \leq \frac{1}{8}|x_m|$. Then $|z| \sim |x|$, $|z_m| \sim |x_m|$.

We introduce the set $\widetilde{D}_1 = B(x, 2|x-z|)$, $\widetilde{D}_2 = B(z, 3|x-z|)$, $\widetilde{D}_3 = B(z, \frac{1}{2}|x_m| - |x-z|)$. Clearly, $\widetilde{D}_1 \subset \widetilde{D}_2 \subset \widetilde{D}_3 \subset B(x, \frac{1}{2}|x_m|) = D_1$. We have the representation

$$v(x) - v(z) = \int_{\mathbb{R}^m} [K(x, x-y) - K(z, z-y)]u(y)dy =$$
$$= \int_{\mathbb{R}^m} [K(x, x-y) - K(x, z-y)]u(y)dy +$$
$$+ \int_{\mathbb{R}^m} [K(z, x-y) - K(z, z-y)]u(y)dy \equiv I_1(x,z) + I_2(x,z).$$

By virtue of the first inequality of the condition (c) the term $I_1(x,z)$ is evaluated exactly in the same manner as $v(x)$ and we obtain the estimate

$$|I_1(x,y)| \leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{10}$$

Rewrite the term $I_2(x,z)$ in the form

$$I_2(x,z) = \int_{D_1} [K(z, x-y) - K(z, z-y)]u(y)dy +$$

$$+\int_{\mathbb{R}^m\setminus D_1}[K(z,x-y)-K(z,z-y)]u(y)dy=$$

$$=\int_{D_1}K(z,x-y)[u(y)-u(x)]dy-\int_{D_1\setminus\widetilde{D}_3}[K(z,z-y)u(y)dy-$$

$$-\int_{\widetilde{D}_3\setminus\widetilde{D}_2}K(z,z-y)[u(y)-u(x)]dy-\int_{\widetilde{D}_2}K(z,z-y)[u(y)-u(z)]dy+$$

$$+\int_{\mathbb{R}^m\setminus D_1}[K(z,x-y)-K(z,z-y)]u(y)dy=$$

$$=\Big(\int_{\widetilde{D}_2}+\int_{D_1\setminus\widetilde{D}_3}\Big)K(z,x-y)[u(y)-u(x)]dy-$$

$$-\int_{\widetilde{D}_2}K(z,z-y)[u(y)-u(z)]dy-$$

$$-\int_{\widetilde{D}_3\setminus\widetilde{D}_2}[K(z,x-y)-K(z,z-y)][u(y)-u(x)]dy-$$

$$-\int_{D_1\setminus\widetilde{D}_3}K(z,z-y)u(y)dy+$$

$$+\int_{\mathbb{R}^m\setminus D_1}[K(z,x-y)-K(z,z-y)]u(y)dy\equiv\sum_{i=1}^{5}J_i(x,z).$$

In evaluating $J_1(x,z)$, note that $\widetilde{D}_2\subset B(x,4|x-z|)$, $B(x,\frac{1}{2}|x_m|-2|x-z|)\subset\widetilde{D}_3$. Therefore by virtue of the condition (b) and the inequalities (1) we obtain

$$|J_1(x,z)|\le\int_{B(x,4(x-z))}|K(z,x-y)||u(y)-u(x)|dy+$$

$$+\int_{D_1\setminus B(x,\frac{1}{2}|x_m|)-2|x-z|)}|K(z,x-y)||u(y)-u(x)|dy\le$$

$$\le c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|, \tag{11}$$

since $\big(\frac{1}{2}|x_m|\big)^{\nu}-\big(\frac{1}{2}|x_m|-2|z-z|\big)^{\nu}\le c|x-z|^{\nu}$. Similarly, if $y\in\widetilde{D}_2$, then $|y-z|\le 3|x-z|\le\frac{3|x_m|}{8}\le\frac{3}{7}|z_m|$. Therefore

$$|J_2(x,z)|\le c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{12}$$

It is clear that $B(x,\frac{1}{2}|x_m|)\subset B(z,\frac{1}{2}|x_m|+|x-z|)$ and hence

$$|J_4(x,z)|\le c|x_m|^{-\alpha}(1+|x|)^{-\beta}\int_{D_1\setminus\widetilde{D}_3}|K(z,z-y)|dy\|u\|\le$$

$$\le c|x_m|^{-\alpha}(1+|x|)^{-\beta}\ln\frac{|x_m|+2|x-z|}{|x_m|-2|x-z|}\|u\|\le$$

$$\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\frac{|x-z|}{|x_m|-2|x-z|}\|u\| \leq$$
$$\leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{13}$$

Note that if $y \notin \widetilde{D}_2$, then

$$|x-y| > 2|x-z|, \quad |z-y| > 3|x-z|,$$
$$|x-y| < |x-z| + |z-y| < \frac{4}{3}|z-y|, \quad |z-y| < \frac{3}{2}|x-y|.$$

Taking these inequalities into account, the inequality (2) readily implies that for $y \notin \widetilde{D}_2$

$$|K(z,x-y) - K(z,z-y)| \leq$$
$$\leq c\frac{|x-z|^{\nu_1}}{|x-y|^{m+\nu_1}}\Big(\frac{|x-y|^{\nu_1+\sigma}}{|x_m-y_m|^{\nu_1+\sigma}} + \frac{|z-y|^{\nu_1+\sigma}}{|z_m-y_m|^{\nu_1+\sigma}}\Big), \tag{14}$$

using which we obtain

$$|J_3(x,z)| \leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu_1} \times$$
$$\times \int_{\widetilde{D}_3\setminus\widetilde{D}_2} \frac{1}{|x-y|^{m+\nu_1-\nu}}\Big(\frac{|x-y|^{\nu_1+\sigma}}{|x_m-y_m|^{\nu_1+\sigma}} + \frac{|z-y|^{\nu_1+\sigma}}{|z_m-y_m|^{\nu_1+\sigma}}\Big)dy\|u\| \leq$$
$$\leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu_1} \times$$
$$\times\Big(\int_{D_1\setminus\widetilde{D}_1} \frac{1}{|x-y|^{m+\nu_1-\nu}}\Big(\frac{|x-y|}{|x_m-y_m|}\Big)^{\nu_1+\sigma}dy +$$
$$+\int_{\widetilde{D}_3\setminus\widetilde{D}_2} \frac{1}{|z-y|^{m+\nu_1-\nu}}\Big(\frac{|z-y|}{|z_m-y_m|}^{\nu_1+\sigma}\Big)dy\Big).$$

Passing to the spherical coordinates and keeping in mind that $\frac{x_m-y_m}{|x-y|}$, $\frac{z_m-y_m}{|z-y|}$ do not depend on the radius, we have

$$|J_3(x,z)| \leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu_1}\big(|x_m|^{\nu-\nu_1} + |x-z|^{\nu-\nu_1}\big)\|u\| \leq$$
$$\leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{15}$$

In deriving the estimate, we took into account that $\nu < \nu_1$, $\nu_1+\sigma < 1$. By virtue of the inequality (14)

$$|J_5(x,z)| \leq c|z-x|^{\nu_1}\Big(\int_{\mathbb{R}^m\setminus D_1} |x-y|^{\sigma-m}|x_m-y_m|^{-\sigma-\nu_1}|u(y)|dy +$$
$$+\int_{\mathbb{R}^m\setminus B(z,\frac{1}{3}|z_m|)} |z-y|^{\sigma-m}|z_m-y_m|^{-\sigma-\nu_1}|u(y)|dy\Big).$$

We evaluate the obtained integral expression by the same technique as was used to evaluate the integral expression

$$\int_{\mathbb{R}^m \setminus D_1} |K(x, x-y)||u(y)|dy$$

(see the estimates (5)–(8)) and finally obtain

$$|J_5(x,z)| \leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{16}$$

The estimates (10)–(13), (15), (16) show that

$$|v(x)-v(z)| \leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|,$$

which, with the equality (9) taken into account, proves the theorem. □

Corollary 1. *Under the conditions of Theorem* 1 *the operator*

$$\int_{\mathbb{R}^m_+} K(x, x-y)u(y)dy$$

is bounded when acting from the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m_+)$ *into the space* $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m \setminus \Gamma)$.

Definition 2. Let M be a closed set in $\mathbb{R}^m$. The set M is called an $(m-1)$-dimensional manifold without a boundary of the class $C^{1,\delta}$ $(0 \leq \delta \leq 1)$, if for each $x \in M$ there exist a positive number r_x and a neighborhood $Q(x)$ of the point x in $\mathbb{R}^m$, which is mapped by means of the orthogonal transform T_x onto the cylinder $\Omega_0 = \{\xi : \xi \in \mathbb{R}^m, |\xi'| < r_x, |\xi_m| < r_x\}$ and if the following conditions are fulfilled: $T_x(x) = 0$, the set $T_x(M \cap Q(x))$ is given by the equation $\xi_m = \varphi_x(\xi')$, $|\xi'| < r_x$; $\varphi_x \in C^{1,\delta}$ in the domain $|\xi'| < r_x$ and $\partial_{\xi_i}\varphi_x(0) = 0$, $i = 1, \dots, m-1$.

Clearly, $Q(x)$ is the cylinder to be denoted by $C(x, r_x)$.

In what follows the manifold M will be assumed compact.

We introduce the notation

$$d(x) \equiv d(x, M) = \inf_{y \in M} |x-y|, \quad M(\tau) = \{x \in \mathbb{R}^m, d(x) < \tau\}.$$

Note some properties of the function $d(x)$:

$$\begin{gathered} d(x) \leq c(1+|x|), \quad |d(x)-d(y)| \leq c|x-y|; \\ \forall x \in \mathbb{R}^m \setminus M, \forall y \in B(x, \tfrac{1}{2}d(x)) \\ d(y) \leq \frac{3}{2}d(x) \leq 3d(y), \quad 1+|x| \sim 1+|y|; \\ \forall x \in M, \forall y \in C(x, \tfrac{1}{3}r_x) \\ d(y, M) = d(y, M \cap C(x, r_x)) \quad \text{and if} \quad y = T_x^{-1}(\eta), \quad \text{then} \end{gathered} \tag{17}$$

$$d(y) \leq |\eta_m - \varphi_x(\eta')| \leq 2(1 + a_x)d(y), \tag{18}$$

where a_x is the Lipshitz constant of the function φ_x.

Definition 3. A function u defined on $\mathbb{R}^m \backslash M$ belongs to the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m \backslash M)$ $(0 < \nu, \alpha < 1,\ \beta \geq 0,\ \alpha + \beta < m)$, if:

(i) $\forall x \in \mathbb{R}^m \backslash M \quad |u(x)| \leq cd^{-\alpha}(x)(1 + |x|)^{-\beta}$,

(ii) $\forall x \in \mathbb{R}^m \backslash M,\ \forall y \in B(x, \frac{1}{2}d(x))$

$$|u(x) - u(y)| \leq cd^{-(\alpha+\nu)}(x)(1 + |x|)^{-\beta}|x - y|^{\nu}.$$

The norm in the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m \backslash M)$ is defined by the equality

$$\|u\| = \sup_{x \in \mathbb{R}^m \backslash M} d^{\alpha}(x)(1 + |x|)^{\beta}|u(x)| + \\ + \sup_{\substack{x \in \mathbb{R}^m \backslash M \\ y \in B(x, \frac{1}{2}d(x))}} d^{\nu+\alpha}(x)(1 + |x|)^{\beta}\frac{|u(x) - u(y)|}{|x - y|^{\nu}}.$$

The space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m \backslash M)$ is the Banach one.

Theorem 2. *Let $M \in C^{1,\delta}$, and let the characteristic f of the singular operator A be defined on $(\mathbb{R}^m \backslash M) \times S(0, 1)$ and satisfy the conditions:*

(a) $\forall x \in \mathbb{R}^m \backslash M,\ \forall z \in S(0, 1)$

$$|f(x, z)| \leq c, \quad \int_{S(0,1)} f(x, z)d_z S = 0;$$

(b) $\forall x, y \in \mathbb{R}^m \backslash M,\ \forall \theta, \omega \in S(0, 1)$

$$|f(x, \theta) - f(y, \theta)| \leq c|x - y|^{\nu}\big(\min(d(x), d(y))\big)^{-\nu},$$
$$|f(x, \theta) - f(x, \omega)| \leq c|\theta - \omega|^{\nu_1}, \quad \nu < \nu_1.$$

Then the operator A is bounded in the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m \backslash M)$.

Proof. Let $M \subset B(0, r_0)$ and e_1 be an infinitely differentiable function such that $e_1(x) = 1$ for $|x| \leq r_0 + 1$, $e_1(x) = 0$ for $|x| \geq r_0 + 2$. Setting $e_2 = 1 - e_1$, we have

$$v(x) = \int_{\mathbb{R}^m} K(x, x - y)e_1(y)u(y)dy + \\ + \int_{\mathbb{R}^m} K(x, x - y)e_2(y)u(y)dy \equiv v_1(x) + v_2(x).$$

Let us evaluate the integral

$$v_1(x) = \int_{\mathbb{R}^m} K(x, x - y)u_1(y)dy \quad (u_1 = e_1u).$$

Choose a constant r^* $(0 < r^* < 1)$ such that the system $\{C(\overset{i}{x}, \frac{1}{4}r^*)\}_{i=1}^l$ ($\overset{i}{x}$, $i = 1, \dots, l$, are points of the manifold M) covers the manifold M and $C(\overset{i}{x}, 4r^*)$, $i = 1, \dots, l$, are again the coordinate neighborhoods.

We introduce the sets

$$D_1 = B(x, \tfrac{1}{2}d(x)), \quad D_2 = (B(x, \tfrac{1}{4}r^*)\backslash D_1) \cap B(0, r_0 + 2)$$
$$D_3 = B(0, r_0 + 2)\backslash(D_1 \cup D_2), \quad D_4 = \{y : \ y \in \mathbb{R}^m, \ d(y) < \tfrac{1}{2}r^*\}$$

Now

$$v_1(x) = \int_{D_1} K(x, x-y)[u_1(y) - u_1(x)]dy + \\ + \sum_{i=2}^{3} \int_{D_i} K(x, x-y)u_1(y)dy \equiv \sum_{i=1}^{3} I_i(x).$$

By virtue of the inequality (17) and the condition (ii) we obtain

$$|I_1(x)| \leq cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u_1\| \leq cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{19}$$

Next,

$$|I_2(x)| \leq \int_{D_2 \cap D_4} |K(x, x-y)||u_1(y)|dy + \\ + \int_{D_2 \backslash D_4} |K(x, x-y)||u_1(y)|dy \equiv I_2'(x) + I_2''(x).$$

If $y \in D_2$, then $1 + |x| \sim 1 + |y| \sim c$. Moreover, $d(y) \leq \frac{1}{2}r^*$ for $y \in D_2 \cap D_4$ and therefore there exists i $(i = 1, \dots, e)$ such that $x, y \in C(\overset{i}{x}, \frac{3+\sqrt{2}}{2}r^*)$. Let $y = T_{\overset{i}{x}}^{-1}(\eta)$, $x = T_{\overset{i}{x}}^{-1}(\xi)$. By virtue of the inequality (18)

$$|\eta_m - \varphi_{\overset{i}{x}}(\eta')| \leq 2(1 + a_{\overset{i}{x}})d(y), \quad |\xi_m - \varphi_{\overset{i}{x}}(\xi')| \leq 2(1 + a_{\overset{i}{x}})d(x).$$

Taking into account that $|x - y| > \frac{1}{2}d(x)$, we therefore obtain

$$\begin{aligned} |x-y| = |\xi - \eta| &\geq \tfrac{1}{4}(|\xi' - \eta'| + |\xi_m - \eta_m| + d(x)) \geq \\ &\geq \frac{1}{16}(1 + a_{\overset{i}{x}})^{-1}\big(4(1 + a_{\overset{i}{x}})|\xi' - \eta'| + |\xi_m - \eta_m| + 4(1 + a_{\overset{i}{x}})d(x)\big) \geq \\ &\geq \frac{1}{16}(1 + a_{\overset{i}{x}})^{-1}\big(|\xi' - \eta'| + |\eta_m - \varphi_{\overset{i}{x}}(\eta')| + d(x)\big), \end{aligned} \tag{20}$$

since $|\xi_m - \eta_m| \geq |\eta_m - \varphi_{\overset{i}{x}}(\eta')| - |\varphi_{\overset{i}{x}}(\eta') - \varphi_{\overset{i}{x}}(\xi')| - |\xi_m - \varphi_{\overset{i}{x}}(\xi')|$. Thus

$$|I_2'(x)| \leq c \int\limits_{|\xi' - \eta'| \leq 4r^*} d\eta' \int_{-2r^*}^{2r^*} |\eta_m - \varphi_{\overset{i}{x}}(\eta')|^{-\alpha} \times$$

$$\times\big(|\xi'-\eta|+|\eta_m-\varphi_{\underset{x}{i}}(\eta')|+d(x)\big)^{-m}d\eta_m\|u\|.$$

Using the transform $\eta_m-\varphi_{\underset{x}{i}}(\eta')=\widetilde{\eta}_m$, we obtain

$$|I_2'(x)|\le c\int_0^{4r^*}r^{m-2}dr\int_{-4r^*}^{4r^*}|\eta_m|^{-\alpha}\big(r+|\eta_m|+d(x)\big)^{-m}d\eta_m\|u\|,$$

which, upon applying the transform $r=d(x)\widetilde{r}$, $\eta_m=d(x)\widetilde{\eta}_m$, gives

$$|I_2'(x)|\le cd^{-\alpha}(x)\|u\|\le cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{21}$$

If $y\in D_2\backslash D_4$, then $d(y)\ge\frac{1}{2}r^*$, $\frac{1}{2}d(x)\le|x-y|<\frac{1}{4}r^*$ by virtue of which $d(x)\ge d(y)-|x-y|\ge\frac{1}{4}r^*$, $|x-y|\ge\frac{1}{8}r^*$. Therefore

$$|I_2''(x)|\le c\|u\|\le cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{22}$$

Finally, if $y\in D_3$, then $1+|x|\le 1+|y|+|x-y|\le c|x-y|$. Therefore

$$|I_3(x)|\le c(1+|x|)^{-m}\|u\|\le cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{23}$$

The inequalities (19), (21)–(23) show that

$$|v_1(x)|\le cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{24}$$

In evaluating the difference $v_1(x)-v_1(z)$, it will be assumed that $|x-z|<\frac{1}{8}d(x)$. Then $1+|z|\sim 1+|x|$, $d(x)\sim d(y)$.

We introduce the set

$$\widetilde{D}_1=B(x,2|x-y|),\quad \widetilde{D}_2=B(z,3|x-z|),\quad \widetilde{D}_3=B(z,\tfrac{1}{2}d(x)-|x-z|).$$

Proceeding as in proof of Theorem 1, we obtain

$$|v_1(x)-v_1(z)|\le cd^{-(\alpha+\nu)}(x)(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{25}$$

To evaluate the integral

$$v_2(x)=\int_{\mathbb{R}^m}K(x,x-y)u_2(y)dy\quad(u_2=e_2u)$$

note that the function u_2 is defined on $\mathbb{R}^m$ and satisfies the conditions of Definition 3, if the function $d(x)$ is replaced by the function $1+|x|$. Therefore, after introducing the sets

$$D_1=B(x,\tfrac{1}{2}(1+|x|)),\quad D_2=B(0,2|x|+1)\backslash D_1,\quad D_3=\mathbb{R}^m\backslash(D_1\cup D_2),$$

we readily obtain the estimate

$$|v_2(x)|\le cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{26}$$

Now, considering the sets

$$\widetilde{D}_1=B(x,2|x-z|),\;\; \widetilde{D}_2=B(z,3|x-z|),\;\; \widetilde{D}_3=B(z,\tfrac{1}{2}(1+|x|)-|x-z|)$$

it is easy to show that

$$|v_2(x) - v_2(z)| \le c d^{-(\alpha+\nu)}(x)(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{27}$$

The estimates (24)–(27) prove the theorem. □

A result close to the one presented here is obtained in [9] (see also [10]).

Definition 4. A function u defined on $\mathbb{R}^m$ belongs to the space $H^{\nu}_{\lambda}(\mathbb{R}^m)$ $(\nu, \lambda > 0)$, if

$$|u(x)| \le c(1+|x|)^{-\beta}, \quad |u(x) - u(y)| \le c|x-y|^{\nu} \rho_{xy}^{-\nu-\lambda},$$

where $\rho_{xy} = \min(1+|x|, 1+|y|)$.

Theorem 3. *Let the characteristic f of the singular operator A satisfy the conditions of Theorem 1, assuming that $\sigma < \alpha$ and the first inequality of the condition* (c) *is fulfilled in the strong form*

$$|f(x,z) - f(y,z)| \le c|x-z|^{\nu_1}(\min(|x_m|, |y_m|))^{\nu_1}|z_m|^{-\sigma}.$$

It is also assumed that $a \in C(\dot{\mathbb{R}}^m)$ $(\dot{\mathbb{R}}^m = \mathbb{R}^m \cup \infty)$ and $(a - a(\infty)) \in H^{\nu_1}_{\delta}(\mathbb{R}^m)$. Then the integral operator

$$v(x) = B(u)(x) = \int_{\mathbb{R}^m} [a(x) - a(y)] K(x, x-y) u(y) dy$$

is completely continuous in the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m \backslash \Gamma)$.

Proof. From the proof of Theorem 1 it follows that B is the bounded operator from the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m \backslash \Gamma)$ into the space $H^{\nu+\gamma}_{\alpha-\gamma,\beta+2\gamma}(\mathbb{R}^m \backslash \Gamma)$, where γ is an arbitrary positive number satisfying the condition

$$\gamma < \min\{\lambda, \nu_1 - \nu, \alpha - \sigma, \tfrac{1}{2}(m - \beta - \alpha)\}.$$

Indeed, it is clear that

$$|v(x)| \le \int_{D_1 \cup D_2 \cup D_4} |a(x) - a(y)||K(x, x-y)||u(y)|dy +$$
$$+ \int_{D_3} (|a(x) - a(\infty)| + |a(y) - a(\infty)|)|K(x, x-y)||u(y)|dy.$$

Taking into account that $(a - a(\infty)) \in H^{\gamma}_{\gamma}(\mathbb{R}^m)$ and repeating the proof of Theorem 1, we obtain

$$|v(x)| \le c|x_m|^{\gamma-\alpha}(1+|x|)^{-\beta-2\gamma}\|u\|. \tag{28}$$

Let us now assume that $|x-z| \le \frac{1}{8}|x_m|$ and evaluate the difference $v(x)-v(z)$. We have

$$|v(x)v(y)| \le \int_{\widetilde{D}_2} |a(x)-a(y)||K(x,x-y)||u(y)|dy +$$
$$+\int_{\widetilde{D}_2} |a(z)-a(y)||K(z,z-y)||u(y)|dy +$$
$$+\int_{\mathbb{R}^m\setminus\widetilde{D}_1} |a(x)-a(z)||K(x,x-y)||u(y)|dy +$$
$$+\int_{\mathbb{R}^m\setminus\widetilde{D}_2} |a(z)-a(y)||K(x,x-y)-K(z,x-y)||u(y)|dy \equiv \sum_{i=1}^{4} I_i(x,z).$$

Hence

$$|I_i(x,z)| \le c|x_m|^{-\alpha}(1+|x|)^{-\beta}(1+|x|)^{-\nu_1-\lambda}|x-z|^{\nu_1}\|u\| \le$$
$$\le c|x_m|^{-\alpha-\gamma}(1+|x|)^{-\beta-2\gamma}\|u\| \quad (i=1,2). \tag{29}$$

Write the term I_3 in the form

$$I_3(x,z) = |a(x)-a(z)|\Big(\int_{D_1\setminus\widetilde{D}_1} |K(x,x-y)||u(y)|dy +$$
$$+\int_{\mathbb{R}^m\setminus\widetilde{D}_1} |K(x,x-y)||u(y)|dy\Big).$$

This representation gives

$$|I_3(x,z)| \le c|x-z|^{\nu_1}(1+|x|)^{-\nu_1-\lambda}|x_m|^{-\alpha}(1+|x|)^{-\beta} \times$$
$$\times\Big(\ln\frac{|x_m|}{|x-z|}+c_1\Big)\|u\| \le c|x_m|^{-\alpha-\nu}(1+|x|)^{-\beta-2\gamma}|x-z|^{\nu+\gamma}\|u\|. \tag{30}$$

To evaluate the integral term I_4 note that for $y \notin \widetilde{D}_1$ we have

$$|K(x,x-y)-K(z,z-y)| \le c\frac{|x-z|^{\nu_1}}{|x-y|^m}|x_m|^{-\nu_1}\frac{|x-y|^{\sigma}}{|x_m-y_m|^{\sigma}} +$$
$$+c\frac{|x-z|^{\nu_1}}{|x-y|^{m+\nu_1}}\Big(\frac{|x-y|^{\nu_1+\sigma}}{|x_m-y_m|^{\nu_1+\sigma}}+\frac{|z-y|^{\nu_1+\sigma}}{|z_m-y_m|^{\nu_1+\sigma}}\Big).$$

Using this estimate in the same manner as in proving Theorem 1, we obtain

$$|I_4(x,z)| \le c|x_m|^{-\alpha-\nu}(1+|x|)^{-\beta-2\gamma}|x-z|^{\nu+\gamma}\|u\|. \tag{31}$$

The estimates (28)–(31) show that the operator B is bounded from the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m\setminus\Gamma)$ into the space $H^{\nu+\gamma}_{\alpha-\gamma,\beta+2\gamma}(\mathbb{R}^m\setminus\Gamma)$. The validity of the theorem now follows from the complete continuity of the operator of the embedding of the space $H^{\nu+\gamma}_{\alpha-\gamma,\beta+2\gamma}(\mathbb{R}^m\setminus\Gamma)$ into the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m\setminus\Gamma)$. □

In a similar manner we prove

Theorem 4. *Let $m \in C^{1,\delta}$, let the characteristic f of the operator A satisfy the conditions of Theorem* 2*, the first inequality of the condition* (b) *being replaced by a stronger inequality*

$$|f(x,\theta)-f(y,\theta)| \leq c|x-y|^{\nu_1}\big(\min(d(x),d(y))\big)^{-\nu_1},$$

and let the function a satisfy the conditions of Theorem 3*. Then the operator B is completely continuous in the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m\backslash M)$.*

2. We shall consider the matrix-function $A(\xi)=\|A_{ij}(\xi)\|_{n\times n}$. Let

$$A(\lambda\xi)=A(\xi)\quad(\lambda>0),\quad A_{ij}\in C^{\infty}(\mathbb{R}^m\backslash 0),\quad \det A(\xi)\neq 0\quad(\xi\neq 0).$$

We set $A_0 = A^{-1}(0,\dots,0,-1)A(0,\dots,0,+1)$. It is assumed that λ_j $(j=1,\dots,s)$ is the eigenvalue of the matrix A_0 and r_j is its multiplicity ($\sum_{j=1}^{s} r_j = n$).

We introduce the matrices $B_r(\alpha)\equiv\|B_{\nu k}(\alpha)\|_{r\times r}$ where

$$B_{\nu k}(\alpha)=\begin{cases}0, & \nu<k;\\ 1, & \nu=k;\\ \frac{\alpha^{\nu-k}}{(\nu-k)!}, & \nu>k,\end{cases}$$

$$B(r_i;\alpha)=\operatorname{diag}\big[B_{r_{i1}}(\alpha),\dots,B_{r_{ip_i}}(\alpha)\big]\ \ (r_{i1}+\dots+r_{ip_i}=r_i).$$

By the Jordan theorem the matrix A_0 is representable in the form $A_0 = gBg^{-1}$, where $\det g\neq 0$, B is the modified Jordan form of the matrix A_0, $B=\operatorname{diag}[\lambda_1 B(r_1;1),\dots,\lambda_s B(r_s;1)]$. We introduce the notation

$$\delta'_j=\frac{1}{2\pi i}\ln\lambda_j,\quad \delta_j=\delta'_k\quad\text{for}\quad \sum_{\nu=1}^{k-1}r_\nu<j\leq\sum_{\nu=1}^{k}r_\nu,\quad j=1,\dots,n;$$

$$\alpha_{\pm}(\xi)=\frac{1}{2\pi i}\ln\frac{\xi_m\pm|\xi'|}{|\xi'|}\quad(\xi'=(\xi_1,\dots,\xi_{m-1}));$$

by $\ln z$ we denote a logarithm branch which is real on the positive semi-axis, i.e., $-\pi<\arg z\leq\pi$,

$$\Big(\frac{\xi_m\pm i|\xi'|}{|\xi'|}\Big)^{\delta}\equiv\operatorname{diag}\Big[\Big(\frac{\xi_m\pm i|\xi'|}{|\xi'|}\Big)^{\delta_1},\dots,\Big(\frac{\xi_m\pm i|\xi'|}{|\xi'|}\Big)^{\delta_n}\Big],$$

$$B_{\pm}(\xi)\equiv\operatorname{diag}\big[B(r_1;\alpha_{\pm}(\xi),\dots,B(r_s;\alpha_{\pm}(\xi))\big].$$

Theorem 5. *Let the matrix A be strongly elliptic. Then A admits the factorization $A(\xi)=cgA_-(\xi',\xi_m)D(\xi)A_+(\xi',\xi_m)g^{-1}$, where*

$$c=A(0,\dots,+1),\quad D(\xi)=B_-(\xi)(\xi_m-i|\xi'|)^{\delta}(\xi_m+i|\xi'|)^{-\delta}B_+^{-1}(\xi),$$

$$A_{\pm}(\lambda\xi)=A_{\pm}(\xi)\quad(\lambda>0),\quad \det A_{\pm}(\xi)\neq 0.$$

For $|\xi'| \neq 0$ the matrices A_+, A_+^{-1} (accordinly, A_-, A_-^{-1}) admit analytic continuations with respect to ξ_m into the upper (lower) complex half-plane and these continuations are bounded.

Moreover, for any natural number k the matrix $A_\pm$ admits the expansion

$$A_\pm(\xi', \xi_m) = I + \\ + \sum_{p=1}^{k} \sum_{q=0}^{(p+1)(2n-1)} c^{pq}\Big(\frac{\xi'}{|\xi'|}\ln^q \frac{\xi_m \pm |\xi'|}{|\xi'|}\Big(\frac{\xi_m \pm |\xi'|}{|\xi'|}\Big)^{-p} + \overset{k}{A}(\xi), \qquad (32)$$

where $c^{pq} \in C^\infty(\mathbb{R}^{m-1}\backslash 0)$, $\overset{k}{A}(\lambda\xi) = \overset{k}{A}(\xi)$ $(\lambda > 0)$, $\overset{k}{A} \in C^k(\mathbb{R}^m\backslash 0)$. Similar expansions also hold for the inverse matrices $A_\pm^{-1}$.

Let us outline a scheme for proving the theorem.

We set

$$A_*(\xi) = \Big(\frac{\xi_m - i|\xi'|}{|\xi'|}\Big)^{-\delta} B_-^{-1}(\xi) g^{-1} A^{-1}(0, \dots, 0, +1) \times \\ \times A(\xi) g B_+(\xi)\Big(\frac{\xi_m + i|\xi'|}{|\xi'|}\Big)^{\delta},$$

$$Z_+ = \{z = x_1 + ix_2, x_2 > 0\}, \quad Z_- = \{z = x_1 + ix_2,\ x_2 < 0\}.$$

Consider the homogeneous Hilbert problem: Find an analytic (in the domain $Z_+ \cup Z_-$) matrix-function $\Phi(\xi', z)$, which is left and right continuously extendable on $\mathbb{R}$, by the boundary condition

$$\Phi^-(\xi', t) = A_*(\xi', t)\Phi^+(\xi', t), \quad \lim_{\bar{Z}_+ \ni z \to \infty} \Phi(\xi', z) = I, \quad \lim_{\bar{Z}_- \ni z \to \infty} \Phi(\xi', z) = I.$$

The solution is to be sought in the form

$$\Phi(\xi', z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\varphi(\xi', z)}{t - z} dt + I.$$

To define the matrix φ we obtain the system of singular integral equations

$$\big(A_*(\xi', t_0) + I\big)\varphi(\xi', t_0) + \frac{1}{\pi i}\big(A_*(\xi', t_0) - I\big) \int_{\mathbb{R}} \frac{\varphi(\xi', t)}{t - t_0} dt = \\ = 2\big(I - A_*(\xi', t_0)\big). \qquad (33)$$

One can prove that the system (33) is unconditionally and uniquely solvable and obtain, after a rather sophisticated reasoning, the desired result.

Remark 1. The fact that partial indices of the strongly elliptic matrix A are equal to zero is proved in [5]. An expansion of the form (32) when A is a scalar function is also obtained therein.

Let D be a finite or infinite domain in $\mathbb{R}^m$ bounded by the compact manifold without a boundary M from the class C^{1,ν_1}.

Consider the matrix singular operator

$$A(u)(x)=a(x)u(x)+\int_D f\big(x,\frac{x-y}{|x-y|}\big)|x-y|^{-m}u(y)dy, \tag{34}$$
$$a(x)=\|a_{ij}(x)\|_{n\times n},\quad f(x,z)=\|f_{ij}(x,z)\|_{n\times n},$$
$$u=(u_1,\dots,u_n)$$

in the spaces $[H^{\nu}_{\alpha,\beta}(D)]^n$ $(0<\alpha,\ \nu<1,\ \nu<\nu_1,\ \beta>0,\ \alpha+\beta<m)$ and $[L_p(D,(1+|x|)^{\gamma})]^n$ $(p>1,\ -\frac{m}{p}<\gamma<\frac{m}{p'},\ p'=\frac{p}{p-1})$, $u\in L_p(D,(1+|x|^{\gamma})\leftrightarrow \int_D |u(x)^p(1+|x|)^{p\gamma}dx<\infty$.

Taking into account the character of the linear bounded operator acting in the spaces with two norms (see [11]), the proved theorems enable us to prove

Theorem 6. *Let $a\in H^{\nu_1}_{\lambda}(\bar D)$, $f(x,\cdot)\in C^{\infty}(\mathbb{R}^m\backslash 0)$, $\int_{S(0,1)} f(x,z)d_zS=0$; $\partial^p_z f(\cdot,z)\in H^{\nu_1}_{\lambda}(\bar D)$, $|p|=0,1,\dots$, if the domain D is bounded; $\lim\limits_{x\to\infty} f(x,z)\equiv f(\infty,z)$ and $\partial^p_z(f(\cdot,z)-f(\infty,z))\in H^{\nu_1}_{\lambda}(\bar D)$, if the domain is unbounded. The determinant of the symbol matrix $\Phi(A)(x,\xi)$ of the integral operator* (34) *is different from zero and either of the following two conditions is fulfilled:* (i) *$\forall x\in M$ the matrix $\Phi(A)(x,\xi)$ is strongly elliptic and Hermitian;* (ii) *$\forall x\in M$ the matrix $\Phi(A)(x,\xi)$ is strongly elliptic and odd with respect to the variable ξ.*

Then the operator A is the Noether operator both in the space $[L_p(D,(1+|x|)^{\gamma})]^n$ and in the space $[H^{\nu}_{\alpha,\beta}(D)]^n$. Any solution of the equation

$$A(u)(x)=g(x),\quad g\in[L_p(D,(1+|x|)^{\gamma})]^n\cap[H^{\nu}_{\alpha,\beta}(D)]^n \tag{35}$$

from the space $[L_p(D,(1+|x|)^{\gamma})]^n$ belongs to the space $[L_p(D,(1+|x|)^{\gamma})]^n\cap[H^{\nu}_{\alpha,\beta}(D)]^n$. For equation (35) *to be solvable it is necessary and sufficient that $(g,v)=0$, where v is an arbitrary solution of the formally conjugate equation $A'(v)=0$ from the space $[L_{p'}(D,(1+|x|)^{-\gamma})]^n$ into $[H^{\nu}_{\alpha,\beta}(D)]^n$.*

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