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ON THE BOUNDEDNESS OF CAUCHY SINGULAR OPERATOR FROM THE SPACE L_p TO L_q, $p > q \geq 1$

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Abstract. It is proved that for a Cauchy type singular operator, given by equality (1), to be bounded from the Lebesgue space $L_p(\Gamma)$ to $L_q(\Gamma)$, as $\Gamma = \cup_{n=1}^{\infty}\Gamma_n$, $\Gamma_n = \{z : |z| = r_n\}$, it is necessary and sufficient that either condition (4) or (5) be fulfilled.

1. Let Γ be a plane rectifiable Jordan curve, $L_p(\Gamma)$, $p \geq 1$, a class of functions summable to the p-th degree on Γ, and S_Γ a Cauchy singular operator

$$S_\Gamma(f)(t) = \frac{1}{\pi i}\int\limits_\Gamma \frac{f(\tau)\,d\tau}{\tau - t}, \quad f \in L_p(\Gamma), \;\; t \in \Gamma. \tag{1}$$

Numerous studies have been devoted to problems of the existence of $S_\Gamma(f)(t)$ and boundedness of the operator $S_\Gamma : f \to S_\Gamma(f)$ in the space $L_p(\Gamma)$ (see, e.g., [1–3]). The final solution of these problems is given in [4,5]. It was proved by G.David that for the operator S_Γ to be bounded in $L_p(\Gamma)$, it is necessary and sufficient that the condition

$$l(t,r) \leq Cr \tag{2}$$

be fulfilled, where $l(t,r)$ is a length of the part of Γ contained in the circle with center at $t \in \Gamma$ and radius r, and C is a constant.[1]

The present paper is devoted to the problem of boundedness of the operator S_Γ from $L_p(\Gamma)$ to $L_q(\Gamma)$, $p > q \geq 1$ (see also [6–9]).

2. Throughout the rest of this paper by $\{r_n\}_{n=1}^{\infty}$ is meant a strictly decreasing sequence of positive numbers satisfying the condition $\sum_{k=1}^{n} r_k < \infty$, and by Γ, the family of concentric circumferences on a complex plane $\Gamma_n = \{z : |z| = r_n\}$, $n = 1, 2, \ldots$.

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[1]Following [7,8], the necessity of condition (2) is shown also in [6]. In the same work its sufficiency is proved for special classes of curves.

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It has been shown in [10,11] that for the operator S_Γ to be bounded in $L_p(\Gamma)$, $p>1$, it is necessary and sufficient that the conditions

$$\sum_{k=n}^{\infty} r_k \le C r_n, \quad n=1,2,\dots, \tag{3}$$

be fulfilled, where C is an absolute constant.

We shall prove

Theorem. *Let $p>q\ge 1$ and $\sigma = pq/(p-q)$. Then the following statements are equivalent:*

(A) *operator S_Γ is bounded from $L_p(\Gamma)$ to $L_q(\Gamma)$;*

(B) $$\sum_{n=1}^{\infty}\left(\frac{\sum_{k=n}^{\infty} r_k}{r_n}\right)^{\sigma} r_n < \infty; \tag{4}$$

(C) $$\sum_{n=1}^{\infty} n^{\sigma} r_n < \infty. \tag{5}$$

Remark. A family of concentric circumferences the sum of whose lengths is finite, as a set of integration, principally, "simulates" rectifiable curves with isolated singularities. Analogy of conditions (2) and (3) also indicates this fact. Taking into account the above, we assume that the following statement (an analogue of the theorem from Subsection 2) is valid: for the operator S_Γ to be bounded from $L_p(\Gamma)$ to $L_q(\Gamma)$, where Γ is an arbitrary rectifiable curve, $p>q\ge 1$, it is necessary and sufficient that the condition

$$\int_{\Gamma} [\chi(t)]^{pq/(p-q)} |dt| < \infty$$

be fulfilled, where

$$\chi(t) = \sup_r \frac{l(t,\tau)}{r}, \quad t\in\Gamma.$$

3. In proving this theorem, use will often be made of the well-known Abel equality (see, e.g., [12], p.307)

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n} u_k\Big) v_n = \sum_{n=1}^{\infty} u_n\Big(\sum_{k=n}^{\infty} v_k\Big), \tag{6}$$

where $\{u_n\}$ and $\{v_n\}$ are sequences of positive numbers and $\sum_{k=1}^{\infty} v_k < \infty$, as well as of its particular case

$$\sum_{n=1}^{\infty} n v_n = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} v_k. \tag{7}$$

We shall also need

Lemma. *Let $p > 0$. If f is a function analytic in the circle $|z| < 1$, then for $r < R < 1$,*

$$\int\limits_{|z|=r} |f(z)|^p |dz| \le \frac{r}{R} \int\limits_{|z|=R} |f(z)|^p |dz|. \tag{8}$$

If f is a function analytic in the domain $|z| > 1$ and $f(\infty) = 0$ then for $1 < R < r$

$$\int\limits_{|z|=r} |f(z)|^p |dz| \le \Big(\frac{R}{r}\Big)^{p-1} \int\limits_{|z|=R} |f(z)|^p |dz|. \tag{9}$$

If, in addition, f belongs to the Hardy class H_p in the domains $|z| < 1$ or $|z| > 1$, i.e., $\sup_\rho \int_0^{2\pi} |f(\rho e^{i\vartheta})|^p d\vartheta < \infty$ (in particular, if f is represented by a Cauchy type integral), then we can take $R = 1$ in inequalities (8) *and* (9).

Proof. Since $|dz| = |d\rho e^{i\vartheta}| = \rho d\vartheta$, inequality (8) follows from the fact that the mean value $\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\vartheta})|^p d\vartheta$ of $|f(\rho e^{i\vartheta})|^p$ is a nondecreasing function of ρ (see, e.g., [13], p.9).

Under the conditions of the lemma, if $|z| > 1$, then the function $g(\zeta) = \frac{1}{\zeta} f(\frac{1}{\zeta})$ is analytic in the circle $|\zeta| < 1$. Using inequality (8) for g, we get

$$\int\limits_{|\zeta|=\frac{1}{r}} \Big|f\Big(\frac{1}{\zeta}\Big)\Big|^p |d\zeta| \le \Big(\frac{R}{r}\Big)^{p+1} \int\limits_{|\zeta|=\frac{1}{R}} \Big|f\Big(\frac{1}{\zeta}\Big)\Big|^p |d\zeta|.$$

Applying the transformation of $\zeta = \frac{1}{z}$, the latter inequality reduces to (9).

If $f \in H_p$, then by the Riesz theorem

$$\lim_{\rho\to 1} \int\limits_0^{2\pi} |f(\rho e^{i\vartheta})|^p d\vartheta = \int\limits_0^{2\pi} |f(e^{i\vartheta})|^p d\vartheta$$

(see, e.g., [13], p.21), which enables us to suppose that $R = 1$. ■

4. Let us prove the equivalence of conditions (B) and (C). This follows from equality (7) for $\sigma = 1$ and therefore we shall assume that $\sigma > 1$.

(C) follows from (B). We use Abel–Dini's theorem (see, e.g., [12], p. 292): if a series with positive terms $\sum_{n=1}^\infty a_n$ diverges and S_n means its n-th partial sum, then the series $\sum_{n=1}^\infty \frac{a_n}{S_n}$ also diverges, while the series $\sum_{n=1}^\infty \frac{a_n}{S_n^{1+\varepsilon}}$ ($\varepsilon > 0$) converges. Assume that the series $\sum_{n=1}^\infty n^\sigma r_n$ diverges. Then, setting $a_n = n^\sigma r_n$ and $\omega_n = 1\big/\sum_{k=1}^n k^\sigma r_k$, we shall see by this theorem that the series $\sum_{n=1}^\infty \omega_n n^\sigma r_n$ diverges while the series $\sum_{n=1}^\infty \omega_n^{\sigma'} r_n n^\sigma$ converges, where $\sigma' = \frac{\sigma}{\sigma-1} > 1$.

Using equality (6) and the Hölder inequality, we obtain

$$\sum_{n=1}^{\infty}\omega_n n^{\sigma} r_n \le 2\sum_{n=1}^{\infty}\omega_n\Big(\sum_{k=1}^{n}k^{\sigma-1}\Big)r_n \le 2\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\omega_k k^{\sigma-1}\Big)r_n =$$
$$= 2\sum_{n=1}^{\infty}\omega_n n^{\sigma-1}\Big(\frac{\sum_{k=n}^{\infty}r_k}{r_n}\Big)r_n \le$$
$$\le 2\left[\sum_{n=1}^{\infty}\Big(\frac{\sum_{k=n}^{\infty}r_k}{r_n}\Big)^{\sigma}r_n\right]^{1/\sigma}\Big(\sum_{n=1}^{\infty}\omega_n^{\sigma'}n^{\sigma}r_n\Big)^{1/\sigma'} < \infty.$$

The obtained contradiction shows that (C) follows from (B).

Let us now show that (B) follows from (C). If $m \le n$, then

$$A_m = \frac{\sum_{k=m}^{\infty}r_k}{r_m} = \frac{r_m + r_{m+1} + \cdots + r_{n-1}}{r_m} + $$
$$+ \frac{\sum_{k=n}^{\infty}r_k}{r_m} \le (n-m) + A_n. \tag{10}$$

Let $1 \le s \le \sigma$. Using equality (6) and inequality (10), we get

$$\sum_{n=1}^{\infty}n^{s-1}A_n^{\sigma-s+1}r_n = \sum_{n=1}^{\infty}n^{s-1}A_n^{\sigma-s}\sum_{k=n}^{\infty}r_k =$$
$$= \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}k^{s-1}A_k^{\sigma-s}\Big)r_n \le \sum_{n=1}^{\infty}\Big(k^{s-1}\big[A_n+(n-k)\big]^{\sigma-s}r_n\Big) \le$$
$$\le 2^{\sigma}\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}k^{s-1}\big[A_n^{\sigma-s}+(n-k)^{\sigma-s}\big]\Big)r_n \le$$
$$\le 2^{\sigma}\sum_{n=1}^{\infty}A_n^{\sigma-s}\Big(\sum_{k=1}^{n}k^{s-1}\Big)r_n + 2^{\sigma}\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}k^{s-1}(n-k)^{\sigma-s}\Big)r_n \le$$
$$\le 2^{\sigma}\sum_{n=1}^{\infty}A_n^{\sigma-s}n^s r_n + 2^{\sigma}\sum_{n=1}^{\infty}n^{\sigma}r_n. \tag{11}$$

Let $[\sigma]$ be the integer part of σ and $\alpha = \sigma - [\sigma]$. Using inequality (11) successively $[\sigma]$ times for $s = 1, 2, \dots, [\sigma]$, we arrive at the inequality

$$\sum_{n=1}^{\infty}A_n^{\sigma}r_n \le C_1\sum_{n=1}^{\infty}A_n^{\alpha}n^{[\sigma]}r_n + C_2, \tag{12}$$

where the constants C_1 and C_2 depend on σ only.

If σ is an integer, then $\alpha = 0$, and consequently the proof is completed. Let $\alpha > 0$. Then making use of the Hölder inequality and equality (7), we obtain

$$\sum_{n=1}^{\infty} A_n^{\alpha} n^{[\sigma]} r_n = \sum_{n=1}^{\infty} A_n^{\alpha} n^{\alpha(\sigma-1)} n^{\sigma(1-\alpha)} r_n \le$$
$$\le \Big(\sum_{n=1}^{\infty} A_n n^{\sigma-1} r_n\Big)^{\alpha} \Big(\sum_{n=1}^{\infty} n^{\sigma} r_n\Big)^{1-\alpha} =$$
$$= \Big(\sum_{n=1}^{\infty} n^{\sigma} r_n\Big)^{1-\alpha} \Big(\sum_{n=1}^{\infty} n^{\sigma-1} \sum_{k=n}^{\infty} r_k\Big)^{\alpha} \le$$
$$\le \Big(\sum_{n=1}^{\infty} n^{\sigma} r_n\Big)^{1-\alpha} \Big(\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} k^{\sigma-1} r_k\Big)^{\alpha} =$$
$$= \Big(\sum_{n=1}^{\infty} n^{\sigma} r_n\Big)^{1-\alpha} \Big(\sum_{n=1}^{\infty} n(n^{\sigma-1} r_n)\Big)^{\alpha} = \sum_{n=1}^{\infty} n^{\sigma} r_n < \infty,$$

which completes the proof.

5. Let us show that (A) follows from (B) or (C). Consider first the case when $q = 1$ and show that if $p > 1$ and $\sigma = p' = p/(p-1)$, then S_Γ is bounded from $S_p(\Gamma)$ to $L_1(\Gamma)$.

Let ϕ_n^i and ϕ_n^l be the functions determined respectively in $\operatorname{Int}\Gamma_n$ and $\operatorname{Ext}\Gamma_n$ by the Cauchy type integral

$$\frac{1}{2\pi i} \int\limits_{\Gamma_n} \frac{\varphi_n(t)\,dt}{t-z}, \quad \varphi_n \in L_p(\Gamma_n), \;\; p \ge 1, \;\; z \notin \Gamma_n. \tag{13}$$

Using the Sokhotsky–Plemelj formula

$$\phi_n^i(t) - \phi_n^e(t) = \varphi_n(t), \quad \phi_n^i(t) + \phi_n^e(t) = S_\Gamma(\varphi_n)(t)$$

and the Cauchy formula

$$\frac{1}{2\pi i} \int\limits_{\Gamma_n} \frac{\varphi_n(t)\,dt}{t-z} = \frac{1}{2\pi i} \int\limits_{\Gamma_n} \frac{\phi_n^i(t) - \phi_n^e(t)}{t-z}\,dt = \begin{cases} \phi_n^i(z), & z \in \operatorname{Int}\Gamma_n, \\ \phi_n^e(z), & z \in \operatorname{Ext}\Gamma_n, \end{cases}$$

we obtain by direct calculations

$$S_\Gamma(\varphi)(t) = 2\sum_{k=1}^{n-1} \phi_k^i(t) + [\phi_n^i(t) + \phi_n^e(t)] + 2\sum_{k=n+1}^{\infty} \phi_k^e(t) \tag{14}$$

for $t \in \Gamma_n$.

Let us evaluate the integrals of the sums

$$S_1(t)=2\sum_{k=1}^{n-1}\phi_k^i(t)+\phi_n^i(t),\quad S_2(t)=\phi_n^e(t)+2\sum_{k=n+1}^{\infty}\phi_k^e(t).$$

Using the lemma from Subsection 3 and the Hölder inequality, we can write

$$\int\limits_{\Gamma_n}|S_1(t)|\,ds\le 2\sum_{k=1}^{n}\int\limits_{\Gamma_n}|\phi_k^i(t)|\,ds\le 2\sum_{k=1}^{n}\frac{r_n}{r_k}\int\limits_{\Gamma_k}|\phi_k^i(t)|\,ds\le$$
$$\le 2(2\pi)^{1/p'}\sum_{k=1}^{n}\frac{r_k}{r_k^{1/p}}\Big(\int\limits_{\Gamma_k}|\phi_k^i(t)|^p\,ds\Big)^{1/p},$$

where ϕ_k^i is a limiting function of the Cauchy type integral (13) on Γ_k, $k=1,2,\dots,n$.

Next, changing the order of summation and using the Riesz's inequality for the Cauchy singular operator in the case of the circle as well as the Hölder inequality, we get

$$\int\limits_{\Gamma}|S_1(t)|\,ds=\sum_{n=1}^{\infty}\int\limits_{\Gamma_n}|S_1(t)|\,ds\le$$
$$\le 2(2\pi)^{1/p'}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{r_n}{r_k^{1/p}}\Big(\int\limits_{\Gamma_k}|\phi_k^i(t)|^p ds\Big)^{1/p}=$$
$$=2(2\pi)^{1/p'}\sum_{n=1}^{\infty}\frac{\sum_{k=n}^{\infty}r_k}{r_n^{1/p}}\Big(\int\limits_{\Gamma_n}|\phi_n^i(t)|^p ds\Big)^{1/p}\le$$
$$\le 2(2\pi)^{1/p'}C_p\sum_{n=1}^{\infty}\frac{\sum_{k=n}^{\infty}r_k}{r_n^{1/p}}\Big(\int\limits_{\Gamma_n}|\varphi_n^i(t)|^p ds\Big)^{1/p}\le$$
$$\le 2(2\pi)^{1/p'}C_p\left[\sum_{n=1}^{\infty}\Big(\frac{\sum_{k=n}^{\infty}r_k}{r_n}\Big)^{p'}r_n\right]^{1/p'}\Big(\int\limits_{\Gamma}|\varphi(t)|^p\,ds\Big)^{1/p},\qquad(15)$$

where C_p is the constant from the Riesz inequality (which depends on p only).

The integral of $S_2(t)$ can be evaluated analogously. Using inequality (9), as well as the Hölder and Riesz inequalities, we obtain

$$\int\limits_{\Gamma_n}|S_2(t)|dt\le 2\sum_{k=n}^{\infty}\int\limits_{\Gamma_n}|\phi_k^e(t)|ds\le 2(2\pi)^{1/p'}\sum_{k=n}^{\infty}\Big(\int\limits_{\Gamma_n}|\phi_k^e(t)|^p ds\Big)^{1/p}r_n^{1/p'}\le$$

$$\leq 2(2\pi)^{1/p'} \sum_{k=n}^{\infty} \Big(\frac{r_k}{r_n}\Big)^{\frac{p-1}{p}} \Big(\int_{\Gamma_k} |\phi_k^e(t)|^p\,ds\Big)^{1/p} r_n^{1/p'} \leq$$

$$\leq 2(2\pi)^{1/p'} C_p \sum_{k=n}^{\infty} r_k^{1/p'} \Big(\int_{\Gamma_k} |\varphi_k(t)|^p\,ds\Big)^{1/p}.$$

Next, changing the order of summation and using the Hölder inequality, we can write

$$\int_{\Gamma} |S_2(t)|\,dt = \sum_{n=1}^{\infty} \int_{\Gamma_n} |S_2(t)|\,ds \leq 2(2\pi)^{1/p'} C_p \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} r_k^{1/p'} \times$$

$$\times \Big(\int_{\Gamma_k} |\varphi_k(t)|^p\,ds\Big)^{1/p} = 2(2\pi)^{1/p'} C_p \sum_{n=1}^{\infty} n r_n^{1/p'} \Big(\int_{\Gamma_n} |\varphi_n(t)|^p\,ds\Big)^{1/p} \leq$$

$$\leq 2(2\pi)^{1/p'} C_p \Big(\sum_{n=1}^{\infty} n^{p'} r_n\Big)^{1/p'} \Big[\sum_{n=1}^{\infty} \Big(\int_{\Gamma_n} |\varphi_n(t)|^p\,ds\Big)^{\frac{1}{p}\cdot p}\Big]^{\frac{1}{p}} =$$

$$= 2(2\pi)^{1/p'} C_p \Big(\sum_{n=1}^{\infty} n^{p'} r_n\Big)^{1/p'} \Big(\int_{\Gamma} |\varphi(t)|^p\,ds\Big)^{1/p}. \tag{16}$$

It follows from (14),(15) and (16) that if conditions (B) and (C) are fulfilled for $\sigma = p'$, then the operator S_Γ is bounded from $L_p(\Gamma)$ to $L_1(\Gamma)$.

Let us now consider the general case. Let conditions (B) and (C) be fulfilled for $p > q \geq 1$ and $\sigma = pq/(p-q)$. Then, by virtue of the above arguments, S_Γ is continuous from $L_{\sigma'}(\Gamma)$, $\sigma' = \frac{\sigma}{\sigma-1}$, to $L_1(\Gamma)$. But then S_Γ is also continuous from $L_\infty(\Gamma)$ to $L_\sigma(\Gamma)$ ($L_\infty(\Gamma)$ is a class of functions essentially bounded on Γ). This statement can be proved by the well-known method using the Riesz equality

$$\int_{\Gamma} \varphi S_\Gamma \psi\,dt = -\int_{\Gamma} \psi S_\Gamma \varphi\,dt, \quad \varphi \in L_{\sigma'}(\Gamma), \;\; \psi \in L_\infty(\Gamma),$$

whose validity in our case can be immediately verified.

Further, since S_Γ is bounded from $L_{\sigma'}(\Gamma)$ and $L_\infty(\Gamma)$ to $L_1(\Gamma)$ and $L_\sigma(\Gamma)$, respectively, according to Riesz–Torin's theorem on interpolation of linear operators (see, e.g., [14], p.144), it follows that S_Γ is bounded from $L_\alpha(\Gamma)$, $\sigma' \leq \alpha \leq \infty$, to $L_{\alpha\sigma/(\alpha+\sigma)}(\Gamma)$. Letting $\alpha = p$, we get that S_Γ is bounded from $L_p(\Gamma)$ to $L_q(\Gamma)$.

6. Let us now show that (C) and consequently (B) follow from (A). Let for a pair p and q, $p > q \geq 1$, $\sigma = pq/(p-q)$, the series $\sum_{n=1}^{\infty} n^\sigma r_n$

diverge. Then, according to the above-mentioned Abel–Dini's theorem, if $\omega_n = (\sum_{k=1}^{n} k^\sigma r_k)^{-1/q}$, then

$$\sum_{n=1}^{\infty} \omega_n^p n^\sigma r_n = \sum_{n=1}^{\infty} \frac{n^\sigma r_n}{S_n^{p/q}} < \infty, \quad S_n = \sum_{k=1}^{n} k^\sigma r_k,$$

$$\sum_{n=1}^{\infty} \omega_n^q n^\sigma r_n = \sum_{n=1}^{\infty} \frac{n^\sigma r_n}{S_n} = \infty.$$

Consider, on Γ, the function $\varphi(t) = \omega_n n^{\sigma/p}$ for $t \in \Gamma_n$, $n = 1, 2, \dots$. Then

$$\int_\Gamma |\varphi(t)|^p \, |dt| = \sum_{n=1}^{\infty} \int_{\Gamma_n} |\varphi(t)|^p \, ds = 2\pi \sum_{n=1}^{\infty} \omega_n^p n^\sigma r_n < \infty. \tag{17}$$

Next, by equality (14) we have

$$S_\Gamma(\varphi)(t) = 2\sum_{k=1}^{n-1} \omega_k k^{\sigma/p} + \omega_n n^{\sigma/p} > \sum_{k=1}^{n} \omega_k k^{\sigma/p}$$

for $t \in \Gamma_n$. Consequently,

$$\int_\Gamma |S_\Gamma(\varphi)(t)|^q \, |dt| = \sum_{n=1}^{\infty} \int_{\Gamma_n} |S_\Gamma(\varphi)(t)|^q \, |dt| > 2\pi \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \omega_k k^{\sigma/p} \Big)^q r_n >$$

$$> 2\pi \sum_{n=1}^{\infty} \omega_n^q \Big(\sum_{k=1}^{n} k^{\sigma/p} \Big)^q r_n \geq 2\pi \sum_{n=1}^{\infty} \omega_n^q n^{(\frac{\sigma}{p}+1)q} r_n =$$

$$= 2\pi \sum_{n=1}^{\infty} \omega_n^q n^\sigma r_n = \infty. \tag{18}$$

It follows from (17) and (18) that if condition (C) is not fulfilled for $p > q \geq 1$, then there exists a function $\varphi \in L_p(\Gamma)$ for which $S_\Gamma(\varphi) \notin L_q(\Gamma)$. Consequently, for condition (A) to be fulfilled, it is necessary that condition (C) (and hence (B)) be fulfilled.

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