

## ON THE DIMENSION MODULO CLASSES OF TOPOLOGICAL SPACES AND FREE TOPOLOGICAL GROUPS

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**Abstract.** Functions of dimension modulo a (rather wide) class of spaces are considered and the conditions are found, under which the dimension of the product of spaces modulo these classes is equal to zero. Based on these results, the sufficient conditions are established, under which spaces of free topological semigroups (in the sense of Marxen) and spaces of free topological groups (in the sense of Markov and Graev) are zero-dimensional modulo classes of compact spaces.

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The question when the product of two spaces is  $\pi$ -compact (i.e., has an open base, each element of which has a compact boundary) was studied in various particular cases in [17], [10] and in a general case in [5].

According to A. Lelek and J. de Groot (see, e.g., [1] and [14]), if a function of dimension type is considered modulo a class of compact spaces which is defined by the method of induction analogously to the function  $ind$ , then  $\pi$ -compact spaces turn out to be zero-dimensional (modulo a class of compact spaces).

In this paper, functions of dimension modulo a (rather wide) class of spaces are considered and the conditions are found, under which the dimension of the product of spaces modulo these classes is equal to zero. These results can therefore be regarded as a generalization of the results of [17], [10] and [5]. Moreover, based on the same results of [17], the sufficient conditions are established, under which spaces of free topological semigroups (in the sense of Marxen) and spaces of free topological groups (in the sense of Markov and Graev) are zero-dimensional modulo the class of compact spaces.

The paper consists of three parts. Part 1 contains the notation. In Part 2 the necessary definitions and statements are formulated. In Part 3 the conditions, under which the dimension modulo classes of a product of spaces is equal to zero, are established. Based on these results the structures of free topological semi-groups and free topological groups are determined.

### 1. NOTATION

All spaces considered in this paper are assumed to be Hausdorff and completely regular (Tychonoff). By  $\mathfrak{M}$  (with or without an index) we denote topologically closed subclasses of the class of all completely regular spaces (that  $\mathfrak{M}$

is topologically closed means the following: if  $X \in \mathfrak{M}$  and  $X$  is homeomorphic to  $Y$ , then  $Y \in \mathfrak{M}$ ).

For each topological space  $X$  and for any of its subset  $U$  by  $[U]_X$  we denote the closure of  $U$  in  $X$ , and by  $Fr_X U$  the boundary of  $U$  in  $X$ .

Also, the following notation is used:

$\mathfrak{M}_c$  is a class of all compact spaces;

$\mathfrak{M}_{mc}$  is the class of all metrizable compact spaces;

$\mathfrak{M}_{dc}$  is the class of all Dugundji compacts [2];

$\mathfrak{M}_{cor}$  is the class of all Corson compacts [2];

$\mathfrak{M}_{\aleph_0}$  is the class of all countable spaces;

$\mathfrak{M}_{cn}$  is the class of spaces with countable networks in the sense of Arkhangel'skii [2];

$\mathfrak{M}_{con}$  is the class of all connected spaces;

$\mathfrak{M}_{ps}$  is the class of all pseudocompact spaces;

$\mathfrak{M}_{cc}$  is the class of all countably compact spaces;

$\mathfrak{M}_{par}$  is the class of all paracompact spaces;

$\mathfrak{M}_{pp}$  is the class of all weakly paracompact spaces;

by  $N$  we denote the set of all integers  $\geq -1$ , i.e.,  $N = \{-1, 0, 1, 2, \dots\}$ .

## 2. DEFINITIONS AND NECESSARY AUXILIARY PROPOSITIONS

**Definition 2.1.** Let  $\mathfrak{M}$  be some topologically closed subclass of the class of completely regular spaces. Denote by  $loc \mathfrak{M}$  the class of all spaces possessing the following property:  $X \in loc \mathfrak{M}$  if and only if for each point  $x$  of the space  $X$  and each open set  $U_x$  such that  $x \in U_x$  there exists an open set  $V_x$  in  $X$  such that  $x \in V_x \subseteq [V_x]_X \subseteq U_x$  and  $[V_x]_X \in \mathfrak{M}$ .

*Remark 2.1.* It is obvious that for each topologically closed class  $\mathfrak{M}$ ,  $loc \mathfrak{M}$  is also topologically closed.

**Definition 2.2.** The class  $\mathfrak{M}$  is called  $c$ -monotone if  $\mathfrak{M}$  is topologically closed and any closed subspace of each space from the class  $\mathfrak{M}$  also belongs to  $\mathfrak{M}$ .

**Definition 2.3.** Let  $\mathfrak{M}$  be some class of spaces. For each space  $X$  (from the class of all completely regular spaces) the small inductive dimension  $in(X, \mathfrak{M})$  modulo  $\mathfrak{M}$  is defined as follows:

(i)  $in(X, \mathfrak{M}) = -1$  if  $X \in \mathfrak{M}$ ;

(ii) for each  $n \geq 0$ ,  $in(X, \mathfrak{M}) \leq n$  if for each point  $x$  of the space  $X$  and for each neighborhood of this point  $U_x$  there exists an open neighborhood  $V_x$  in  $X$  such that  $x \in V_x \subseteq U_x$  and  $in(Fr_X V_x, \mathfrak{M}) \leq n - 1$ ;  $in(X, \mathfrak{M}) = n$  if  $in(X, \mathfrak{M}) \leq n$  and  $in(X, \mathfrak{M}) \not\leq n - 1$  and finally,  $in(X, \mathfrak{M}) = +\infty$  if  $in(X, \mathfrak{M}) \not\leq n$  for each  $n \in N$ .

*Remark 2.2.* If  $\mathfrak{M}$  is the empty class, then for any space  $X$  we have:  $in(X, \mathfrak{M}) = ind X$  (where  $ind X$  is the small inductive dimension of  $X$  [14]).

*Remark 2.3.*  $in(X, \mathfrak{M}) \leq 0$  if and only if either 1)  $X \in \mathfrak{M}$  or 2) there exists in  $X$  an open base  $\{U_a\}_{a \in M}$  such that  $Fr_X U_a \in \mathfrak{M}$  for each  $a \in M$ . In

particular, for each regular space  $X$  we have:  $in(X, \mathfrak{M}_c) \leq 0$  if and only if  $X$  is semicompact ( $\equiv \pi$ -compact), see [8].

**Definition 2.4.** The class  $\mathfrak{M}$  is called stable if the following conditions are fulfilled:

- 1)  $\mathfrak{M}$  is topologically closed;
- 2)  $\mathfrak{M}$  is  $c$ -monotone;
- 3) if  $X \in \mathfrak{M}$  and  $Y$  is a continuous image of the space  $X$ , then  $Y \in \mathfrak{M}$ .

**Definition 2.5.** The class  $\mathfrak{M}$  is called multiplicative if the following conditions are fulfilled:

- 1)  $\mathfrak{M}$  is topologically closed;
- 2) if  $X \in \mathfrak{M}$  and  $Y \in \mathfrak{M}$ , then  $X \times Y \in \mathfrak{M}$ , where  $X \times Y$  denotes the usual product of topological spaces  $X$  and  $Y$ .

The classes  $\mathfrak{M}_c, \mathfrak{M}_{mc}, \mathfrak{M}_{dc}, \mathfrak{M}_{cor}, \mathfrak{M}_{\aleph_0}, \mathfrak{M}_{cn}$  are examples of stable multiplicative classes.

**Definition 2.6.** The class  $\mathfrak{M}$  is called weakly stable if it has the following property: If  $X \in \mathfrak{M}$  and  $f : X \rightarrow Y$  is a continuous closed mapping “onto”, then  $Y \in \mathfrak{M}$ .

*Remark 2.4.* One can easily show that if  $\mathfrak{M}$  is a stable class and  $f : X \rightarrow Y$  is an open continuous mapping of a space  $X$  onto a space  $Y$  and  $X \in loc \mathfrak{M}$  then  $Y \in loc \mathfrak{M}$ .

Next, we formulate some needed definitions.

**Definition 2.7** ([12]). Let  $X$  be a completely regular topological space. A pair  $(S(X), \Theta)$  is called a free topological semigroup generated by the space  $X$  if the following conditions are fulfilled:

- 1)  $S(X)$  is a topological semigroup the space of which is completely regular;
- 2)  $\Theta : X \rightarrow S(X)$  is a topological embedding of the space  $X$  in  $S(X)$ ;
- 3)  $\Theta(X)$  generates  $S(X)$  algebraically;
- 4) for each completely regular topological semigroup  $T$  and for each continuous mapping  $\omega : X \rightarrow T$  of the space  $X$  to  $T$ , there exists a unique continuous homomorphism  $\Omega : S(X) \rightarrow T$  such that the composition  $\Omega\Theta$  is equal to  $\omega$ .

In [12] and [4] it is shown that for the completely regular topological space  $X$ , a free topological semigroup generated by  $X$  exists and is unique.

**Definition 2.8** (A. A. Markov [11]). Let  $X$  be a completely regular space. A Hausdorff topological group  $FM(X)$  (a Hausdorff abelian topological group  $AM(X)$ ) is called a free topological group in the sense of Markov (a free abelian topological group in the sense of Markov) of the space  $X$  if  $X$  is a subspace of  $FM(X)$  (resp. of  $AM(X)$ ) and for each continuous mapping  $\Phi : X \rightarrow G$  of the space  $X$  into an arbitrary topological group (into an arbitrary abelian topological group)  $G$  there exists a unique continuous homomorphism  $\Phi_F : FM(X) \rightarrow G$  (resp.  $\Phi_A : AM(X) \rightarrow G$ ) such that  $\Phi_F|_X = \Phi$  (resp.  $\Phi_A|_X = \Phi$ ).

**Definition 2.9** (M.N. Graev [9]). Let  $X$  be a completely regular space with a distinguished point  $e$ . A Hausdorff topological group  $FG(X)$  (a Hausdorff abelian topological group  $AG(X)$ ) is called a free topological group in the sense of Graev (a free abelian topological group in the sense of Graev) of the space  $X$  if the following conditions are fulfilled:

- 1)  $X$  is a subspace of  $FG(X)$  (resp. of  $AG(X)$ );
- 2)  $X$  algebraically generates  $FG(X)$  (resp.  $AG(X)$ );
- 3) for each continuous mapping  $\varphi : X \rightarrow Q$  of the space  $X$  into an arbitrary topological group (an arbitrary topological abelian group)  $Q$ , which sends the point  $e$  of  $X$  to the unit (to the zero element) of the group  $Q$ , there exists a continuous homomorphism  $\Phi_F : FG(X) \rightarrow Q$  (resp.  $\Phi_A : AG(X) \rightarrow Q$ ) such that  $\Phi_F|_X = \varphi$  (resp.  $\Phi_A|_X = \varphi$ ).

A. A. Markov and M. N. Graev showed ([11], [9]) the existence and uniqueness (in the corresponding categories) of the groups  $FM(X)$ ,  $AM(X)$ ,  $FG(X)$  and  $AG(X)$ .

### 3. ON THE ZERO-DIMENSIONALITY BY MODULO CLASS OF TOPOLOGICAL SPACES TOPOLOGICAL PRODUCTS, FREE TOPOLOGICAL GROUPS AND SEMIGROUPS

The following theorem is valid:

**Theorem 3.1.** *If  $Z = X \times Y$ , where  $X$  and  $Y$  are topological spaces,  $Y$  is a nontrivial connected compact and in  $(X \times Y, \mathfrak{M}) = 0$ , where  $\mathfrak{M}$  is a weakly stable class, then  $X \in \text{loc } \mathfrak{M}$ .*

*Proof.* Let us consider a point  $x \in X$  and an open neighborhood  $O_x$  of  $x$ . Let  $U$  be some open in  $Y$  set with  $U \neq \emptyset$  and  $U \neq Y$ . Take some point  $y \in U$ . Let  $z = (x, y)$ . Denote  $U_z = O_x \times U \subseteq W_x$ . Since in  $(X \times Y, \mathfrak{M}) \leq 0$ , there exists open  $H_z$  such that  $z \in H_z \subseteq [H_z] \subseteq U_z$  and  $\text{Fr}_Z H_z \in \mathfrak{M}$ . Denote  $T_x = \text{Pr}_X(H_z)$ , where  $\text{Pr}_X : X \times Y \rightarrow X$  is the projection. Obviously,  $x \in T_x \subseteq [T_x]_X \subseteq O_x$ . Show that  $T_x \subseteq \text{Pr}_X(\text{Fr}_Z H_z)$ . Indeed, take any  $x' \in T_x$  and consider the set  $\{x'\} \times Y = A_{x'}$ . Denote  $A_{x'} \cap H_z = B_{x'}$ . Show that  $B_{x'} \neq \emptyset$ . Indeed, if  $B_{x'} = \emptyset$ , then  $\text{Pr}_X(B_{x'}) = \emptyset$ . But  $x' \in \text{Pr}_X(B_{x'})$ , hence, it is not empty. On the other hand, since  $H_z$  is open in  $Z$ , the set  $B_{x'}$  is open in  $A_{x'}$ . Now let us show that 1)  $\text{Fr}_{A_{x'}} B_{x'} \subseteq \text{Fr}_Z H_z$  and 2)  $\text{Fr}_{A_{x'}} B_{x'} \neq \emptyset$ .

First let us show 1).  $\text{Fr}_{A_{x'}} B_{x'} = [B_{x'}]_{A_{x'}} \setminus B_{x'}$ , where  $[B_{x'}]_{A_{x'}} = [B_{x'}]_Z$  (because  $A_{x'}$  is closed in  $Z$ ). But  $[B_{x'}]_Z \subseteq [A_{x'}]_Z \cap [H_z]_Z = A_{x'} \cap [H_z]_Z$  (here we use the inclusion  $[A \cap B] \subseteq [A] \cap [B]$  and the fact that  $A_{x'}$  is closed in  $Z$ ).

Furthermore  $[B_{x'}]_Z \setminus B_{x'} \subseteq (A_{x'} \cap [H_z]_Z) \setminus B_{x'}$  (here we use the following: if  $A \subseteq B$ , then for any  $C$  we have  $A \setminus C \subseteq B \setminus C$ ). Let us show that  $(A_{x'} \cap [H_z]_Z) \setminus (A_{x'} \cap H_z) \subseteq [H_z]_Z \setminus H_z$  (i.e.,  $(A_{x'} \cap [H_z]_Z) \setminus B_{x'} \subseteq [H_z]_Z \setminus H_z$ ). Indeed if  $x'' \in A_{x'} \cap [H_z]_Z \setminus A_{x'} \cap H_z$ , then  $x'' \in A_{x'} \cap [H_z]_Z$  and  $x'' \notin A_{x'} \cap H_z$ . Then  $x'' \in A_{x'}$ ,  $x'' \in [H_z]_Z$  and  $x'' \notin H_z$ . Hence, in particular,  $x'' \in [H_z]_Z \setminus H_z$ . Thus  $[B_{x'}]_Z \setminus B_{x'} \subseteq [H_z]_Z \setminus H_z$ . So we have  $\text{Fr}_{A_{x'}} B_{x'} = [B_{x'}]_{A_{x'}} \setminus B_{x'} = [B_{x'}]_Z \setminus B_{x'} \subseteq [H_z]_Z \setminus H_z = \text{Fr}_Z H_z$ . Thus 1) is proved.

Now let us prove 2). Let us consider the mapping  $\pi_{x'} : A_{x'} \rightarrow Y$  determined as follows:  $\pi_{x'}(x', y) = y$ . This mapping is one-to-one and surjective. The mapping  $\pi_{x'}$  is continuous because it is the restriction of the continuous mapping. Since  $Y$  is compact and  $\{x'\} \times Y = A_{x'}$ ,  $A_{x'}$  is compact and so the mapping  $\pi_{x'}$  is a homeomorphism (since continuous and one-to-one mapping from a compact space onto a Hausdorff space is a homeomorphism). If  $Fr_{A_{x'}} B_{x'} = \emptyset$ , then  $B_{x'}$  is closed and open in  $A_{x'}$ . Let us note that  $\pi_{x'}(B_{x'}) \subseteq U$ . Since  $\pi_{x'}$  is a homeomorphism, the set  $\pi_{x'}(B_{x'})$  is closed and open in  $Y$ . But this is impossible because  $\pi_{x'}(B_{x'}) \subset U \neq Y$ ,  $\pi_{x'}(B_{x'}) \neq \emptyset$  and  $Y$  is connected. Therefore our assumption is false. So  $Fr_{A_{x'}} B_{x'} \neq \emptyset$ .

Thus we see that for every  $x' \in T_x$  there exists a point  $z' \in Fr_z H_z$  such that  $x' = Pr_x(z')$ . This means that  $T_x \subseteq Pr_X(Fr_Z H_z)$ .  $H_z$  is dense in  $[H_z]$ . Therefore  $T_x = Pr_X(H_z)$  is dense in  $Pr_X([H_z]_Z)$ . Hence,  $[T_x]_X = Pr_X([H_z]_Z)$ . But since  $Y$  is compact,  $Pr_X : X \times Y \rightarrow X$  is the closed mapping, i.e.,  $Pr_X([H_z]_Z)$  is closed in  $X$ . On the other hand,  $Fr_Z H_z \subseteq [H_z]_Z$ . Thus we have  $T_x \subseteq Pr_X(Fr_Z H_z) \subseteq Pr_X([H_z]_Z)$ . But  $[T_x]_X = Pr_X([H_z]_Z)$ , so  $[T_x]_X \subseteq [Pr_X(Fr_Z H_z)]_X \subseteq [Pr_X[H_z]_Z]_X$ . Thus  $[T_x]_X = [Pr_X(Fr_Z H_z)]_X = Pr_X(Fr_Z H_z)$ . Since  $Pr_X : X \times Y \rightarrow X$  is the closed mapping and  $Fr_Z H_z \in \mathfrak{M}$ , by weak stability,  $Pr_X(Fr_Z H_z) \in \mathfrak{M}$ . So we have found a neighborhood  $T_x$  of the point  $x$  such that  $T_x \subset [T_x]_X \subset O_x$  and  $[T_x]_X \in \mathfrak{M}$ . Thus  $X \in loc \mathfrak{M}$ . □

**Corollary 3.1.** *If  $Z = X \times Y$  where  $X$  and  $Y$  are topological spaces,  $Y$  is a connected compact and  $in(X \times Y, \mathfrak{M}_{con}) = 0$ , then  $X \in loc \mathfrak{M}_{con}$ .*

**Corollary 3.2.** *If  $Z = X \times Y$  where  $X$  and  $Y$  are topological spaces,  $Y$  is a connected compact and  $in(X \times Y, \mathfrak{M}_{ps}) = 0$ , then  $X \in loc \mathfrak{M}_{ps}$ .*

**Corollary 3.3.** *If  $Z = X \times Y$  where  $X$  and  $Y$  are topological spaces,  $Y$  is a connected compact and  $in(X \times Y, \mathfrak{M}_{cor}) = 0$ , then  $X \in loc \mathfrak{M}_{cor}$ .*

**Corollary 3.4.** *If  $Z = X \times Y$ , where  $X$  and  $Y$  are topological spaces,  $Y$  is a connected compact and  $in(X \times Y, \mathfrak{M}_{par}) = 0$ , then  $X \in loc \mathfrak{M}_{par}$ .*

**Corollary 3.5.** *If  $Z = X \times Y$ , where  $X$  and  $Y$  are topological spaces,  $Y$  is a connected compact and  $in(X \times Y, \mathfrak{M}_{pp}) = 0$ , then  $X \in loc \mathfrak{M}_{pp}$ .*

**Corollary 3.6.** *If  $Z = X \times Y$ , where  $X$  and  $Y$  are topological spaces,  $Y$  is a connected compact and  $in(X \times Y, \mathfrak{M}_c) = 0$ , then  $X \in loc \mathfrak{M}_c$ .*

As it is shown in [1], the following theorem holds.

**Theorem 3.2.** *If  $X \times Y$  is a topological product of the spaces  $X$  and  $Y$  such that  $in(X \times Y, \mathfrak{M}) \leq 0$ , where  $\mathfrak{M}$  is a stable class and  $ind X > 0$ , then  $Y \in loc \mathfrak{M}$ .*

In the sequel we shall need the following propositions which are easy to prove.

**Proposition 3.1.** *If  $\mathfrak{M}$  is a  $c$ -monotone class and  $X \in loc \mathfrak{M}$ , then  $in(X, \mathfrak{M}) \leq 0$ .*

*Proof.* Let  $x \in X$  and  $O_x$  be some open neighborhood of  $x$  in  $X$ . Since  $X \in \text{loc } \mathfrak{M}$ , there is an open neighborhood  $V_x$  such that  $x \in V_x \subseteq [V_x]_X \subseteq O_x$  and  $[V_x]_X \in \mathfrak{M}$ . But  $\text{Fr}_X V_x \subseteq [V_x]_X$  and since  $[V_x]_X \in \mathfrak{M}$  and  $\text{Fr}_X V_x$  is closed in  $X$  (and therefore in  $[V_x]_X$  too), then, by  $c$ -monotony,  $\text{Fr}_X V_x \in \mathfrak{M}$ , which means that  $\text{in}(X, \mathfrak{M}) \leq 0$ .  $\square$

**Proposition 3.2.** *If  $\mathfrak{M}$  is a multiplicative class, then the class  $\text{loc } \mathfrak{M}$  is also multiplicative.*

*Proof.*  $X \in \text{loc } \mathfrak{M}$  means that for every point  $x$  of  $X$  and for every its open neighborhood  $O_x$  there exists an open neighborhood  $V_x$  of  $x$  such that  $x \in V_x \subseteq [V_x]_X \subseteq O_x$  and  $[V_x]_X \in \mathfrak{M}$ . Also,  $Y \in \text{loc } \mathfrak{M}$  means that for every point  $y \in Y$  and any its open neighborhood  $O_y$  there is an open neighborhood  $V_y$  of  $y$  such that  $y \in V_y \subseteq [V_y]_Y \subseteq O_y$  and  $[V_y]_Y \in \mathfrak{M}$ . Denote  $Z = X \times Y$ . Consider the point  $z = (x, y) \in Z$  and its open neighborhood  $O_z = O_x \times O_y$ . We have  $z = (x, y) \in V_x \times V_y \subseteq [V_x]_X \times [V_y]_Y \subseteq O_x \times O_y$ . Since  $\mathfrak{M}$  is multiplicative,  $[V_x]_X \times [V_y]_Y \in \mathfrak{M}$ , i.e.,  $[V_x \times V_y]_Z \in \mathfrak{M}$ . Since the point  $z = (x, y)$  was selected arbitrarily in  $(X \times Y)$  and since the family of neighborhoods of the form  $O_z = O_x \times O_y$  is the base of the space  $X \times Y$  at the point  $z = (x, y)$ , then  $X \times Y \in \text{loc } \mathfrak{M}$ , i.e.,  $\text{loc } \mathfrak{M}$  is multiplicative.  $\square$

**Proposition 3.3.** *If  $\mathfrak{M}$  is a  $c$ -monotone class,  $A$  is closed in  $X$  and  $\text{in}(X, \mathfrak{M}) \leq 0$ , then  $\text{in}(A, \mathfrak{M}) \leq 0$ .*

*Proof.* Let us consider a point  $x \in A$  and an open in  $A$  neighborhood  $O_x$  of  $x$ . Then there exists an open neighborhood  $O'_x$  of  $x$  in  $X$  such that  $O'_x \cap A = O_x$ . Then there is  $V'_x$ , open in  $X$ , such that  $x \in V'_x \subseteq O'_x$  and  $\text{Fr}_X V'_x \in \mathfrak{M}$ . Let us consider  $V'_x \cap A = V_x$ .

Clearly,  $V_x$  is open in  $A$ ,  $V_x \subseteq O'_x$  and  $\text{Fr}_A V_x \subseteq \text{Fr}_X V'_x$ . Furthermore  $\text{Fr}_A V_x$  is closed in  $\text{Fr}_X V'_x$ . Since  $\text{Fr}_X V'_x \in \mathfrak{M}$ , by  $c$ -monotony,  $\text{Fr}_A V_x \in \mathfrak{M}$ . Thus we obtain that for every  $x \in A$  and any its neighborhood  $O_x$  there is an open neighborhood  $V_x$  of  $x$  such that  $x \in V_x \subseteq O_x$  and  $\text{Fr}_A V_x \in \mathfrak{M}$ . This means that  $\text{in}(A, \mathfrak{M}) \leq 0$ .  $\square$

The following is true.

**Theorem 3.3.** *Let  $\mathfrak{M}$  be a stable multiplicative class. Then the inequality,  $\text{in}(X \times Y, \mathfrak{M}) \leq 0$  takes place if and only if one of the following four conditions is fulfilled:*

- 1)  $\text{ind } X \leq 0, \text{ind } Y \leq 0$ ;
- 2)  $X \in \text{loc } \mathfrak{M}, Y \in \text{loc } \mathfrak{M}$ ;
- 3)  $X \in \text{loc } \mathfrak{M}, \text{ind } X = 0$  and  $\text{in}(Y, \mathfrak{M}) \leq 0$ ;
- 4)  $Y \in \text{loc } \mathfrak{M}, \text{ind } Y = 0$  and  $\text{in}(X, \mathfrak{M}) \leq 0$ .

*Proof. Necessity.* Let us show that if  $\text{in}(X \times Y, \mathfrak{M}) \leq 0$ , then one of the conditions given above is fulfilled. Let us consider four possible cases:

1.  $\text{ind } X = 0, \text{ind } Y = 0$ ;
2.  $\text{ind } X > 0, \text{ind } Y > 0$ ;
3.  $\text{ind } X > 0, \text{ind } Y = 0$ ;

4.  $ind X = 0, ind Y > 0$ .

1. If  $ind X = 0$  and  $ind Y = 0$ , then there is nothing to prove.

2. If  $ind X > 0$  and  $ind Y > 0$ , then, by Theorem 3.2,  $Y \in loc \mathfrak{M}$  and  $X \in loc \mathfrak{M}$ .

3. If  $ind X > 0$  and  $ind Y = 0$ , then, by Theorem 3.1,  $Y \in loc \mathfrak{M}$  and  $ind Y = 0$ .

4. The proof is similar to the proof in case 3.

*Sufficiency.* 1) If  $ind X \leq 0, ind Y \leq 0$ , then  $ind (X \times Y) \leq 0$  (see [7]). Then since (by  $c$ -monotony)  $\emptyset \in \mathfrak{M}$ , we have  $in (X \times Y, \mathfrak{M}) \leq 0$ .

2) If  $X \in loc \mathfrak{M}$  and  $Y \in loc \mathfrak{M}$ , then, by Proposition 3.2,  $X \times Y \in loc \mathfrak{M}$ . Then by Proposition 3.1, we have:  $in (X \times Y, \mathfrak{M}) \leq 0$ . Suppose  $X \in loc \mathfrak{M}$ ,  $ind X = 0$  and  $in (Y, \mathfrak{M}) \leq 0$ . Let us show that  $in (X \times Y, \mathfrak{M}) \leq 0$ .

3) Suppose  $z = (x, y)$  is any point of the space  $X \times Y$  and  $U_z$  is any neighborhood of this point. Then there is a neighborhood  $O_x$  of  $x$  and there is a neighborhood  $O_y$  of  $y$  such that  $z \in O_x \times O_y \subseteq U_z$ . Since  $X \in loc \mathfrak{M}$ , there exists  $V_x$  such that  $x \in V_x \subseteq [V_x] \subseteq O_x$  and  $[V_x]_X \in \mathfrak{M}$ . Since  $ind X = 0$ , there is  $W_X$  such that  $x \in W_x \subseteq V_x \subseteq [V_x] \subseteq O_x$  and  $W_X$  is closed and open in  $X$  (i.e.,  $Fr_X W_x = \emptyset$ ). Thus  $W_x$  is closed and open in  $[V_x]_X$  too. Since  $\mathfrak{M}$  is  $c$ -monotone,  $W_x \in \mathfrak{M}$ . So  $W_x$  is closed and open and  $W_x \in \mathfrak{M}$ . Since  $in (Y, \mathfrak{M}) \leq 0$ , there exists an open set  $W_y$  such that  $y \in W_y \subseteq [W_y]_Y \subseteq O_y$  and  $Fr_Y W_y \in \mathfrak{M}$ .

Let us consider  $W_z = W_x \times W_y$ . Then  $z \in W_z = W_x \times W_y \subseteq O_x \times O_y \subseteq U_z$  and  $Fr_Z W_z = ([W_x]_X \times Fr_Y W_y) \cap (Fr_X W_x \times [W_y]_Y)$ . Since  $Fr_X W_x = \emptyset$ ,  $(Fr_X W_x \times [W_y]_Y) = \emptyset$ , i.e.,  $Fr_Z W_z = [W_x]_X \times Fr_Y W_y = W_x \times Fr_Y W_y$  because  $[W_x]_X = W_x$  (since it is closed and open). But  $W_x \in \mathfrak{M}$  and  $Fr_Y W_y \in \mathfrak{M}$ . Since  $\mathfrak{M}$  is multiplicative, we have  $W_x \times Fr_Y W_y \in \mathfrak{M}$ . So  $Fr_Z W_z \in \mathfrak{M}$ . Since the point  $x$  and its open neighborhood  $O_z$  were chosen arbitrarily,  $in (X \times Y, \mathfrak{M}) \leq 0$ .

The fourth case is similar to the third one. □

**Corollary 3.7.**  $in (X \times Y, \mathfrak{M}_c) \leq 0$  if and only if one of the following four conditions is fulfilled:

- 1)  $ind X = 0, ind Y = 0$ ;
- 2)  $X \in loc \mathfrak{M}_c, X \in loc \mathfrak{M}_c$ ;
- 3)  $X \in loc \mathfrak{M}_c, ind X = 0$  and  $in (Y, \mathfrak{M}_c) \leq 0$ ;
- 4)  $Y \in loc \mathfrak{M}_c, ind Y = 0$  and  $in (X, \mathfrak{M}_c) \leq 0$ .

**Corollary 3.8.**  $in (X \times Y, \mathfrak{M}_{mc}) \leq 0$  if and only if one of the following four conditions is fulfilled:

- 1)  $ind X = 0, ind Y = 0$ ;
- 2)  $X \in loc \mathfrak{M}_{mc}, X \in loc \mathfrak{M}_{mc}$ ;
- 3)  $X \in loc \mathfrak{M}_{mc}, ind X = 0$  and  $in (Y, \mathfrak{M}_{mc}) \leq 0$ ;
- 4)  $Y \in loc \mathfrak{M}_{mc}, ind Y = 0$  and  $in (X, \mathfrak{M}_{mc}) \leq 0$ .

**Corollary 3.9.**  $in (X \times Y, \mathfrak{M}_{dc}) \leq 0$  if and only if one of the following four conditions is fulfilled:

- 1)  $ind X = 0, ind Y = 0$ ;
- 2)  $X \in loc \mathfrak{M}_{dc}, X \in loc \mathfrak{M}_{dc}$ ;

- 3)  $X \in \text{loc } \mathfrak{M}_{dc}$ ,  $\text{ind } X = 0$  and  $\text{in } (Y, \mathfrak{M}_{dc}) \leq 0$ ;
- 4)  $Y \in \text{loc } \mathfrak{M}_{dc}$ ,  $\text{ind } Y = 0$  and  $\text{in } (X, \mathfrak{M}_{dc}) \leq 0$ .

**Corollary 3.10.**  $\text{in } (X \times Y, \mathfrak{M}_{cor}) \leq 0$  if and only if one of the following four conditions is fulfilled:

- 1)  $\text{ind } X = 0$ ,  $\text{ind } Y = 0$ ;
- 2)  $X \in \text{loc } \mathfrak{M}_{cor}$ ,  $X \in \text{loc } \mathfrak{M}_{cor}$ ;
- 3)  $X \in \text{loc } \mathfrak{M}_{cor}$ ,  $\text{ind } X = 0$  and  $\text{in } (Y, \mathfrak{M}_{cor}) \leq 0$ ;
- 4)  $Y \in \text{loc } \mathfrak{M}_{cor}$ ,  $\text{ind } Y = 0$  and  $\text{in } (X, \mathfrak{M}_{cor}) \leq 0$ .

**Corollary 3.11.**  $\text{in } (X \times Y, \mathfrak{M}_{\aleph_0}) \leq 0$  if and only if one of the following four conditions is fulfilled:

- 1)  $\text{ind } X = 0$ ,  $\text{ind } Y = 0$ ;
- 2)  $X \in \text{loc } \mathfrak{M}_{\aleph_0}$ ,  $X \in \text{loc } \mathfrak{M}_{\aleph_0}$ ;
- 3)  $X \in \text{loc } \mathfrak{M}_{\aleph_0}$ ,  $\text{ind } X = 0$  and  $\text{in } (Y, \mathfrak{M}_{\aleph_0}) \leq 0$ ;
- 4)  $Y \in \text{loc } \mathfrak{M}_{\aleph_0}$ ,  $\text{ind } Y = 0$  and  $\text{in } (X, \mathfrak{M}_{\aleph_0}) \leq 0$ .

**Corollary 3.12.**  $\text{in } (X \times Y, \mathfrak{M}_{cn}) \leq 0$  if and only if one of the following four conditions is fulfilled:

- 1)  $\text{ind } X = 0$ ,  $\text{ind } Y = 0$ ;
- 2)  $X \in \text{loc } \mathfrak{M}_{cn}$ ,  $X \in \text{loc } \mathfrak{M}_{cn}$ ;
- 3)  $X \in \text{loc } \mathfrak{M}_{cn}$ ,  $\text{ind } X = 0$  and  $\text{in } (Y, \mathfrak{M}_{cn}) \leq 0$ ;
- 4)  $Y \in \text{loc } \mathfrak{M}_{cn}$ ,  $\text{ind } Y = 0$  and  $\text{in } (X, \mathfrak{M}_{cn}) \leq 0$ .

Based on the obtained results, we determine the structure of free topological semigroups (in the sense of Marxen) and free topological groups (in the sense of Markov [11] and Graev [9]) in the case where these objects are zero-dimensional modulo a certain class.

**Theorem 3.4.** *Let  $\mathfrak{M}$  be a stable multiplicative class. Then the space  $S(X)$  of free topological semigroups of the space  $X$  has  $\text{in } (S(X), \mathfrak{M}) \leq 0$  if and only if either  $\text{ind } X = 0$  or  $X \in \text{loc } \mathfrak{M}$ .*

*Proof.* Marxen showed in [12] that the space of the topological semi-group  $S(X)$  of any space  $X$  is of the form  $S(X) = \bigvee_{n=1}^{\infty} X^n$ , where  $\bigvee$  is a symbol for a topological sum and  $X^n$  is the  $n^{\text{th}}$  power of  $X$ .

As is known, if  $\text{ind } X = 0$ , then for every  $n \geq 1$  we have  $\text{ind } X^n = 0$ , but the topological sum of zero-dimensional (in sense of  $\text{ind}$ ), spaces is still zero-dimensional (in sense of  $\text{ind}$ ). Thus  $\text{ind } S(X) = 0$ . Since  $\mathfrak{M}$  is a stable class, we have  $\text{in } (S(X), \mathfrak{M}) \leq 0$ . If  $X \in \text{loc } \mathfrak{M}$ , then for every  $n \geq 1$  (by multiplicity of  $\mathfrak{M}$ ) we have  $X^n \in \text{loc } \mathfrak{M}$ . Now let us show that  $S(X) \in \text{loc } \mathfrak{M}$ . Suppose  $x \in S(X)$  is a point chosen arbitrarily and  $O_x$  is any its neighborhood in  $S(X)$ . Then there is unique  $n$  such that  $x \in X^n$ . Let us consider  $O'_x = O_x \cap X^n$ . Since  $X^n \in \text{loc } \mathfrak{M}$ , there is a neighborhood  $V'_x$  such that  $x \in V'_x \subseteq [V'_x]_{X^n} \subseteq O'_x$  and  $[V'_x]_{X^n} \in \mathfrak{M}$ .  $V'_x$  is open in  $X^n$  and  $X^n$  is open in  $S(X)$ . So  $V'_x$  is open in  $S(X)$ . Thus we have found an open neighborhood  $V_x$  in  $S(X)$  such that  $x \in V_x \subseteq [V_x]_{X^n} = [V_x]_{S(X)} \subseteq O'_x \subseteq O_x$  and  $[V_x]_{S(X)} \in \mathfrak{M}$ . Consequently,  $S(X) \in \text{loc } \mathfrak{M}$ . Therefore, by Proposition 3.1,  $\text{in } (S(X), \mathfrak{M}) \leq 0$ .



And vice versa suppose  $in(S(X), \mathfrak{M}) \leq 0$ . We have to show that either  $ind X = 0$  or  $X \in loc \mathfrak{M}$ . Since  $S(X) = \bigvee_{n=1}^{\infty} X^n$ ,  $X \times X$  is the closed subset of  $S(X)$ . So, by Proposition 3.3,  $in(X \times X, \mathfrak{M}) \leq in(S(X), \mathfrak{M}) \leq 0$ . Thus  $in(X \times X, \mathfrak{M}) \leq 0$ . Since  $\mathfrak{M}$  is a stable multiplicative class, by Theorem 3.2, either  $ind X = 0$  or  $X \in loc \mathfrak{M}$ , which was to be proved.  $\square$

**Definition 3.1.** Let  $\mathfrak{M}$  be a topologically closed class. The uncountable space  $X$  is called  $\mathfrak{M}$ -dispersed if for every uncountable and closed subset  $B$  we have  $AM(B) \notin loc \mathfrak{M}$ , where  $AM(B)$  is a free abelian topological group of  $B$  in the sense of Markov.

*Remark 3.1.* One can show that every uncountable compact space is  $\mathfrak{M}_c$ ,  $\mathfrak{M}_{dc^-}$ ,  $\mathfrak{M}_{mc^-}$ ,  $\mathfrak{M}_{\aleph_0}$ -dispersed (see, e.g., [6]).

**Theorem 3.5.** Let  $X$  be an uncountable compact and  $\mathfrak{M}$ -dispersed space, where  $\mathfrak{M}$  is a stable and multiplicative class. Then the following conditions are equivalent: 1)  $ind X \leq 0$ , 2)  $ind AG(X) \leq 0$ , 3)  $ind AM(X) \leq 0$ , 4)  $in(AG(X), \mathfrak{M}) \leq 0$ , 5)  $in(AM(X), \mathfrak{M}) \leq 0$ , 6)  $Ind AM(X) \leq 0$ , 7)  $Ind AG(X) \leq 0$ .

*Proof.* The proof will follow the scheme: 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$  5) $\Rightarrow$ 6) $\Rightarrow$ 7) $\Rightarrow$ 1).

First let us show that 1) $\Rightarrow$ 2). Since  $X$  is a compact space,  $AG(X)$  is a  $K_\omega$ -space (see [9]), i.e.,  $AG(X) = \bigcup_{n=1}^{\infty} A_n(X)$ , where  $A_n(X)$  symbolizes a subspace of the group  $AG(X)$ , which consists of words the length of which is  $\leq n$ . It is known (see [9]) that for every  $n = 1, 2, \dots$ ,  $A_n(X)$  is a compact subspace of  $AG(X)$ . Let us show by induction that  $dim A_n(X) = 0$ . Indeed, when  $n = 1$  we have  $A_1(X) = X \vee (-X) \vee \{0\}$ . Since  $ind X = 0$ , we have  $dim X = 0$ . Since  $(-X)$  is homeomorphic to  $X$ , we have  $dim(-X) = 0$ , i.e.,  $dim A_1(X) = dim X = ind X = 0$ . Suppose for every  $p \leq n - 1$  it is known that  $dim A_p(X) = 0$ . Let us discuss  $A_n(X)$ . It is obvious that  $A_n(X) = A_{n-1}(X) \cup (A_n(X) \setminus A_{n-1}(X))$ . Since  $A_{n-1}(X)$  is closed in  $AG(X)$  (see [9]), it is closed in  $A_n(X)$ . Therefore  $A_n(X) \setminus A_{n-1}(X)$  is open in  $A_n(X)$ . Now let us show that  $ind(A_n(X) \setminus A_{n-1}(X)) \leq 0$ . Suppose  $\alpha \in A_n(X) \setminus A_{n-1}(X)$ . This means that  $\alpha$  is a non-cancellable word of length  $n$ , i.e.,  $\alpha = \varepsilon_1 x_1 + \dots + \varepsilon_n x_n$ , where  $\varepsilon_i$  is equal to 1 or  $-1$  for every  $i = 1, 2, \dots, n$ .

Let us consider any open neighborhood  $O_\alpha$  of the point  $\alpha$  in  $A_n(X)$ . According to Arkhangel'skii–Joiner's lemma (see, e.g., [3]) there exist closed and open subsets  $U_1, U_2, \dots, U_n$  which satisfy the following:

1.  $x_i \in U_i$  for every  $i = 1, 2, \dots, n$ ;
2.  $U_i \cap U_j = \emptyset$  as soon as  $i \neq j$  ( $i, j = 1, 2, \dots, n$ );
3. the set  $\varepsilon_1 U_1 + \varepsilon_2 U_2 + \dots + \varepsilon_n U_n$  is an open neighborhood of the point  $\alpha$  in  $A_n(X)$ ;
4.  $\varepsilon_1 U_1 + \varepsilon_2 U_2 + \dots + \varepsilon_n U_n \subseteq O_\alpha$  because for every  $i$ ,  $U_i$  is compact. Therefore  $\varepsilon_1 U_1 + \varepsilon_2 U_2 + \dots + \varepsilon_n U_n$  is an open compact neighborhood of the point  $\alpha \in A_n(X) \setminus A_{n-1}(X)$ .

Since the point  $\alpha$  and its neighborhood  $O_\alpha$  were chosen arbitrarily, then  $\text{ind}(A_n(X) \setminus A_{n-1}(X)) \leq 0$ . Furthermore, for any closed subset  $F$  of  $A_n$ , where  $A_n(X) \subseteq A_n(X) \setminus A_{n-1}(X)$ , we have  $\text{dim } F \leq \text{ind } F \leq \text{ind}(A_n(X) \setminus A_{n-1}(X)) \leq 0$ . Then, by the inductive assumption and according to Dowker's theorem [7], we obtain  $\text{dim } A_n(X) \leq 0$ . Since  $AG(X) = \bigcup_{n=1}^{\infty} A_i(X)$ , we have  $\text{dim } AG(X) \leq 0$ .

Thus  $\text{ind } AG(X) \leq 0$ .

1) $\Rightarrow$ 2) is proved.

2) $\Rightarrow$ 3). According to F. Ward's theorem  $AM(X) \approx AG(X) \times Z$ , where  $Z$  is the set of integers with discrete topology. Therefore  $AM(X) = \bigvee_{n=1}^{\infty} AG(X)_i$  and

thus 2) $\Rightarrow$ 3).

3) $\Rightarrow$ 4). Since  $\mathfrak{M}$  is a stable class,  $\emptyset \in \mathfrak{M}$  and according to Nishiura's Theorem 1.8 from [14] we have  $\text{in}(AM(X), \mathfrak{M}) \leq \text{ind } AM(X) = \text{ind } AG(X) = 0$ . 3) $\Rightarrow$ 4) is proved.

4) $\Rightarrow$ 5) follows also from [16]. Indeed,  $\text{in}(AG(X), \mathfrak{M}) = 0 \Rightarrow \text{in}(AM(X), \mathfrak{M}) = \text{in}(AG(X) \times Z, \mathfrak{M}) = \text{in}(\bigvee_{n=1}^{\infty} AG(X)_i, \mathfrak{M}) = 0$ .

5) $\Rightarrow$ 6). We have to prove that  $\text{in}(AM(X), \mathfrak{M}) \leq 0 \Rightarrow \text{Ind } AM(X) \leq 0$ . Suppose, on the contrary, that  $\text{Ind } AM(X) > 0$ . Then  $\text{ind } X > 0$  because if  $\text{ind } X \leq 0$ , then  $\text{ind } AG(X) \leq 0$  and  $\text{ind } AM(X) \leq 0$ . Since  $AM(X)$  is  $\sigma$ -compact and thus, finally compact, we have  $\text{dim } AM(X) \leq \text{Ind } AM(X) = 0$ . This contradicts our assumption. Therefore  $\text{ind } X > 0$ . Since  $X$  is compact and  $\text{ind } X > 0$ , we have  $\text{dim } X > 0$ . By Proposition 2.6.3 from [17], there are compact subspaces  $F_1$  and  $F_2$  of the compact  $X$  such that  $F_1 \cap F_2 = \emptyset$ ,  $\text{dim } F_1 > 0$ ,  $\text{dim } F_2 > 0$ . Let us consider the compact space  $F = F_1 \cup F_2$ . Since  $F_1 \cap F_2 = \emptyset$ , the compact  $F$  is homeomorphic to the topological sum of  $F_1$  and  $F_2$ , i.e.,  $F = F_1 \vee F_2$ .

According to Ward's theorem from [16], the free topological group  $AM(X)$  is topologically isomorphic to  $AG(X) \times Z$ . Since  $\text{in}(AM(X), \mathfrak{M}) \leq 0$  we have  $\text{in}(AG(X), \mathfrak{M}) \leq 0$ . Denote by  $G(F)$  the subgroup of  $AG(X)$  which is generated algebraically by the compact subspace  $F \subseteq X$ . It is well-known that [9]  $G(F)$  is closed in  $AG(X)$  and  $G(F)$  is topologically isomorphic to  $AG(F)$ .

Thus  $\text{in}(G(F), \mathfrak{M}) = \text{in}(AG(F), \mathfrak{M}) \leq 0$ .

By Ward's theorem,  $AM(F) = AG(F) \times Z$ , i.e.,  $AM(F) = \bigvee_{n=1}^{\infty} AG(F)_i$ , where for every  $i = 1, 2, \dots$ ,  $AG(F)_i$  is homeomorphic to  $AG(F)$ . It is easy to show that we have  $\text{in}(AM(F), \mathfrak{M}) \leq 0$ . But  $F = F_1 \vee F_2$ . It is known that  $AM(F_1 \vee F_2) \cong AM(F_1) \times AM(F_2)$ . Then  $\text{in}(AM(F_1) \times AM(F_2), \mathfrak{M}) \leq 0$ . Consequently, by Theorem 3.3, one of the following four cases will take place:

1.  $\text{ind } AM(F_1) \leq 0$ ,  $\text{ind } AM(F_2) \leq 0$ ,
2.  $AM(F_1) \in \text{loc } \mathfrak{M}$ ,  $AM(F_2) \in \text{loc } \mathfrak{M}$ ,
3.  $AM(F_1) \in \text{loc } \mathfrak{M}$ ,  $\text{ind } AM(F_1) = 0$  and  $\text{in } AM(F_2), \mathfrak{M} \leq 0$ ,
4.  $AM(F_2) \in \text{loc } \mathfrak{M}$ ,  $\text{ind } AM(F_2) = 0$  and  $\text{in } AM(F_1), \mathfrak{M} \leq 0$ .

But in accordance with the condition it is impossible for  $AM(F_1)$  or  $AM(F_2)$  to belong to  $loc \mathfrak{M}$ . Therefore only case 1) remains.

Since  $F_1 \subseteq AM(F_1)$  and  $F_2 \subseteq AM(F_2)$ , we have  $ind F_1 \leq 0$  and  $ind F_2 \leq 0$ . Then  $dim F_1 \leq 0$  and  $dim F_2 \leq 0$ . But this contradicts  $dim F_1 > 0$ ,  $dim F_2 > 0$ .

Thus our assumption that  $Ind AM(X) > 0$  is false. So 5)  $\Rightarrow$  6) is proved.

6)  $\Rightarrow$  7) follows from Ward's theorem.

7)  $\Rightarrow$  1). Since  $Ind AG(X) \leq 0$ , we have  $Ind X \leq 0$  because  $X$  is the closed subspace of  $AG(X)$ . Hence  $ind X \leq 0$ . □

**Theorem 3.6.** *Let  $X$  be an uncountable  $K_\omega$ -space [13] and an  $\mathfrak{M}$ -dispersed space, where  $\mathfrak{M}$  is a stable multiplicative class. Then the following conditions are equivalent: 1)  $ind X \leq 0$ , 2)  $ind FG(X) \leq 0$ , 3)  $ind FM(X) \leq 0$ , 4)  $in (FG(X), \mathfrak{M}) \leq 0$ , 5)  $in (FM(X), \mathfrak{M}) \leq 0$ .*

*Proof.* The proof will follow the scheme 1)  $\Rightarrow$  2)  $\Rightarrow$  3)  $\Rightarrow$  4)  $\Rightarrow$  5)  $\Rightarrow$  1).

1)  $\Rightarrow$  2). Since  $X$  is a  $K_\omega$ -space, there exists a  $K_\omega$ -decomposition  $\{X_i\}$  of  $X$ , i.e., there exists a system  $\{X_i\}$ ,  $i = 1, 2, \dots$ , of subspaces of the space  $X$  such that:

1.  $X = \bigcup_{i=1}^{\infty} X_i$ ;
2.  $X_i$  is compact for every  $i = 1, 2, \dots$ ;
3.  $X_i \subseteq X_j$ , for every  $i < j$  ( $i, j = 1, 2, \dots$ );
4. the set  $A$  is closed in  $X$  if and only if  $A \cap X_i$  is compact for every  $i = 1, 2, \dots$

(If  $\{X_i\}$  is a  $K_\omega$ -decomposition of  $X$ , we write  $X \stackrel{w}{=} \bigcup_{i=1}^{\infty} X_i$ ).

As we know (see [13]), if  $X \stackrel{w}{=} \bigcup_{i=1}^{\infty} X_i$ , then  $FG(X) \stackrel{w}{=} \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} F_j(X_i)$ , where

$F_j(X_i)$  denotes the set of words generated by elements of  $X_i$ , the length of which is  $\leq j$ . It is clear that  $F_j(X_i) \subseteq FG(X_i)$  for every  $j = 1, 2, \dots$ . Let us show that  $ind FG(X_i) = 0$  for every  $i = 1, 2, \dots$ . We know that  $F_n(X_i)$  is compact for every  $n$ . Let us show by induction that  $dim F_n(X_i) = 0$  for every  $i$ . For  $n = 1$  we have  $dim X_i = 0$ . Then  $dim F_1(X_i) = 0$ , where  $F_1(X_i) = X_i \vee X_i^{-1} \vee \{1\}$ . Suppose for every  $p \leq n - 1$  the assertion is already proved and that  $dim F_p(X_i) = 0$ . Let us consider  $F_n(X_i) = F_{n-1}(X_i) \cup (F_n(X_i) \setminus F_{n-1}(X_i))$ . Since  $F_{n-1}(X_i)$  is closed in  $FG(X_i)$ , then it is also closed in  $F_n(X_i)$ . Thus  $F_n(X_i) \setminus F_{n-1}(X_i)$  is open in  $F_n(X_i)$ . Let us show that  $ind (F_n(X_i) \setminus F_{n-1}(X_i)) \leq 0$ . Indeed, suppose  $\alpha \in F_n(X_i) \setminus F_{n-1}(X_i)$ . Thus  $\alpha = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$ , where  $\varepsilon_i$  is equal to 1 or  $-1$ . Let us consider any open neighborhood  $O_\alpha$  of the point  $\alpha$  in  $F_n(X_i)$ . Then according to Arkhangel'skii-Joiner's lemma (which is also true for the group  $FG(X_i)$ ), there exist open compact sets  $U_1, U_2, \dots, U_n$  which satisfy the following conditions:

1.  $x_i \in U_i$ ,  $i = 1, 2, \dots$ ;
2.  $U_i \cap U_j = \emptyset$ ,  $i \neq j$  ( $i, j = 1, 2, \dots$ );
3. the set  $U_1^{\varepsilon_1} U_2^{\varepsilon_2} \dots U_n^{\varepsilon_n}$  is an open neighborhood of the point  $\alpha$  in  $F_n(X_i)$ ;
4.  $U_1^{\varepsilon_1} U_2^{\varepsilon_2} \dots U_n^{\varepsilon_n} \subseteq O_\alpha$ .

Since  $U_i$  is compact for every  $i = 1, 2, \dots, n$ ,  $U_1^{\varepsilon_1} U_2^{\varepsilon_2} \dots U_n^{\varepsilon_n}$  is an open compact neighborhood of the point  $\alpha \in F_n(X_i) \setminus F_{n-1}(X_i)$ . Since the point  $\alpha$  and its neighborhood  $O_\alpha$  where chosen arbitrarily, we have  $\text{ind}(F_n(X_i) \setminus F_{n-1}(X_i)) \leq 0$ .

For every compact  $F \subseteq F_n(X_i) \setminus F_{n-1}(X_i)$  we have:  $\text{dim } F \leq 0$ . Since according to the inductive assumption,  $\text{dim } F_{n-1}(X_i) \leq 0$ , by Dowker's theorem (see [7]) we conclude that  $\text{dim } F_n(X_i) \leq 0$ , i.e., for every  $i$  we have shown that  $\text{dim } F_n(X_i) \leq 0$ . Hence  $FG(X_i) \leq 0$ . Since  $FG(X)$  is compact and  $FG(X) = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} F_n(X_i)$ , we have  $\text{dim } FG(X) \leq 0$ . Thus  $\text{ind } FG(X) \leq 0$ .

2) $\Rightarrow$ 3). Suppose  $\text{ind } FG(X) = 0$ . Then  $\text{ind } X = 0$  because  $X \subseteq FG(X)$ . Let us consider the space  $Y = X \vee \{p\}$ , where  $p \notin X$ , which is a topological sum of the space  $X$  and the singleton  $\{p\}$ . It is clear that since  $X$  is the  $K_\omega$ -space, we have  $\text{ind } Y = 0$ . So by means of 1) $\Rightarrow$ 2) we get  $\text{ind } FG(Y) = 0$ . But it is known (see [9]) that  $FM(X)$  is topologically isomorphic to  $FG(Y)$ . Thus  $\text{ind } FM(X) = 0$ .

So  $\text{ind } FM(X) = 0$ . According to Morris' theorem (see [13]), the group  $FM(X)$  is topologically isomorphic to the coproduct of the group  $FG(X)$  and the discrete group  $Z$  of integers i.e.,  $FM(X) \cong FG(X) * Z$ . Hence  $FG(X)$  is a closed subspace of  $FM(X)$ . Thus  $\text{ind } FG(X) = 0$  is proved.

3) $\Rightarrow$ 4). Since  $\mathfrak{M}$  is the stable class and  $\emptyset \in \mathfrak{M}$ , according to Nishiura's theorem (see [14]),  $\text{in}(FM(X), \mathfrak{M}) \leq \text{in}(FM(X), \emptyset) = \text{ind } FM(X) \leq 0$ .

4) $\Rightarrow$ 5). Suppose  $\text{in}(FM(X), \mathfrak{M}) \leq 0$ . We have to show that  $\text{in}(FG(X), \mathfrak{M}) \leq 0$ . According to Morris' theorem,  $FM(X) \cong FG(X) * Z$ . Since  $FG(X)$  is a topological group the space of which is a  $K_\omega$ -space, it is complete in the sense of A. Weil (see [9]). Moreover,  $FG(X)$  is a closed subgroup of  $FM(X)$ . Therefore, by  $c$ -monotonicity of the function  $\text{in}(X, P)$ , we get  $\text{in}(FG(X), \mathfrak{M}) \leq 0$ .

Now let us prove 5) $\Rightarrow$ 1). Suppose  $\text{in}(FG(X), \mathfrak{M}) \leq 0$ . We have to show that  $\text{ind } X = 0$ .

Let us assume that the opposite is true, i.e.,  $\text{ind } X > 0$ . Then  $\text{dim } X > 0$ . Since  $X$  is the  $K_\omega$ -space, there exists its  $\omega$  decomposition  $X \stackrel{\omega}{=} \bigcup_{i=1}^{\infty} X_i$ , where each  $X_i$  is compact.

Suppose  $i_0$  is a natural number such that  $\text{dim } X_{i_0} > 0$  (if there is no such  $i_0$  and for every  $i$   $\text{dim } X_i$  is equal to 0, then  $\text{dim } X$  is also equal to 0).

Since  $X_{i_0}$  is compact, then, based on the well-known result that  $\text{dim } X_{i_0}$  is equal to the supremum of  $\text{dim}$  dimensions of the components of  $X_{i_0}$ , there exists a connected compact  $\phi^{i_0} \subseteq X_{i_0}$  such that  $\text{dim } \phi^{i_0} > 0$ .

According to Zambakhidze's Theorem 2.6.3 (see [17]) there exist subsets  $F^{i_0(1)}$  and  $F^{i_0(2)}$  closed in  $\phi^{i_0}$  (and therefore in  $X$  too) such that  $F^{i_0(1)} \cap F^{i_0(2)} = \emptyset$ ,  $\text{dim } F^{i_0(1)} > 0$ ,  $\text{dim } F^{i_0(2)} > 0$ . Since  $\phi^{i_0}$  is connected and compact, there exists a point  $z = \phi^{i_0} \setminus F^{i_0(1)} \cup F^{i_0(2)}$  (otherwise  $\phi^{i_0}$  would be decomposed into the sum of two separate closed subsets, which contradicts the connectedness of  $\phi^{i_0}$ ). Suppose  $Y_{i_0} = F^{i_0(1)} \cup F^{i_0(2)}$  and  $Y'_{i_0} = F^{i_0(1)} \cup F^{i_0(2)} \cup \{z\}$ . Since  $F^{i_0(1)} \cap F^{i_0(2)} = \emptyset$  and  $z \notin F^{i_0(1)} \cup F^{i_0(2)}$ , we have  $Y_{i_0} = F^{i_0(1)} \vee F^{i_0(2)}$ ,  $Y'_{i_0} = F^{i_0(1)} \vee F^{i_0(2)} \vee \{z\}$ . Obviously,  $Y_{i_0}$  and  $Y'_{i_0}$  are compact subspaces of

the space  $X$ . Then according to Graev's theorem (see [9]),  $FM(Y_{i_0})$  is topologically isomorphic to the group  $FG(Y'_{i_0})$ . Since  $Y'_{i_0}$  is a compact subspace of the  $K_\omega$ -space, the subgroup  $G(Y'_{i_0})$  of the group  $FG(X)$  (which is generated algebraically from the compact  $Y'_{i_0}$ ) is closed in  $FG(X)$  and is topologically isomorphic to the free topological group  $FG(Y'_{i_0})$ . By the condition,  $in(FG(X), \mathfrak{M}) \leq 0$ . So  $in(FG(Y'_{i_0}), \mathfrak{M}) \leq 0$  and thus  $in(FG(Y_{i_0}), \mathfrak{M}) \leq 0$ .

Furthermore,  $Y_{i_0}$  is homeomorphic to the topological sum of the spaces  $F^{i_0(1)}$  and  $F^{i_0(2)}$ . According to Morris' theorem (see [13]) the group  $FM(Y_{i_0})$  is topologically isomorphic to the free product  $FM(F^{i_0(1)}) * FM(F^{i_0(2)})$  of topological groups  $FM(F^{i_0(1)})$  and  $FM(F^{i_0(2)})$ . Thus  $in(FM(F^{i_0(1)}) * FM(F^{i_0(2)}), \mathfrak{M}) \leq 0$ . According to Ordman's theorem (see [15]), in the sequence  $FM(F^{i_0(1)}) \times FM(F^{i_0(2)}) \xrightarrow{i} FM(F^{i_0(1)}) * FM(F^{i_0(2)}) \xrightarrow{\rho} FM(F^{i_0(1)}) \times FM(F^{i_0(2)})$ , where  $i$  is a homeomorphic inclusion of  $FM(F^{i_0(1)}) \times FM(F^{i_0(2)})$  into  $FM(F^{i_0(1)}) * FM(F^{i_0(2)})$  and  $\rho$  is a continuous mapping of  $FM(F^{i_0(1)}) * FM(F^{i_0(2)})$  onto  $FM(F^{i_0(1)}) \times FM(F^{i_0(2)})$ , the composition  $i\rho$  is the identical mapping. Hence  $i(FM(F^{i_0(1)}) \times FM(F^{i_0(2)}))$  is closed in  $FM(F^{i_0(1)}) * FM(F^{i_0(2)})$ . Therefore  $in(FM(F^{i_0(1)}) \times FM(F^{i_0(2)}), \mathfrak{M}) \leq 0$ .

According to Theorem 3.3  $in(FM(F^{i_0(1)}) \times FM(F^{i_0(2)}), \mathfrak{M}) \leq 0$  is true if and only if one of the following conditions is fulfilled:

1.  $ind FM(F^{i_0(1)}) = 0, ind FM(F^{i_0(2)}) = 0$ ;
2.  $FM(F^{i_0(1)}) \in loc \mathfrak{M}, FM(F^{i_0(2)}) \in loc \mathfrak{M}$ ;
3.  $FM(F^{i_0(1)}) \in loc \mathfrak{M}, ind FM(F^{i_0(1)}) = 0$  and  $in(FM(F^{i_0(2)}), \mathfrak{M}) \leq 0$ ;
4.  $FM(F^{i_0(2)}) \in loc \mathfrak{M}, ind FM(F^{i_0(2)}) = 0$  and  $in(FM(F^{i_0(1)}), \mathfrak{M}) \leq 0$ .

As we know (see [11]), abelian free topological groups  $AM(F^{i_0(1)})$  and  $AM(F^{i_0(2)})$  are topologically isomorphic to the factor groups  $FM(F^{i_0(1)})/K_1$  and  $FM(F^{i_0(2)})/K_2$ , where  $K_1$  and  $K_2$  are commutants. The mappings

$$\begin{aligned} \pi_1 : FM(F^{i_0(1)}) &\rightarrow FM(F^{i_0(1)})/K_1 = AM(F^{i_0(1)}), \\ \pi_2 : FM(F^{i_0(2)}) &\rightarrow FM(F^{i_0(2)})/K_2 = AM(F^{i_0(2)}) \end{aligned}$$

are open. If  $FM(F^{i_0(1)}) \in loc \mathfrak{M}$  and  $FM(F^{i_0(2)}) \in loc \mathfrak{M}$ , then  $AM(F^{i_0(1)}) \in loc \mathfrak{M}$  and  $AM(F^{i_0(2)}) \in loc \mathfrak{M}$ , respectively. But this is impossible because the space  $X$  is  $\mathfrak{M}$ -dispersed. Thus we exclude cases 2), 3) and 4). So we have  $ind FM(F^{i_0(1)}) = 0$  and  $ind FM(F^{i_0(2)}) = 0$ . On the other hand, we have  $dim FM(F^{i_0(1)}) > 0$  and  $dim FM(F^{i_0(2)}) > 0$  (since  $dim F^{i_0(1)} > 0$  and  $dim F^{i_0(2)} > 0$ ). So,  $FM(F^{i_0(1)})$  and  $FM(F^{i_0(2)})$  are  $\sigma$ -compacts and  $ind FM(F^{i_0(1)}) > 0, ind FM(F^{i_0(2)}) > 0$ . This is the contradiction. Thus our assumption that  $ind X > 0$  is not true. Therefore  $ind X \leq 0$ . □

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