Georgian Mathematical Journal
1(1994), No. 5, 537-559

FRACTIONAL TYPE OPERATORS IN WEIGHTED GENERALIZED HÖLDER SPACES

S.G. SAMKO AND Z.U. MUSSALAEVA

Abstract. Weighted Zygmund type estimates are obtained for the continuity modulus of some convolution type integrals. In the case of fractional integrals this is strengthened to a result on isomorphism between certain weighted generalized Hölder type spaces.

1. Introduction

A great number of results is known concerning boundedness of convolution type operators in spaces of summable functions, including the weighted case. In the spaces of continuous functions such as $H_0^\omega(\rho)$ the convolution type operators are less investigated. The goal of this paper is to fill a gap to a certain extent in investigations of such a kind.

We consider here the Volterra convolution type operators

$$K\varphi = \int_a^x k(x-t)\varphi(t)\,dt, \quad a < x < b, \tag{1.1}$$

in the weighted generalized Hölder spaces $H_0^\omega(\rho)$ (see definitions in Sec.2), $-\infty < a < b < \infty$. The kernel $k(x)$ is assumed to be close in a sense to a power function.

The result of the type

$$K : H_0^\omega(\rho) \to H_0^{\omega_1}(\rho) \tag{1.2}$$

for certain characteristic functions $\omega(h)$ and $\omega_1(h)$ was earlier known in the case of the power kernel $k(x) = x^{\alpha-1}$, $0 < \alpha < 1$, and a power weight function $\rho(t)$. We deal here with arbitrary kernels and weights, i.e. not necessarily power ones.

We introduce a certain class V_λ of kernels and the class w_μ of weight functions for which we manage to give the weighted Zygmund type estimate, that is, to estimate the modulus of continuity $\omega(\rho K_\varphi, h)$ by the modulus of

1072-947X/1994/0900-0537$07.00/0

continuity $\omega(\rho\varphi, h)$. This estimate provides the general result of the type (1.2).

In the case of purely power kernel, i.e. in the case of the fractional integration operator

$$I_{a+}^{\alpha}\varphi = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \quad 0 < \alpha < 1, \tag{1.3}$$

the result (1.2) is extended to isomorphism:

$$I_{a+}^{\alpha}[H_0^{\omega}(\rho)] = H_0^{\omega_\alpha}(\rho) \tag{1.4}$$

with $\omega_\alpha(h) = h^\alpha \omega(h)$. This is achieved by the preliminary derivation of Zygmund type estimate for fractional differentiation. The latter is treated in a difference form due to A.Marchaud [9] and G.H.Hardy and J.E.Littlewood [2]: (See [17], Sec.13, in this connection.)

$$D_{a+}^{\alpha} f(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}} dt, \quad 0 < \alpha < 1. \tag{1.5}$$

The paper is organized as follows. In Sec.2 we give necessary preliminaries. Sec.3 contains Zygmund type estimates for the operator (1.2) in the case of kernels in V_λ in the non-weighted case first (Theorem 1) and afterwards in the weighted case (Theorem 2). In Theorem 3 we give conditions of Zygmund–Bari–Stechkin type on a characteristic function $\omega(h)$ guaranteeing the result (1.2) for $k(x) \in V_\lambda$ and weighted functions in w_μ. The characteristic function $\omega_1(h)$ in (1.2) proves to be equal to $hk(h)\omega(h)$. We note corollary of Theorem 3 for $k(x) = x^{\alpha-1}(\ln \frac{\gamma}{x})^\beta$, $\gamma > b-a$.

In Sec.4 we establish the weighted Zygmund type estimate for $D_{a+}^{\alpha} f$ with a weight function in w_μ (Theorem 4). We prove the assertion D_{a+}^{α} : $H_0^{\omega}(\rho) \to H_0^{\omega_{-\alpha}}(\rho)$ with $\omega_{-\alpha}(h) = h^{-\alpha}\omega(h)$ under appropriate assumptions on $\omega(h)$ and $\rho(x)$ (Theorem 5).

As a corollary of Theorems 3 and 5 we give conditions for validity of the second index law of E.R.Love within the framework of the spaces $H_0^\lambda(x^\mu)$

Finally, in Sec.5 we prove the isomorphism (1.4) (Theorem 6).

Presented theorems generalize the results of the papers [10]–[12],[18], where the power case for both $k(x) = x^{\alpha-1}$ and $\rho(x) = (x-a)^\alpha$ was considered. The presentation of the results of [10] in the non-weighted case can be also found in [18], Sec.13. Note that in [12] the case $\rho(x) = (x-a)^\mu(b-x)^v$ was also considered, not contained in the results of the present paper. The origin of the statement (1.4) is the classical result by G.H.Hardy and J.E.Littlewood [2] for the fractional integration concerning the case

$\omega(h) = h^\lambda$, $\rho(x) \equiv 1$, $\alpha + \lambda < 1$. (As for the case $\omega(h) = \prod_{k=1}^n |x - x_k|^{\mu_k}$, see [13] and [17]. Sec.13.)

We also note the papers [4], [5] where Zygmund type estimates are given for the fractional integrodifferentiation in the case of L_p-moduli of continuity.

The question we finally note as open is whether $I^\alpha_{a+}[H^\omega_0(\rho)] = H^\omega_0(\rho)$ in the case of purely imaginary α, under the appropriate assumptions on $\omega(h)$ and $\rho(x)$. We refer to the paper [7] by E.R.Love concerning such fractional integrals (see also [17], Sec.2, $n^0 4$).

2. Preliminaries

We follow the papers [14], [15] in the definitions below.

Definition 1. We say that $\psi(x) \in W_\mu = W_\mu([0,l])$ if $\psi(x) \in C([0,l])$, $\psi(0) = 0$, $\psi(x) > 0$ for $x > 0$, $\psi(x)$ is almost increasing, while $\psi(x)/x^\mu$ is almost decreasing and there exists a constant $c > 0$ such that

$$\Big|\frac{\psi(x) - \psi(y)}{x - y}\Big| \le c\frac{\psi(x^*)}{x^*}, \quad x^* = \max(x, y). \tag{2.1}$$

We remind that a non-negative function $\psi(x)$, $0 \le x \le l$, $0 < l \le \infty$, is called almost increasing (decreasing) if $\psi(x) \le c\psi(y)$ for all $x \le y$ ($x \ge y$, resp.), this notion being due to S.Bernstein.

Definition 2. We say that $\psi(x) \in W^*_\mu$ if $\psi(x) \in W_\mu$ and $\psi(x)/x^{\mu-\varepsilon}$ is almost increasing for all $\varepsilon > 0$.

We shall also need the following modification of the Definition 2.

Definition 3. We say that a non-negative function $k(x)$ on $[0,l]$ belongs to the class V_λ, $\lambda > 0$ if

i) $k(x) \not\equiv 0$, $x^\lambda k(x)$ is almost increasing and $x^\lambda k(x)\big|_{x=0} = 0$;

ii) there exists ε, $0 < \varepsilon < \lambda$, such that $x^{\lambda-\varepsilon}k(x)$ is almost decreasing;

iii) there exists $c > 0$ such that

$$\Big|\frac{k(x) - k(y)}{x - y}\Big| \le c\frac{k(x^*)}{x^*}, \quad x^* = \max(x, y). \tag{2.2}$$

Remark 1. $x^\lambda k(x) \in W^*_\lambda \Rightarrow k(x) \in V_\lambda$ and $k(x) \in V_\lambda \Rightarrow k(x) \in W_\lambda$.

Remark 2. If the almost monotonicity in Definitions 1 and 3 is replaced by the usual monotonicity, then conditions (2.1) and (2.2) are satisfied automatically.

Indeed, let us prove e.g. (2.1), following [14]. If $\varphi(x)/x^\mu$ is decreasing, so that $1 - \varphi(x)/\varphi(y) \le 1 - x^\mu/y^\mu$ for $y \ge x$, then $\varphi(y) - \varphi(x) \le \frac{y^\mu - x^\mu}{y^\mu}\varphi(y)$. Since $y^\mu - x^\mu \le c(y - x)y^{\mu-1}$, we obtain (2.1).

Definition 4 ([1]). A non-negative function $\varphi(t)$ on $[0,l]$ belongs to Zygmund class $Z = Z([0,l])$ if $\int_0^h \frac{\varphi(t)}{t}\,dt \le c\varphi(h)$, $0 \le h \le l$.

Definition 5 ([1]). A non-negative function $\varphi(t)$ belongs to Zygmund class $Z_1 = Z_1([0,l])$ if $\int_h^l \frac{\varphi(t)}{t^2}\,dt \le c\frac{\varphi(h)}{h}$.

Definition 6. A function $\varphi(x)$ belongs to the generalized Hölder space $H^\omega = H^\omega([a,b])$ if

$$\omega(\varphi,h) \overset{\text{def}}{=} \sup_{0<t<h} \sup_{x,x+t\in[a,b]} |\varphi(x+t)-\varphi(x)| \le c\omega(h), \tag{2.3}$$

where $\omega(h)$ is a given positive function on $[0,l]$, $\omega(0)=0$; we set $\|\varphi\|_{H^\omega} = \|\varphi\|_C + \sup_{h>0}[\omega(\varphi,h)/\omega(h)]$.

By H_0^ω we denote the subspace of functions in H^ω which vanish at $x=a$. The function $\omega(h)$ is called a characteristic of the space H^ω.

Definition 7. By $H_0^\omega(\rho)$ we denote the space of functions $f(x)$ such that $\rho(x)f(x) \in H_0^\omega$, $\|f\|_{H_0^\omega(\rho)} = \|\rho f\|_{H_0^\omega}$, where $\rho(x)$ is a non-negative weight function.

In the sequel we shall use the following inequalities:

1) if $\omega(\varphi,h)$ is the continuity modulus (2.3), then

$$\frac{\omega(\varphi,x)}{x} \le c\frac{\omega(\varphi,y)}{y}, \quad x \ge y; \tag{2.4}$$

2) if $0<\alpha<1$, then

$$\frac{\omega(\varphi,h)}{h^\alpha} \le c\int_0^h \frac{\omega(\varphi,t)}{t^{1+\alpha}}\,dt; \tag{2.5}$$

3) if $\psi(x) \in W_\mu$, then

$$\psi(x) \le c\Big(\frac{x}{y}\Big)^\mu \psi(y), \quad x \ge y; \tag{2.6}$$

4) if $\psi(x)\in W_\mu$ with $0<\mu<1$, then (2.1) holds with x^* replaced both by x and y:

$$\begin{gathered} \Big|\frac{\psi(x)-\psi(y)}{x-y}\Big| \le c\frac{\psi(x)}{x}, \\ \Big|\frac{\psi(x)-\psi(y)}{x-y}\Big| \le c\frac{\psi(y)}{y}; \end{gathered} \tag{2.7}$$

5) if $k(x)\in V_\lambda$, then

$$k(x) \le c\Big(\frac{y}{x}\Big)^\lambda k(y), \quad x \le y, \tag{2.8}$$

and there exists $\varepsilon>0$ such that

$$k(x) \le c\Big(\frac{y}{x}\Big)^{\lambda-\varepsilon} k(y), \quad x \ge y; \tag{2.9}$$

6) if $\lambda \le 1$, then

$$|x^\lambda - y^\lambda| \le c(x-y)y^{\lambda-1}, \quad x \ge y > 0, \tag{2.10}$$

and if $\lambda \ge 0$, then

$$|x^\lambda - y^\lambda| \le c(x-y)x^{\lambda-1}, \quad x \ge y > 0. \tag{2.11}$$

Lemma 1. *Let $k(x) \in V_\lambda$, $\lambda > 0$ and let $\omega(x) \ge 0$ be an almost increasing function. Then $\omega(x)k(x) \le c\int_x^l \frac{\omega(t)k(t)}{t}dt$ for $0 < x < l/2$.*

Proof. By (2.8) we have

$$\int_x^l \frac{\omega(t)k(t)}{t}dt \ge c\omega(x)x^\lambda k(x)\int_x^l \frac{dt}{t^{1+\lambda}} \ge c\omega(x)x^\lambda k(x)\int_x^{2x} \frac{dt}{t^{1+\lambda}} = c\omega(x)k(x). \quad \blacksquare$$

3. Mapping Properties of Convolution Operators in the Space $H_0^\omega(\rho)$

The following theorem provides a Zygmund type estimate for the integral (1.1).

Theorem 1. *Let $k(x) \in V_\lambda$, $0 < \lambda < 1$ and $\varphi(x) \in C([a,b])$, $\varphi(a) = 0$. Then*

$$\omega(K\varphi, h) \le chk(h)\omega(\varphi, h) + ch\int_h^{b-a} \frac{k(t)\omega(\varphi,t)}{t}\,dt. \tag{3.1}$$

Proof. Let $a = 0$ for simplicity. We denote $g(x) = \varphi(x) - \varphi(0)$ and $f(x) = \int_0^x k(x-t)g(t)dt$. For all $x, x+h \in [0,b]$ we have

$$f(x+h) - f(x) = \int_{-h}^{x} [g(x-t) - g(x)]k(t+h)dt -$$

$$-\int_0^x [g(x-t) - g(x)]k(t)dt + g(x)\Big[\int_{-h}^{x} k(t+h)dt - \int_0^x k(t)dt\Big].$$

So,

$$|f(x+h) - f(x)| \le \Big|\int_{-h}^{0} [g(x-t) - g(x)]k(t+h)dt\Big| +$$

$$+\Big|\int_0^x [g(x-t)-g(x)][k(t)-k(t+h)]dt\Big| + \Big|g(x)\int_x^{x+h} k(t)dt\Big| = A_1 + A_2 + A_3.$$

Taking (2.8) and increasing of $\omega(\varphi,t)$ into account, we have for A_1:

$$A_1 \le \int_0^h \omega(\varphi,t)k(h-t)dt \le c\omega(\varphi,h)k(h)\int_0^h \Big(\frac{h}{h-t}\Big)^\lambda dt \le chk(h)\omega(h). \quad (3.2)$$

For A_2, applying (2.2) and (2.9), we obtain in the case $h \ge x$:

$$\begin{aligned} A_2 &\le ch\int_0^x \frac{\omega(\varphi,t)k(t+h)}{t+h}\,dt \le chk(h)h^{\lambda-\varepsilon}\int_0^x \frac{\omega(\varphi,t)dt}{(t+h)^{1+\lambda-\varepsilon}} = \\ &= chk(h)\int_0^{x/h} \frac{\omega(\varphi,ht)dt}{(t+1)^{1+\lambda-\varepsilon}} \le chk(h)\omega(\varphi,h)\int_0^1 \frac{dt}{(t+1)^{1+\lambda-\varepsilon}} \le \\ &\le chk(h)\omega(\varphi,h). \end{aligned} \quad (3.3)$$

In the case $h < x$ we write $A_2 \le \int_0^h + \int_h^x = B_1 + B_2$. For B_1 the estimate (3.3) is valid, while for B_2 we have

$$B_2 \le ch\int_h^b \frac{\omega(\varphi,t)k(t)}{t}\,dt. \quad (3.4)$$

As regards A_3, we have in the case $h \ge x$:

$$\begin{aligned} A_3 &\le c\omega(\varphi,h)k(x+h)(x+h)^\lambda \int_x^{x+h} \frac{dt}{t^\lambda} \le \\ &\le c\omega(\varphi,h)k(h)h^\lambda \int_0^{2h} \frac{dt}{t^\lambda} \le c\omega(\varphi,h)hk(h). \end{aligned} \quad (3.5)$$

If $h < x$, we use Lemma 1 to obtain

$$A_3 \le c\omega(\varphi,x)hk(x) \le ch\int_x^b \frac{\omega(\varphi,t)k(t)dt}{t} \le ch\int_h^b \frac{\omega(\varphi,t)k(t)}{t}\,dt. \quad (3.6)$$

Gathering all the estimates for A_i, $i = 1,2,3$, we arrive at (3.1). □

Theorem 2. *Let $k(x) \in V_\lambda$, $0 < \lambda < 1$, $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2$. Assume that*

i) *$\rho(x)\varphi(x) \in C([a,b])$ and $\rho(x)\varphi(x)\big|_{x=a} = 0$;*

ii) $\int_0^{b-a} t^{-\gamma}\omega(\rho\varphi,t)dt < \infty, \quad \gamma = \max(1,\mu)$. *Then the following Zygmund type estimate holds:*

$$\omega(\rho K\varphi, h) \le ch^{\gamma}k(h)\int_0^h \frac{\omega(\rho\varphi,t)}{t^{\gamma}}\,dt + ch\int_h^{b-a}\frac{\omega(\rho\varphi,t)k(t)}{t}\,dt, \qquad (3.7)$$

if $0 < \mu < 1+\lambda$ *and*

$$\omega(\rho K\varphi, h) \le ch\int_0^h \frac{\omega(\rho\varphi,t)}{t^{\mu}}\,dt + h\int_h^{b-a}\frac{\omega(\rho\varphi,t)}{t}\,dt, \qquad (3.8)$$

if $1+\lambda \le \mu < 2$.

Proof. Let $\varphi_0(x) = \rho(x)\varphi(x)$ and $a = 0$ for simplicity. We have

$$\rho(x)(K\varphi)(x) = \int_0^x k(x-t)\varphi_0(t)dt + \int_0^x \frac{\psi(x)-\psi(t)}{\psi(t)}\,k(x-t)\varphi_0(t)dt = \\ = f_1(x) + f_2(x).$$

Since $\varphi_0 \in C([0,b])$ and $\varphi_0(0) = 0$, the first term $f_1(t)$ is covered by Theorem 1. To estimate $\omega(f_2,h)$ we represent the difference $f_2(x+h)-f_2(x)$ as

$$f_2(x+h) - f_2(x) = \int_x^{x+h}\frac{\psi(x+h)-\psi(t)}{\psi(t)}\,\varphi_0(t)k(x+h-t)dt + \\ + \int_0^x \frac{\psi(x+h)-\psi(x)}{\psi(t)}\,\varphi_0(t)k(x+h-t)dt + \\ + \int_0^x \frac{\psi(x)-\psi(t)}{\psi(t)}\,[k(x-t+h)-k(x-t)]\varphi_0(t)dt = I_1 + I_2 + I_3.$$

Estimate for I_1. A) Let $0 < \mu < 1$ at first. Taking (2.4), (2.7) and (2.8) into account, we have

$$|I_1| \le \int_x^{x+h}\frac{(x+h-t)k(x+h-t)\omega(\varphi_0,t)dt}{t} \le \\ \le chk(h)\int_x^{x+h}\frac{\omega(\varphi_0,t-x)dt}{t-x} \le chk(h)\int_0^h \frac{\omega(\varphi_0,t)dt}{t}. \qquad (3.9)$$

B) If $1 \leq \mu < 2$, then by (2.4),(2.6) and (2.8) we obtain

$$|I_1| \leq c \int_0^h \frac{(h-t)k(h-t)(x+h)^{\mu-1}}{(x+t)^{\mu-1}} \frac{\omega(\varphi_0,t)}{x+t}\, dt \leq$$

$$\leq chk(h) \int_0^h \frac{(x+h)^{\mu-1}}{t(x+h)^{\mu-1}} \omega(\varphi_0,t) dt. \tag{3.10}$$

In the case $h < x$ we derive from (3.10)

$$|I_1| \leq chk(h) \int_0^h \frac{x^{\mu-1}\omega(\varphi_0,t)dt}{t(x+t)^{\mu-1}} \leq chk(h) \int_0^h \frac{\omega(\varphi_0,t)dt}{t}. \tag{3.11}$$

In the case $h > x$ the inequality (3.10) yields

$$|I_1| \leq chk(h)h^{\mu-1} \int_0^h \frac{\omega(\varphi_0,t)dt}{t(x+t)^{\mu-1}} \leq ch^{\mu}k(h) \int_0^h \frac{\omega(\varphi_0,t)dt}{t^{\mu}}. \tag{3.12}$$

So from (3.9), (3.11) and (3.12) there follows the estimate

$$|I_1| \leq ch^{\gamma}k(h) \int_0^h \frac{\omega(\varphi_0,t)dt}{t^{\gamma}}, \quad \gamma = \max(1,\mu). \tag{3.13}$$

Estimate for I_2. A) Let $0 < \mu < 1$. By (2.6) and (2.7) we have

$$|I_2| \leq ch \int_0^x \frac{k(x+h-t)\omega(\varphi_0,t)dt}{t}. \tag{3.14}$$

In the case $h < x$ we represent (3.14) as $|I_2| \leq \int_0^h + \int_h^{(x+h)/2} + \int_{(x+h)/2}^x = I_2' + I_2'' + I_2'''$. It is clear that

$$I_2' \leq chk(h) \int_0^h \frac{\omega(\varphi_0,t)dt}{t}. \tag{3.15}$$

Since $x+h-t \geq t$ in I_2'', we obtain

$$I_2'' \leq ch \int_h^b \frac{k(t)\omega(\varphi_0,t)dt}{t}. \tag{3.16}$$

Further, $x+h-t \leq t$ in I_2''', so by (2.4)

$$I''' \leq ch \int\limits_{(x+h-t)/2}^{x} \frac{k(x+h-t)\omega(\varphi_0, x+h-t)dt}{x+h-t} \leq$$

$$\leq ch \int\limits_{h}^{b} \frac{\omega(\varphi_0,t)k(t)dt}{t}. \tag{3.17}$$

If $h \geq x$,then (3.14) immediately yields

$$|I_2| \leq chk(h) \int\limits_{0}^{h} \frac{\omega(\varphi_0,t)dt}{t}. \tag{3.18}$$

B) Let now $1 \leq \mu < 2$. Taking (2.1) into account, we have

$$|I_2| \leq ch \int\limits_{0}^{x} \frac{(x+h)^{\mu-1}k(x+h-t)\omega(\varphi_0,t)dt}{t^\mu}. \tag{3.19}$$

Hence

$$|I_2| \leq ch^\mu k(h) \int\limits_{0}^{h} \frac{\omega(\varphi_0,t)dt}{t^\mu}$$

in the case $h \geq x$. If $h < x$, we represent (3.19) as

$$|I_2| \leq \int\limits_{0}^{h} + \int\limits_{h}^{(x+h)/2} + \int\limits_{(x+h)/2}^{x} = B_1 + B_2 + B_3.$$

Taking into account that $x+h \leq 2(x+h-t)$ in B_1, in the case $0 < \mu-1 < \lambda$ we obtain

$$|B_1| \leq ch \int\limits_{0}^{h} \frac{(x+h-t)^{\mu-1}k(x+h-t)\omega(\varphi_0,t)dt}{t^\mu} \leq$$

$$\leq ch^\mu k(h) \int\limits_{0}^{h} \frac{\omega(\varphi_0,t)dt}{t^\mu} \tag{3.20}$$

by (2.9). If $\mu - 1 > \lambda$, the function $t^{\mu-1}k(t)$ is bounded. So

$$B_1 \leq ch \int\limits_{0}^{h} \frac{\omega(\varphi_0,t)dt}{t^\mu}. \tag{3.21}$$

Since $x+h \leq 2(x+h-t)$ again, we have

$$B_2 \leq ch \int_h^{(x+h)/2} \frac{(x+h-t)^{\mu-1}k(x+h-t)}{t^\mu}\, dt.$$

Here $x+h-t \geq t$, so that

$$B_2 \leq ch \int_h^b \frac{k(t)\omega(\varphi_0,t)dt}{t} \tag{3.22}$$

by (2.9), if $\mu-1<\lambda$. If $\mu-1 \geq \lambda$, by boundedness of $t^{\mu-1}k(t)$ we have

$$B_2 \leq ch \int_h^b \frac{\omega(\varphi_0,t)dt}{t^\mu}. \tag{3.23}$$

To estimate B_3 we notice that $t \geq x+h-t$ in B_3. So by (2.4) we have

$$B_3 \leq ch \int_{x+h-t/2}^{x} \frac{k(x+h-t)\omega(\varphi_0,x+h-t)dt}{x+h-t} \leq ch\int_h^b \frac{k(t)\omega(\varphi_0,t)dt}{t}. \tag{3.24}$$

Thus, I_2 admits the estimate

$$|I_2| \leq ch^\gamma k(h) = \int_0^h \frac{\omega(\varphi_0,t)dt}{t^\mu} + ch\int_h^b \frac{k(t)\omega(\varphi_0,t)dt}{t}, \quad \gamma = \max(1,\mu), \tag{3.25}$$

if $\mu < 1+\lambda$ and

$$|I_2| \leq ch \int_0^h \frac{\omega(\varphi_0,t)dt}{t^\mu} + ch \int_h^b \frac{\omega(\varphi_0,t)dt}{t^\mu} \tag{3.26}$$

if $\mu \geq 1+\lambda$.

Estimate for I_3. Let $0<\mu<1$. By (2.2) and (2.4) we have

$$|I_3| \leq ch \int_0^x \frac{k(x+h-t)\omega(\varphi_0,t)dt}{t}$$

which coincides with the estimate in (3.14). If $1 \leq \mu < 2$, we derive from (2.1),(2.2) and (2.6):

$$|I_3| \leq ch\int_0^x \frac{x^{\mu-1}k(x+h-t)\omega(\varphi_0,t)dt}{t^\mu} \leq ch\int_0^x \frac{(x+h)^{\mu-1}k(x+h-t)\omega(\varphi_0,t)dt}{t^\mu}.$$

The latter coincides with (3.19). Gathering estimates for I_1, I_2 and I_3, we obtain (3.7)–(3.8). □

Theorem 3. *Let $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2$, $k(t) \in V_\lambda$, $0 < \lambda < 1$. Assume that*

i) $\mu < \lambda + 1$;

ii) $t^{-\max(0,\mu-1)}\omega(t) \in Z$, $tk(t)\omega(t) \in Z_1$.

Then the operator K is bounded from $H_0^\omega(\rho)$ into $H_0^{\omega_k}(\rho)$ with $\omega_k(h) = hk(h)\omega(h)$.

Proof. Let $f = K\varphi$ with $\varphi \in H_0^\omega(\rho)$ and let $a = 0$. To prove that $f \in H_0^{\omega_k}(\rho)$ we remark at first that

$$\int_0^b \frac{\omega(\rho\varphi, t)}{t^\gamma}\, dt < \infty, \quad \gamma = \max(1, \mu). \tag{3.27}$$

Therefore Zygmund type estimate (3.27) concerning the case $0 < \mu < 1 + \lambda$ holds which gives

$$\frac{\omega(\rho f, h)}{hk(h)\omega(h)} \le \|\rho\varphi\|_{H_0^\omega} \left\{ \frac{h^{\gamma-1}}{\omega(h)} \int_0^h \frac{\omega(t)}{t^\gamma}\, dt + \frac{1}{k(h)\omega(h)} \int_h^b \frac{\omega(t)k(t)dt}{t} \right\}. \tag{3.28}$$

Hence by the condition ii) we have

$$\frac{\omega(\rho f, h)}{hk(h)\omega(h)} \le c\|\varphi\|_{H_0^\omega(\rho)}. \tag{3.29}$$

It remains to prove that $\rho(x)f(x)\big|_{x=0} = 0$. After the change of variable $t = x - \xi x$ we have

$$|\rho(x)f(x)| \le x\psi(x) \int_0^1 \frac{|\varphi_0(x-x\xi)|k(x\xi)d\xi}{\psi(x-x\xi)}.$$

Since $\varphi_0(0) = 0$, this yields

$$|\rho(x)f(x)| \le cx\psi(x) \int_0^1 \frac{\omega(\varphi_0(, 1-\xi)d\xi}{\psi(x-x\xi)}. \tag{3.30}$$

According to (2.6) and (2.8) we see that

$$|\rho(x)f(x)| \le cxk(x) \int_0^1 \frac{\omega(\varphi_0, 1-\xi)d\xi}{\xi^\lambda(1-\xi)^\gamma} = c_1xk(x) \to 0 \tag{3.31}$$

as $x \to 0$ in view of (3.27). So $\rho(x)f(x)\big|_{x=0} = 0$. □

Corollary 1. *The operator* (1.1) *with the kernel* $k(t) = t^{\alpha-1}(\ln \frac{\gamma}{t})^{\beta}$, $\gamma > b-a$, $0 < \alpha < 1$, $\beta \geq 0$ *is bounded from* $H_0^{\omega}(\rho)$ *into* $H_0^{\omega_{\alpha,\beta}}(\rho)$, *where* $\rho(x) = \psi(x-a)$, $\psi(x) \in W_{\mu}$ *and* $\omega\alpha, \beta(h) = \omega(h)h^{\alpha}(\ln \frac{\gamma}{h})^{\beta}$ *under the assumption that* $0 < \mu < 2-\alpha$ *and* $h^{-\max(0,\mu-1)}\omega(h) \in Z$, $h^{\alpha}(\ln \frac{\gamma}{h})^{\beta}\omega(h) \in Z_1$.

In the case $\psi(x) = x^{\mu}$ and $\omega(h) = h^{\lambda}$ the assertion of Corollary 1 was proved in [6] (see [17], Theorem 21.2).

Corollary 2. *The operator of the form* $\int_x^b k(t-x)\varphi(t)dt$ *is bounded from* $H_0^{\omega}(\rho)$ *into* $H_0^{\omega_k}(\rho)$ *under the assumptions of Theorem* 3 *if the requirement* $\rho f\big|_{x=a} = 0$ *in the definition of the space* $H_0^{\omega}(\rho)$ *is replaced by* $\rho f\big|_{x=b} = 0$.

4. Mapping Properties of Fractional Differentiation in the Spaces $H_0^{\omega}(\rho)$

Now we give Zygmund type estimate for the fractional derivative (1.5).

Theorem 4. *Let* $\rho(x) = \psi(x-a)$, $\psi(x) \in W_{\mu}$, $0 < \mu < 2$, *and*

$$\int_0^{b-a} \frac{\omega(\rho f, t)}{t^{\alpha+\mu}}\, dt < \infty, \quad \gamma = \max(1, \mu).$$

Then

$$\omega(\rho D_{a+}^{\alpha} f, h) \leq ch^{\gamma-1} \int_0^h \frac{\omega(\rho f, t)}{t^{\alpha+\gamma}}\, dt. \tag{4.1}$$

Proof. According to (1.5) we have

$$\rho(x)(D_{a+}^{\alpha} f)(x) = \frac{\rho(x)f(x)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \\ + \frac{\alpha}{\Gamma(1-\alpha)}\, \rho(x) \int_0^{x-a} \frac{f(x)-f(x-t)}{t^{1+\alpha}}\, dt. \tag{4.2}$$

We set $a = 0$ and denote $\theta(x) = \psi(x)\int_0^x \frac{f(x)-f(x-t)}{t^{1+\alpha}}dt$. To estimate the difference $\theta(x+h) - \theta(x)$ we represent it in the form $\theta(x+h) - \theta(x) = \sum_{k=1}^{8} A_k(x)$ (as in [13] in the purely power case), where

$$A_1(x) = \Big[1 - \frac{\psi(x)}{\psi(x+h)}\Big] \int_0^x \frac{g(x+h) - g(y)}{(x+h-y)^{1+\alpha}}\, dy,$$

$$A_2(x) = [\psi(x+h) - \psi(x)] \int_0^{x+h} \frac{g(y)}{(x+h-y)^{1+\alpha}} \Big[\frac{1}{\psi(x+h)} - \frac{1}{\psi(y)}\Big]\, dy,$$

$$A_3(x) = \frac{\psi(x)}{\psi(x+h)} \int\limits_x^{x+h} \frac{g(x+h)-g(y)}{(x+h-y)^{1+\alpha}}\, dy,$$

$$A_4(x) = \psi(x) \int\limits_x^{x+h} \frac{g(y)}{(x+h-y)^{1+\alpha}} \Big[\frac{1}{\psi(x+h)} - \frac{1}{\psi(y)}\Big]\, dy,$$

$$A_5(x) = \int\limits_0^x [g(x)-g(y)][(x+h-y)^{-1-\alpha} - (x-y)^{-1-\alpha}]\, dy,$$

$$A_6(x) = \psi(x) \int\limits_0^x g(y) \Big[\frac{1}{\psi(x)} - \frac{1}{\psi(y)}\Big] \Big[(x+h-y)^{-1-\alpha} - (x-y)^{-1-\alpha}\Big] dy,$$

$$A_7(x) = \frac{1}{\alpha} \frac{\psi(x)}{\psi(x+h)} [g(x+h)-g(x)][h^{-\alpha} - (x+h)^{-\alpha}],$$

$$A_8(x) = \frac{1}{\alpha} \psi(x) g(x) \Big[\frac{1}{\psi(x+h)} - \frac{1}{\psi(x)}\Big] \Big[h^{-\alpha} - (x+h)^{-\alpha}\Big].$$

Estimate for A_1. By (2.1) we have

$$|A_1| \le c \frac{h}{x+h} \int\limits_0^{x+h} \frac{\omega(g, x+h-y)}{(x+h-y)^{1+\alpha}} dy = c \frac{h}{x+h} \int\limits_0^{x+h} \frac{\omega(g,t)}{t^{1+\alpha}} dt. \tag{4.3}$$

If $h \ge x$, it is obvious that

$$|A_1| \le c \int\limits_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\, dt. \tag{4.4}$$

If $h < x$ represent (4.3) as $|A_1| \le \int_0^h + \int_h^{x+h} = A_1' + A_1''$. For A_1' the estimate (4.4) holds. As regards A_1'', applying (2.4) and (2.5), we have

$$A_1'' \le ch \int\limits_x^{x+h} \frac{\omega(g,t)}{t} \frac{dt}{t^{1+\alpha}} \le c\omega(g,h) \int\limits_h^{\infty} \frac{dt}{t^{1+\alpha}} \le$$

$$\le c \frac{\omega(g,h)}{h^\alpha} \le c \int\limits_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\, dt. \tag{4.5}$$

Estimate for A_2. A) In the case $0<\mu<1$, using (2.7), we obtain

$$|A_2| \le c\frac{h}{x+h}\int\limits_0^{x+h}\frac{\omega(g,y)}{y(x+h-y)^\alpha}\,dy = \int\limits_0^{h/2} + \int\limits_{h/2}^{x+h} = A_2' + A_2''.$$

Obviously, $A_2' \le c\int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}dt$. Using (2.4) and (2.5), we derive the following estimate for A_2'':

$$A_2'' \le c\frac{\omega(g,h)}{x+h}\int\limits_0^{x+h}\frac{dy}{(x+h-y)^\alpha} = c_1\frac{\omega(g,h)}{(x+h)^\alpha} \le c_2\int\limits_0^h\frac{\omega(g,t)}{t^{1+\alpha}}\,dt.$$

B) If $1\le\mu<2-\alpha$, taking (2.1) and (2.6) into account, we obtain for A_2:

$$|A_2| \le c\frac{h}{(x+h)^{2-\mu}}\int\limits_0^h\frac{\omega(g,y)dy}{y^\mu(x+h-y)^\alpha} = \int\limits_0^{h/2} + \int\limits_{h/2}^{x+h} = B_1 + B_2.$$

It is obvious that

$$B_1 \le ch^{\mu-1}\int\limits_0^h\frac{\omega(g,t)}{t^{\mu+\alpha}}\,dt. \tag{4.6}$$

As regards B_2, we apply (2.4) and obtain

$$B_2 \le c\frac{\omega(g,h)}{(x+h)^{2-\mu}}\int\limits_{h/2}^{x+h}\frac{dy}{y^{\mu-1}(x+h-y)^\alpha} \le$$

$$\le c\frac{\omega(g,h)}{(x+h)^\alpha}\int\limits_0^1\frac{dt}{t^{\mu-1}(1-t)^\alpha} = c_1\frac{\omega(g,h)}{(x+h)^\alpha}.$$

Using then (2.5), we notice that the estimate for B_2 is the same as in (4.6).

Estimate for A_3. Since $\psi(x)$ is almost increasing, we have

$$|A_3| \le c\int\limits_x^{x+h}\frac{\omega(g,x+h-y)}{(x+h-y)^{1+\alpha}}\,dy = c\int\limits_0^h\frac{\omega(g,t)}{t^{1+\alpha}}\,dt.$$

Estimate for A_4. Let $0<\mu<1$ at first. In view of (2.7) we have

$$|A_4| \le c\int\limits_0^h\frac{\omega(g,x+h-t)}{t^\alpha(x+h-t)}\,dt. \tag{4.7}$$

In the case $h < x$ we have $t \le x+h-t$ in (4.7). So by (2.4) we obtain $|A_4| \le c\int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt$. If $h \ge x$, we represent (4.7) as $|A_4| \le \int_0^{(x+h)/2} + \int_{(x+h)/2}^h$. Since $t \le x+h-t$ and $t \ge x+h-t$ in the first and second terms, respectively, by (2.4) we derive that

$$|A_4| \le c\int_0^{(x+h)/2} \frac{\omega(g,t)}{t^{1+\alpha}}\,dt + c\int_{(x+h)/2}^{h} \frac{\omega(g,x+h-t)}{(x+h-t)^{1+\alpha}}\,dt \le c\int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt.$$

Let now $1 \le \mu < 2-\alpha$. Using (2.1) and (2.6), we get

$$|A_4| \le \frac{c}{(x+h)^{1-\mu}}\int_0^h \frac{\omega(g,x+h-t)}{(x+h-t)^{\mu}t^{\alpha}}\,dt.$$

If $h < x$, by (2.4) we have

$$|A_4| \le c(x+h)^{\mu-1}\int_0^h \frac{\omega(g,t)dt}{(x+h-t)^{\mu-1}t^{1+\alpha}} \le c\int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt.$$

If $h \ge x$, then $|A_4| \le \int_0^{(x+h)/2} + \int_{(x+h)/2}^h$. We use (2.4) in the first term and the inequality $t^\alpha \ge (x+h-t)^\alpha$ in the second. This yields

$$|A_4| \le \int_0^{(x+h)/2} \frac{\omega(g,t)}{t^{1+\alpha}}\,dt\, c(x+h)^{\mu-1}\int_{(x+h)/2}^{h} \frac{\omega(g,x+h-t)}{(x+h-t)^{\alpha+\mu}}\,dt \le$$

$$\le c\int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt + ch^{\mu-1}\int_x^{(x+h)/2} \frac{\omega(g,t)}{t^{\alpha+\mu}}\,dt \le ch^{\mu-1}\int_0^h \frac{\omega(g,t)}{t^{\alpha+\mu}}\,dt.$$

Estimate for A_5. Applying (2.11), we have

$$|A_5| \le ch\int_0^h \omega(g,x-y)\,\frac{dy}{(x+h-y)(x-y)^{1+\alpha}} = ch\int_0^x \frac{\omega(g,t)dt}{t^{1+\alpha}(t+h)}.$$

In the case $h \ge x$ it is clear that

$$|A_5| \le c\int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt. \tag{4.8}$$

If $h < x$, then $|A_5| \leq \int_0^h + \int_h^x = A_5' + A_5''$ with the same estimate as in (4.8) for A_5'. As regards A_5'', we have

$$A_5'' \leq c\omega(g,h) \int\limits_h^\infty \frac{dt}{t^{1+\alpha}} \leq c\frac{\omega(g,h)}{h^\alpha} \leq c \int\limits_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt \tag{4.9}$$

by (2.4) and (2.5).

Estimate for A_6. A) Let $0 < \mu < 1$ at first. Applying (2.7) and (2.11) we arrive at

$$|A_6| \leq ch \int\limits_0^x \frac{\omega(g,y)dy}{y(x+h-y)(x-y)^\alpha} = \int\limits_0^{x/2} + \int\limits_{x/2}^x = A_6' + A_6''.$$

For A_6' we have

$$A_6' \leq ch \int\limits_0^{x/2} \frac{\omega(g,y)dy}{(y+h)y^{1+\alpha}}. \tag{4.10}$$

If $h < x$, $A_6' \leq \int_0^{h/2} + \int_{h/2}^{x/2} = K_1 + K_2$. It is evident that K_1 admits the same estimate as in (4.9). For K_2 the application of (2.5) provides the same result: $K_2 \leq c\omega(g,h) \int_{h/2}^\infty \frac{dt}{t^{1+\alpha}} \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt$. If $h \geq x$, then immediately

$$A_6' \leq c \int\limits_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\, dt. \tag{4.11}$$

To estimate A_6' we remark that $y \geq x - y$ so that

$$A_6'' \leq ch \int\limits_{x/2}^x \frac{\omega(g,x-y)dy}{(x-y)^{1+\alpha}(x+h-y)} = ch \int\limits_0^{x/2} \frac{\omega(g,t)dt}{t^{1+\alpha}(t+h)},$$

which is the same as in (4.10) and so A_6'' admits the same estimates as in (4.5).

B) Let $1 \leq \mu < 2 - \alpha$. Using (2.1), (2.6) and (2.11), we have

$$A_6 \leq chx^{\mu-1} \int\limits_0^x \frac{\omega(g,y)dy}{y^\mu(x+h-y)(x-y)^\alpha} = \int\limits_0^{x/2} + \int\limits_{x/2}^x = U_1 + U_2.$$

If $h < x$, we set $U_1 = \int_0^{h/2} + \int_{h/2}^{x/2} = U_1' + U_1''$, whence easy calculations yield the inequality

$$U_1 \le ch \int_0^h \frac{\omega(g,t)dt}{t^{1+\alpha}} + ch^{\mu-1} \int_0^h \frac{\omega(g,t)dt}{t^{\mu+\alpha}} \le ch^{\mu-1} \int_0^h \frac{\omega(g,t)dt}{t^{\mu+\alpha}}. \quad (4.12)$$

For U_2 by (2.5) we have

$$U_2 \le ch \int_{x/2}^{x} \frac{\omega(g,y)dy}{y(x+h-y)(x-y)^\alpha} \le c\omega(g,h) \int_0^{x/2} \frac{dt}{t^\alpha(t+h)} \le$$
$$\le c\frac{\omega(g,h)}{h^\alpha} \int_0^\infty \frac{d\xi}{\xi^\alpha(1+\xi)} \le c \int_0^h \frac{\omega(g,t)dt}{t^{1+\alpha}}.$$

If $h \ge x$, the estimation of U_1 and U_2 is easy and provides the same as in (4.2). Gathering all the estimates, we obtain $|A_6| \le ch^{\gamma-1} \int_0^h \frac{\omega(g,t)}{t^{\alpha+\gamma}}\, dt$, $\gamma = \max(1, \mu)$.

Estimate for A_7. Applying (2.5) and (2.10) and almost increasing of $\psi(x)$, we easily obtain

$$|A_7| \le c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\, dt. \quad (4.13)$$

Estimate for A_8. Using the inequalities (2.1) and (2.10) for $0 < \mu < 2-\alpha$, we make sure of validity of the estimate (4.13) for A_8 as well.

It remains to consider the first term

$$r(x) = \frac{\psi(x-a)f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} = \frac{g(x)}{\Gamma(1-\alpha)(x-a)^\alpha}$$

in (4.2). Since $g(x) \in H_0^\omega$, we have the estimate

$$|r(x+h) - r(x)| \le c \int_0^h \frac{\omega(g,t)dt}{t^{1+\alpha}}, \quad (4.14)$$

which is derived by direct calculations under the assumptions of the theorem.

Collecting all the estimates for A_i, $i = 1, \dots, 8$, and (4.14), we obtain the required inequality (4.1). □

Theorem 5. *Let* $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2-\alpha$ *and let*

$$\begin{aligned} &1)\ \omega(t) \neq 0, \quad t > 0, \\ &2)\ \omega(t)t^{-\alpha}\Big|_{t=0} = 0, \\ &3)\ \omega(t)t^{1-\alpha-\gamma} \in Z, \quad \gamma = \max(1,\mu). \end{aligned} \tag{4.15}$$

Then the operator D^α_{a+} *continuously maps* $H^\omega_0(\rho)$ *into* $H_0^{\omega_{-\alpha}}(\rho)$ *with* $\omega_{-\alpha}(h) = h^{-\alpha}\omega(h)$.

Proof. Let $f(x) \in H^\omega_0(\rho)$ and $\varphi(x) = D^\alpha_{a+}f(x)$. To show that

$$\sup_{0<h\leq b-a} \frac{h^\alpha \omega(\rho\varphi, h)}{\omega(h)} = c < \infty$$

we observe that the inclusion $\omega(t)t^{1-\alpha-\gamma} \in Z$ implies convergence of the integral $\int_0^{b-a} \omega(t)t^{-\alpha-\gamma}dt$, so Theorem 4 is applicable. Using the estimate (4.1) of Theorem 4, we obtain

$$\frac{h^\alpha \omega(\rho\varphi, h)}{\omega(h)} \leq \frac{h^{\alpha+\gamma-1}\int_0^1 \omega(\rho f, t)t^{-\alpha-\gamma}dt}{\omega(h)} \leq c\|f\|_{H^\omega_0(\rho)}. \tag{4.16}$$

It remains to show that $\rho(x)\varphi(x)|_{x=a} = 0$. By (4.2) we have

$$\begin{aligned} |\rho(x)\varphi(x)| \leq \frac{\omega(\rho f, x-a)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{x-a} \frac{\omega(\rho f, t)dt}{t^{1+\alpha}} + \\ + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{x-a} \frac{|\psi(x-a) - \psi(x-a-t)|\omega(\rho f, x-a-t)dt}{t^{1+\alpha}\psi(x-a-t)} = \\ = D_1 + D_2 + D_3. \end{aligned} \tag{4.17}$$

Here $D_1 \leq c\|f\|H^\omega_0(\rho)\frac{\omega(x-a)}{(x-a)^\alpha}$, the condition (4.16) implies $\omega(x)x^{-\alpha}|_{x=0} = 0$. So $\lim_{x\to a} D_1 = 0$. The equality $\lim_{x\to a} D_2 = 0$ is obvious by the existence of the integral in D_2.

For the term D_3 in the case $0 < \mu < 1$ we have by (2.7)

$$D_3 \leq c \int_0^{x-a} \frac{\omega(\rho f, x-a-t)dt}{t^\alpha(x-a-t)}.$$

We evaluate this separately for $x-a-t \geq t$ and $x-a-t \leq t$ by means of (2.4):

$$D_3 \leq c \int_0^{(x-a)/2} \frac{\omega(\rho f, t)dt}{t^{1+\alpha}} + c \int_{(x-a)/2}^{x-a} \frac{\omega(\rho f, x-a-t)}{(x-a-t)^{1+\alpha}}dt,$$

whence $\lim_{x\to a} D_3 = 0$.

If $1 \le \mu \le 2 - \alpha$, we use (2.1) and (2.5) and obtain

$$D_3 \le c \int_0^{x-a} \frac{\omega(\rho f, x-a-t)}{(x-a)^{1-\mu}(x-a-t)^{\mu} t^{\alpha}} = \int_0^{(x-a)/2} + \int_{(x-a)/2}^{x-a}$$

and similar to what we did above we have

$$D_3 \le c \int_0^{(x-a)/2} \frac{(x-a)^{\mu-1}\omega(\rho f, t)}{t^{\alpha+\mu}} dt + c \int_{(x-a)/2}^{x-a} \frac{(x-a)^{\mu-1}\omega(\rho f, x-a-t)}{(x-a-t)^{\alpha+\mu}} dt \le$$

$$\le (x-a)^{\mu-1} \int_0^{(x-a)/2} \frac{\omega(\rho f, t)dt}{t^{\alpha+\mu}},$$

so that $\lim_{x\to a} D_3 = 0$. Therefore, $\lim_{x\to a} \rho(x)\varphi(x) = 0$. $\square$

Corollary 1. *Let $\varphi(x) \in W_\mu$ $0 < \mu < 2 - \alpha$, and let $\omega(t)$ be an almost increasing function on $[0, b-a]$ such that*
1) $\omega(0) = 0$, $\omega(t) \ne 0$ for $t \in (0, b-a]$;
2) $\omega(t)t^{-1-\gamma} \in Z$, $\gamma = \max(1, \mu)$.
Then the operator D^{α}_{a+} of fractional differentiation continuously maps $H_0^{\omega_\alpha}(\rho)$ into H_0^{ω} with $\rho(x) = \varphi(x-a)$, $\omega_\alpha(h) = h^\alpha \omega(h)$.

Another corollary (of Theorems 5 and 3) will be related to the following Love's index law [8]:

$$I^{\gamma}_{0+} x^{\alpha} I^{\beta}_{0+} x^{\gamma} I^{\alpha}_{0+} x^{\beta} f(x) = f(x), \quad \alpha + \beta + \gamma = 0 \tag{4.18}$$

well known in fractional calculus. This corollary will provide conditions guaranteeing validity of (4.18) for functions $f \in H_0^{\omega}(\rho)$. For simplicity we restrict ourselves with the cases $\omega(x) = x^{\lambda}$ and $\rho(x) = x^{\mu}$. The notation

$$\alpha^* = \begin{cases} \alpha, & \alpha \le 1 \\ 1, & \alpha \ge 1 \end{cases}$$

is used below.

Corollary 2. *Relation* (4.18) *is valid for all functions $f(x) \in H_0^{\lambda}(x^{\mu})$ and all $\alpha, \beta, \gamma \in R^1$ such that $\alpha+\beta+\gamma = 0$, if the number $\lambda \in (0,1]$ satisfies the conditions $\lambda > -\alpha, (\lambda+\alpha)^* + \beta > 0$ $[(\lambda+\alpha)^* + \beta]^* + \gamma > 0$ while the weight exponent μ satisfies the conditions*

1) $\mu < (\lambda + \alpha)^* + 1$
2) $\mu < [(\lambda + \alpha)^* + \beta]^* + 1$
3) $\mu < \{[(\lambda + \alpha)^* + \gamma\}^* + 1.$

5. A Theorem on Isomorphism

In Theorem 3 and Corollary 1 of Theorem 5 it was proved that

$$I^{a}_{a+} \;:\; H^{\omega}_0(\rho) \longrightarrow H^{\omega_\alpha}_0(\rho), \tag{5.1}$$

$$D^{a}_{a+} \;:\; H^{\omega_\alpha}_0(\rho) \longrightarrow H^{\omega}_0(\rho) \tag{5.2}$$

under the appropriate assumptions on $\omega(h)$ and $\rho(x)$. To derive the assertion $I^{\alpha}_{a+}[H^{\omega}_0(\rho)] = H^{\omega_\alpha}_0(\rho)$ it remains to show that any function in $H^{\omega_\alpha}_0(\rho)$ is representable by the fractional integral of a function in $H^{\omega}_0(\rho)$. This will be the goal of Theorem 6 below. Preliminarily we state two auxiliary assertions we need.

Lemma 2 ([17], p.185; p.231 in English ed.). *In order a function $f(x)$ to be representable as $f = I^{\alpha}_{a+}\varphi$, $\varphi \in L_p(a,b)$, $-\infty < a < b < \infty$, it is necessary and sufficient that*

i) $f(x)(x-a)^{-\alpha} \in L_p(a,b)$;

ii) $\|\psi_\varepsilon\|_{L_p} \le c < \infty$ *with c not depending on ε, where*

$$\psi_\varepsilon(x) = \int\limits_a^{x-\varepsilon} \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}\,dt,$$

for $a+\varepsilon < x < b$ and $\psi_\varepsilon(x) = 0$ for $a < x < a+\varepsilon$.

A close version of Lemma 2 can be found in [16]. (See also [3] for another version under additional assumptions that $f \in L_p$ and $1 < p < 1/2\alpha$).

Lemma 3. *Let $\omega(t)t^{-\delta} \in Z$. There exists $p > 1$ such that $\omega(t)t^{-1-\delta} \in L_p(0,1)$.*

Proof. It is known that the inclusion $\varphi(t) \in Z$ implies existence of $\varepsilon \in (0,1)$ such that $t^{-\varepsilon}\varphi(t)$ is almost increasing (see, e.g. [1]). Therefore, there exists $\varepsilon \in (0,1)$ such that $t^{-\varepsilon-\delta}\omega(t)$ is bounded. So $\omega(t) \le ct^{\delta+\varepsilon}$ and to have a finite L_p-norm for $\omega(t)/t^{1+\delta}$ we must choose $p < \frac{1}{1-\varepsilon}$. $\square$

Theorem 6 (On isomorphism). *Let $\psi(x) \in W_\mu$, $0 < \mu < 2-\alpha$ and let $\omega(t)$ be a continuous function such that $\omega(t)t^{1-\gamma} \in Z$, $\omega(t)t^{\alpha} \in Z_1$, $\gamma = \max(1,\mu)$, $0 < \alpha < 1$. Then the fractional integration operator I^{α}_{a+} isomorphically maps the weighted space $H^{\omega}_0(\rho)$ with $\rho(x) = \psi(x-a)$ onto the space $H^{\omega_\alpha}_0(\rho)$ with the same weight and the characteristic $\omega_\alpha(h) = h^{\alpha}\omega(h)$.*

Proof. In view of (5.1)–(5.2) it is sufficient to prove the representability of a function $f \in H^{\omega_\alpha}_0(\rho)$ by a fractional integral. Aiming to apply Lemma 2 we shall prove that there exists $p > 1$ such that conditions i)–ii) of Lemma 2 are satisfied.

The estimate

$$\frac{|f(x)|}{(x-a)^{\alpha}} \le c\|f\|_{H_0^{\omega_\alpha}} \frac{\omega(x-a)}{(x-a)^{\mu}} \tag{5.3}$$

is valid for any $f(x) \in H_0^{\omega_\alpha}(\rho)$. Really, by (2.6) we have

$$\frac{|f(x)|}{(x-a)^{\alpha}} \le c\frac{|\rho(x)f(x)|}{(x-a)^{\mu+\alpha}}, \tag{5.4}$$

which immediately provides (5.3).

Since $\omega(x)/x^{\delta-1} \in Z$, from (5.3) and Lemma 3 we conclude that there exists $p > 1$ such that the condition i) of Lemma 2 is satisfied.

For this p we shall show that a constant $c > 0$ exists such that

$$\|\psi_\varepsilon\|_{L_p} \le c \le \infty. \tag{5.5}$$

We set $g(x) = f(x)\psi(x-a)$ and have

$$|\psi_\varepsilon(x)| \le \frac{1}{\psi(x-a)} \int_a^{x-\varepsilon} \frac{|g(x)-g(t)|}{(x-t)^{1+\alpha}}\, dt + \int_a^{x-\varepsilon} \frac{\left|g(t)\left[\frac{1}{\psi(x-a)} - \frac{1}{\psi(t-a)}\right]\right|}{(x-t)^{1+\alpha}}\, dt.$$

To estimate F_1 we use (2.6) and obtain

$$F_1 \le \frac{c}{(x-a)^{\mu}} \int_a^{x} \frac{\omega(g,x-t)}{(x-t)^{1+\alpha}}\, dt \le \frac{c}{(x-a)^{\gamma}} \int_a^{x} \frac{\omega(g,x-t)}{(x-t)^{1+\alpha}}\, dt. \tag{5.6}$$

Since $g(x) \in H_0^{\omega_\alpha}(\rho)$, it is easily proved that (5.6) yields

$$F_1 \le \frac{c}{(x-a)^{\gamma}}. \tag{5.7}$$

For F_2 in the case $0 < \mu < 1$ we use (2.6) and (2.7) and obtain

$$F_2 \le \frac{c}{(x-a)} \int_0^{x-a} \frac{\omega(g,x-t-a)}{t^{\alpha}(x-t-a)}\, dt = \int_0^{(x-a)/2} + \int_{(x-a)/2}^{x-a} .$$

Hence after simple calculations

$$F_2 \le \frac{c}{x-a} \int_0^{(x-a)/2} \frac{\omega(t)dt}{t} + \frac{c}{x-a} \int_{(x-a)/2}^{x-a} \frac{\omega(x-t-a)dt}{x-t-a} \le$$

$$\le c\frac{\omega(x-a)}{x-a} + \frac{c}{x-a} \int_0^{x-a} \frac{\omega(t)}{t}\, dt. \tag{5.8}$$

Since the condition $\omega(t)t^{1-\mu} \in Z$ with $0 < \mu < 1$ implies $\omega(t) \in Z$, we derive from (5.8) the estimate $F_2 \le c\omega(x-a)/(x-a)$.

Let now $1 \le \mu < 2-\alpha$. We use inequalities (2.1) and (2.6) to obtain

$$F_3 \le \frac{c}{x-a} \int_0^{x-a} \frac{\omega(g, x-t-a)}{t^\alpha (x-t-a)^\mu}\, dt \le \frac{c}{x-a} \int_0^{x-a} \frac{\omega(x-t-a)dt}{t^\alpha (x-t-a)^{\mu-a}} =$$

$$= \int_0^{(x-a/2} + \int_{(x-a)/2}^{x-a} .$$

Calculations and arguments similar to those in the case $0 < \mu < 1$ give the estimate $F_2 \le c\frac{\omega(x-a)}{(x-a)^\mu}$. Therefore,

$$F_2 \le c\frac{\omega(x-a)}{(x-a)^\gamma}, \quad \gamma = max(1, \mu). \tag{5.9}$$

So, from (5.8), (5.9) we obtain $|\psi_\varepsilon(x)| \le c\frac{\omega(x-a)}{(x-a)^\gamma} \in L_p$. Hence $\|\psi\|_{L_p} \le c$. □

References

1. N.K. Bari and S.B. Stechkin, Best approximation and differential properties of two conjugate functions. (Russian) *Trudy Mosk. Mat. Obshch.* **5**(1956), 483-522.

2. G.H. Hardy and J.E. Littlewood, Some properties of fractional integrals, I. *Math. Z.* **27**(1928), 565-606.

3. D.L. Herson and P. Heywood, On the range of some fractional integrals. *J. London Math. Soc.* (2), **8**(1974), 607-614.

4. N.K. Karapetyants, Kh.M. Murdaev and A.Ya. Yakubov, Zygmund type estimate for fractional integrodifferentiation in the spaces of functions defined by integral continuity moduli and isomorphism of generalized Hölder spaces. (Russian) *Rostov on Don. Deposited in VINITI (Moscow)* 20.07.90, No. 4096 − B90, 145pp.

5. N.K. Karapetyants, Kh.M. Murdaev and A.Ya. Yakubov, On isomorphism implemented by fractional integrals in generalized Hölder classes. (Russian) *Dokl. Akad. Nauk SSSR* **314**(1990), 288-291. *English translation: Sov. Math. Dokl.* **42**(1991), No.2, 446-450.

6. A.A. Kilbas, Potential type operators with power-logarithmic kernels in weighted Hölder spaces. (Russian) *Izv. Akad. nauk Byel. SSR, Ser. Fiz. Mat. Nauk,* (1978), No.2, 29-37.

7. E.R. Love, Fractional derivatives of imaginary order. *J. London Math. Soc., Ser.* **2**(1971), 3, 241-259.

8. E.R. Love, Two index laws for fractional integrals and derivatives. *J. Austral. Math. Soc.* **14**(1972), 385-410.

9. A. Marchaud, Sur les derivees et sur les differences dans fonctions de variables reels. *J. math. pures et appl.* **6**(1927), 337-425.

10. Kh.M. Murdaev, An estimate for continuity modulus of integrals and derivatives of fractional order. (Russian) *Groznyi. Deposited in VINITI (Moscow)* 14.06.85, No.4209.

11. Kh.M. Murdaev and S.G. Samko, Weighted estimates for continuity moduli of fractional integrals of functions having a prescribed continuity modulus with a weight. (Russian) *Rostov-on-Don, Deposited in VINITI (Moscow)* 11.05.86, No.3351-B, 42pp.

12. Kh.M. Murdaev and S.G. Samko, Mapping properties of fractional integro-differentiation in weighted generalized Hölder spaces $H_0^\omega(\rho)$ with the weight $\rho(x) = (x-a)^\mu (b-x)^\nu$. (Russian) *Rostov-on-Don, Deposited in VINITI (Moscow)* 11.05.86, No.3350-B, 25pp.

13. B.S. Rubin, Fractional integrals in Hölder spaces with a weight and potential type operators. (Russian) *Izv. Akad. Nauk Arm. SSR, Ser. Mat.* **9**(1974), 308-324.

14. N.G. Samko, Weighted Zygmund estimate for the singular operator and a theorem on its boundedness in $H_0^\omega(\rho)$ in the case of general weights. (Russian) *Rostov-on-Don, Deposited in VINITI (Moscow)* 5.12.89, No.7559-B89.

15. —–, A singular operator and operators with a weak singularity in weighted generalized Hölder spaces. (Russian) *Izv. Sev.-Kavk. Nauchn. Tsentra. Vysch. Scholy, Ser. Estestv. Nauk* 1991, No.3, 46-69.

16. S.G. Samko, The space $I^\alpha(L_p)$ of fractional integrals and potential type operators. (Russian) *Izv. Akad. nauk Armyan. SSR, Ser. Mat.* **8**(1973), 359-383.

17. S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional integrals and derivatives and some of their applications. (Russian) *Minsk, Nauka i Tekhnika,* 1980, English translation: *Gordon and Breach, to appear.*

18. S.G. Samko and Kh.M. Murdaev, Weighted Zygmund estimates for fractional differentiation and integration and their applications. (Russian) *Trudy Matem. Inst. Steklov.* **180**(1987), 197-198; English translation: *Proc. Steklov Inst. Math.* 1989, No.3, 233-235.

(Received 15.07.1993)

Author's address:
Rostov University
Department of Mathematics and Mechanics
105, ul. Bol'shaya Sadovaya,
Rostov-on-Don, 344711
Russia