Georgian Mathematical Journal
Volume 10 (2003), Number 3, 595–602

ON TWO-POINT BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS WITH SINGULARITIES

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Abstract. For a differential system

$$\frac{du_1}{dt} = h_0(t,u_1,u_2)u_2, \quad \frac{du_2}{dt} = -h_1(t,u_1,u_2)u_1^{-\lambda} - h_2(t,u_1,u_2),$$

where $\lambda \in]0,1[$ and $h_i :]a,b[\times]0,+\infty[\times\mathbb{R} \to [0,+\infty[$ $(i = 0,1,2)$ are continuous functions, we have established sufficient conditions for the existence of at least one solution satisfying one of the two boundary conditions

$$\lim_{t\to a} u_1(t) = 0, \quad \lim_{t\to b} u_1(t) = 0$$

and

$$\lim_{t\to a} u_1(t) = 0, \quad \lim_{t\to b} u_2(t) = 0.$$

2000 Mathematics Subject Classification: 34B16.
Key words and phrases: Two-dimensional nonlinear differential system with singularities, two-point boundary value problem, existence theorem.

Two-point boundary value problems for second order ordinary differential equations with singularities with respect to one of the phase variable frequently occur in applications (see, for example, [1, 3, 4, 10, 11, 12]) and are investigated with sufficient thoroughness in the works of S. Taliafero [13], J. E. Bouillet and S. M. Gomes [2], Yu. A. Klokov and A. I. Lomakina [7], A. G. Lomtatidze [8, 9], I. Kiguradze and B. Shekhter [6] and others. However analogous problems for two-dimensional singular differential systems still remain little studied. In this paper we investigate the first and second boundary value problems for nonlinear differential system with singularities with respect to both independent and/or one of the phase variables.

We consider the differential system

$$\frac{du_1}{dt} = h_0(t,u_1,u_2)u_2, \quad \frac{du_2}{dt} = -h_1(t,u_1,u_2)u_1^{-\lambda} - h_2(t,u_1,u_2), \tag{1}$$

with the boundary conditions

$$\lim_{t\to a} u_1(t) = 0, \quad \lim_{t\to b} u_1(t) = 0 \tag{2}$$

or

$$\lim_{t\to a} u_1(t) = 0, \quad \lim_{t\to b} u_2(t) = 0. \tag{3}$$

Here $\lambda \in]0,1[$, $h_i :]a,b[\times]0,+\infty[\times\mathbb{R} \to [0,+\infty[$ $(i = 0,1,2)$ are continuous functions. Moreover, there exist a continuous function $h :]a,b[\to [0,+\infty[$ and

ISSN 1072-947X / \$8.00 /

positive numbers l, l_0 such that $0 < \int_a^b h(t)\,dt < \infty$ and the inequalities

$$h(t) \le h_0(t,x,y) \le l\,h(t), \quad h_1(t,x,y) + h_2(t,x,y) \ge l_0 h(t) \tag{4}$$

hold on the set $]a,b[\times]0,+\infty[\times\mathbb{R}$.

Solutions of problems (1), (2) and (1), (3) are sought for in the class of continuously differentiable vector functions $(u_1,u_2) :]a,b[\to]0,+\infty[\times\mathbb{R}$.

We set

$$\delta_1(t) = \int_a^t h(s)\,ds \int_t^b h(s)\,ds, \quad \delta_2(t) = \int_a^t h(s)\,ds,$$
$$\delta(t) = l\,\delta_1(t)\bigg(\int_a^b h(s)\,ds\bigg)^{-1}.$$

Theorem 1. *Let, along with* (4) *the conditions*

$$\delta_1(t) > 0 \quad \textit{for} \quad a < t < b \tag{5}$$

and

$$\limsup_{\rho\to+\infty} \bigg[(2\,l\,l_0^{-1})^\lambda \rho^{\lambda^2-1} \int_a^b \delta_1^{1-\lambda}(t)\,h_1^*(t,\rho)\,dt + \rho^{-1}\int_a^b \delta_1(t)\,h_2^*(t,\rho)\,dt\bigg] < l^{-2}\int_a^b h(t)\,dt \tag{6}$$

be satisfied, where

$$h_i^*(t,\rho) = \sup\big\{h_i(t,x,y) : 0 < x < l,\ \delta(t)|y| < \rho\big\} \quad (i=1,2). \tag{7}$$

Then problem (1), (2) *is solvable.*

Theorem 2. *Let, along with* (4), *the conditions*

$$\delta_2(t) > 0 \quad \textit{for} \quad a < t \le b$$

and

$$\limsup_{\rho\to+\infty} \bigg[\Big(\frac{2}{l_0\,\delta_2(b)}\Big)^\lambda \rho^{\lambda^2-1} \int_a^b \delta_2^{1-\lambda}(t)\,h_1^*(t,\rho)\,dt + \rho^{-1}\int_a^b \delta_2(t)\,h_2^*(t,\rho)\,dt\bigg] < \frac{1}{l}$$

hold, where

$$h_i^*(t,\rho) = \sup\big\{h_i(t,x,y) : 0 < x < \rho,\ \delta_2(t)|y| < \rho\big\} \quad (i=1,2).$$

Then problem (1), (3) *is solvable.*

As an example, consider the differential system

$$\begin{aligned}\frac{du_1}{dt} &= (t-a)^\alpha\,(b-t)^\beta\,\rho_0(t,u_1,u_2)u_2,\\ \frac{du_2}{dt} &= -\frac{p_1(t,u_1,u_2)}{(t-a)^{\alpha_1}\,(b-t)^{\beta_1}}\,u_1^{-\lambda} - \frac{p_2(t,u_1,u_2)}{(t-a)^{\alpha_2}\,(b-t)^{\beta_2}}\,,\end{aligned} \tag{8}$$

where $\lambda \in]0,1[$, and

$$\alpha > -1, \quad \beta > -1, \quad \alpha_i \ge -\alpha, \quad \beta_i \ge -\beta \quad (i=1,2),$$

and $p_i :]a,b[\times]0,+\infty[\times\mathbb{R} \to]r_1,r_2[$ $(i=0,1,2)$ are continuous functions, $r_i = \text{const} > 0$ $(i=1,2)$.

Theorems 1 and 2 give rise to

Corollary 1. *The inequalities*

$$\alpha_1 < (\alpha+1)(1-\lambda)+1,\quad \beta_1 < (\beta+1)(1-\lambda)+1,\quad \alpha_2 < \alpha+2,\quad \beta_2 < \beta+2$$

guarantee the solvability of problem (1), (2), *and the inequalities*

$$\alpha_1 < (\alpha+1)(1-\lambda)+1,\quad \beta_1 < 1,\quad \alpha_2 < \alpha+2,\quad \beta_2 < 1$$

guarantee the solvability of problem (1), (3).

The above example shows that Theorem 1 (Theorem 2) covers the case where the functions h_1 and h_2 have singularities of arbitrary order at $t=a$ and $t=b$ (at $t=a$).

To prove the formulated theorems, we need two lemmas from [5] on the representation of a solution of the differential system

$$\frac{dv_1}{dt} = p(t)\, v_2,\quad \frac{dv_2}{dt} = q(t) \tag{9}$$

under the boundary conditions

$$v_1(a)=0,\quad v_1(b)=0 \tag{10}$$

or

$$v_1(a)=0,\quad v_2(b)=0. \tag{11}$$

Lemma 1. *Let $p,q :]a,b[\to \mathbb{R}$ be continuous functions such that*

$$\int_a^b |p(t)|\,dt < +\infty,\quad \int_a^b \Big(\int_a^t |p(\tau)|\,d\tau \int_t^b |p(\tau)|\,d\tau\Big)|q(t)|\,dt < +\infty\ \ (i=1,2),$$

$$\gamma_0 \overset{\text{def}}{=} \int_a^b p(t)\,dt \neq 0. \tag{12}$$

Then problem (9), (10) *has a unique solution $v=(v_1,v_2)$ and the representations*

$$v_i(t) = \int_a^b g_i(t,s)\, q(s)\,ds \quad (i=1,2) \tag{13}$$

are valid, where

$$g_1(t,s) = \begin{cases} -\dfrac{1}{\gamma_0}\displaystyle\int_a^s p(\tau)\,d\tau \int_t^b p(\tau)\,d\tau & \textit{for}\ \ s\le t,\\[2mm] -\dfrac{1}{\gamma_0}\displaystyle\int_a^t p(\tau)\,d\tau \int_s^b p(\tau)\,d\tau & \textit{for}\ \ s>t,\end{cases}$$

$$g_2(t,s) = \begin{cases} \dfrac{1}{\gamma_0}\displaystyle\int_a^s p(\tau)\,d\tau & \textit{for}\ \ s\le t,\\[2mm] -\dfrac{1}{\gamma_0}\displaystyle\int_s^b p(\tau)\,d\tau & \textit{for}\ \ s>t.\end{cases}$$

Lemma 2. *Let $p, q \in]a,b[\to \mathbb{R}$ be continuous functions such that conditions* (12) *and*

$$\int_a^b p(t)\,dt < +\infty, \quad \int_a^b \Big(\int_a^t |p(\tau)|\,d\tau\Big) q(t)\,dt < +\infty$$

are fulfilled. Then problem (9), (11) *has a unique solution $v = (v_1, v_2)$ and representations* (13) *are valid, where*

$$g_1(t,s) = \begin{cases} -\displaystyle\int_a^s p(\tau)\,d\tau & \textit{for } s \le t, \\ -\displaystyle\int_a^t p(\tau)\,d\tau & \textit{for } s > t, \end{cases} \qquad g_2(t,s) = \begin{cases} 0 & \textit{for } s \le t, \\ -1 & \textit{for } s > t. \end{cases}$$

Proof of Theorem 1. By virtue of condition (6), the constant ρ can be chosen so that the inequalities

$$\rho > 1 + \frac{l_0}{2\,l}\Big(\int_a^b h(s)\,ds\Big)^2, \tag{14}$$

$$\Big(\frac{2\,l}{l_0}\Big)^{\lambda} \rho^{\lambda^2} \int_a^b \delta_1^{1-\lambda}(t)\, h_1^*(t,\rho)\,dt + \int_a^b \delta_1(t)\, h_1^*(t,\rho)\,dt < \frac{\rho}{l^2} \int_a^b h(t)\,dt \tag{15}$$

be fulfilled.

Let $\mathbb{B}$ be a Banach space of two-dimensional continuous vector functions $v = (v_1, v_2) : [a,b] \to \mathbb{R}^2$ with the norm

$$\|v\| = \max\big\{|v_1(t)| + |v_2(t)| : \, a \le t \le b\big\}$$

and $\mathbb{B}_\rho$ be a set of all $v = (v_1, v_2) \in \mathbb{B}$ satisfying the conditions

$$\frac{l_0}{2\,l}\,\rho^{-\lambda}\delta_1(t) \le v_1(t) < \rho, \quad |v_2(t)| \le \rho \quad \text{for} \quad a \le t \le b.$$

In view of inequality (14), $\mathbb{B}_\rho$ is a nonempty closed convex subset of the space $\mathbb{B}$.

For arbitrary $v = (v_1, v_2) \in \mathbb{B}_\rho$ we set

$$\gamma(v_1, v_2) = \int_a^b h_0\Big(\tau, v_1(\tau), \frac{v_2(\tau)}{\sigma(\tau)}\Big)\,d\tau,$$

$$q(v_1, v_2)(t) = -h_1\Big(t, v_1(t), \frac{v_2(t)}{\sigma(t)}\Big) v_1^{-\lambda}(t) - h_2\Big(t, v_1(t), \frac{v_2(t)}{\sigma(t)}\Big),$$

$$g_1(v_1,v_2)(t,s)\cdot\gamma(v_1,v_2) = \begin{cases} -\displaystyle\int_a^s h_0\Big(\tau, v_1(\tau), \frac{v_2(\tau)}{\sigma(\tau)}\Big)\,d\tau & \\ \qquad \times \displaystyle\int_t^b h_0\Big(\tau, v_1(\tau), \frac{v_2(\tau)}{\sigma(\tau)}\Big)\,d\tau & \text{for } s \le t, \\ -\displaystyle\int_a^t h_0\Big(\tau, v_1(\tau), \frac{v_2(\tau)}{\sigma(\tau)}\Big)\,d\tau & \\ \qquad \times \displaystyle\int_s^b h_0\Big(\tau, v_1(\tau), \frac{v_2(\tau)}{\sigma(\tau)}\Big)\,d\tau & \text{for } s > t, \end{cases}$$

$$g_2(v_1,v_2)(t,s)\cdot\gamma(v_1,v_2)=\begin{cases}\displaystyle\int_a^s h_0\Big(\tau,v_1(\tau),\frac{v_2(\tau)}{\sigma(\tau)}\Big)\,d\tau & \text{for } s\le t,\\ \displaystyle-\int_s^b h_0\Big(\tau,v_1(\tau),\frac{v_2(\tau)}{\sigma(\tau)}\Big)\,d\tau & \text{for } s>t.\end{cases}$$

Let us now define the continuous operators $W_i:\mathbb{B}\to C([a,b])$ $(i=1,2)$ and $W:\mathbb{B}\to C([a,b])\times C([a,b])$ by the equalities

$$W_i(v_1,v_2)(t)=(\sigma(t))^{i-1}\int_a^b g_i(v_1,v_2)(t,s)\,q(v_1,v_2)(s)\,ds \quad (i=1,2)$$

and

$$W(v_1,v_2)(t)=\big(W_1(v_1,v_2)(t),W_2(v_1,v_2)(t)\big).$$

Then if problem (1), (2) has a solution $u=(u_1,u_2)$ satisfying the conditions

$$\frac{l_0}{2\,l}\,\rho^{-\lambda}\,\delta_1(t)\le u_1(t)\le\rho,\quad \sigma(t)|u_2(t)|\le\rho\quad\text{for}\quad a\le t\le b, \tag{16}$$

$$\lim_{t\to a}\sigma_1(t)\,u_2(t)=0,\quad \lim_{t\to b}\sigma_1(t)\,u_2(t)=0, \tag{17}$$

the vector function $v=(v_1,v_2)$ with the components

$$v_1(t)=u_1(t),\quad v_2(t)=\sigma(t)\,u_2(t) \tag{18}$$

belongs to the set $\mathbb{B}_\rho$ and, by virtue of Lemma 1, is a solution of the operator equation

$$v(t)=W(v)(t). \tag{19}$$

Conversely, if equation (19) has a solution $v=(v_1,v_2)\in\mathbb{B}_\rho$, then, again by Lemma 1, the function $u=(u_1,u_2)$, the components of which are defined from equalities (18), is a solution of problem (1), (2) satisfying conditions (17). Thus to prove the theorem it is sufficient to establish that the operator W has a fixed point on the set $\mathbb{B}_\rho$.

In the first place, we will show that

$$W(\mathbb{B}_\rho)\subset\mathbb{B}_\rho. \tag{20}$$

Let $(v_1,v_2)\in\mathbb{B}_\rho$. Then by virtue of (7)

$$|q(v_1,v_2)(t)|\le\varphi(t)\quad\text{for}\quad a<t<b,$$

$$\varphi(t)=\Big(\frac{2\,l}{l_0}\Big)^{\lambda}\rho^{\lambda^2}\,\delta_1^{-\lambda}(t)\,h_1^*(t,\rho)+h_2^*(t,\rho),$$

and, also, by virtue of the second of equalities (4)

$$\big|q(v_1,v_2)(t)\big|\ge\rho^{-\lambda}\,l_0\,h(t)\quad\text{for}\quad a<t<b.$$

Taking into account the latter estimates and the first of equalities (4), we obtain

$$W_1(v_1,v_2)(t)\ge\frac{l_0}{l}\,\rho^{-\lambda}\Big(\int_a^b h(\tau)\,d\tau\Big)^{-1}\Big[\int_a^b h(\tau)\,d\tau\int_a^t\Big(\int_a^s h(\tau)\,d\tau\Big)h(s)\,ds$$
$$+\int_a^t h(\tau)\,d\tau\int_t^b\Big(\int_s^b h(\tau)\,d\tau\Big)h(s)\,ds\Big]$$

$$= \frac{l_0}{2\,l}\,\rho^{-\lambda}\,\delta_1(t) \ \text{ for } \ a \le t \le b, \tag{21}$$

$$\begin{aligned} W_1(v_1,v_2)(t) \le l^2\Big(\int_a^b h(\tau)\,d\tau\Big)^{-1}\Big[\int_t^b h(\tau)\,d\tau\int_a^t\Big(\int_a^s h(\tau)\,d\tau\Big)\varphi(s)\,ds \\ + \int_a^t h(\tau)\,d\tau\int_t^b\Big(\int_t^b h(\tau)\,d\tau\Big)\varphi(s)\,ds\Big] \ \text{ for } \ a\le t\le b \end{aligned} \tag{22}$$

and

$$\begin{aligned} W_2(v_1,v_2)(t) \le l^2\Big(\int_a^b h(\tau)\,d\tau\Big)^{-2}\delta_1(t)\Big[\int_a^t\Big(\int_a^s h(\tau)\,d\tau\Big)\varphi(s)\,ds \\ + \int_t^b\Big(\int_s^b h(\tau)\,d\tau\Big)\varphi(s)\,ds\Big] \ \text{ for } \ a\le t\le b. \end{aligned} \tag{23}$$

From equalities (21), (23) with (15) taken into account we obtained

$$W_i(v_1,v_2)(t) \le l^2\Big(\int_a^b h(\tau)\,d\tau\Big)^{-1}\int_a^b \delta_1(\tau)\,\varphi(\tau)\,d\tau < \rho \ \ (i=1,2) \quad \text{for} \quad a\le t\le b. \tag{24}$$

This and (21) imply that inclusion (20) is valid.

Now note that from the definition of the operators W_i $(i=1,2)$ and Lemma 1 it follows that the functions $W_i(v_1,v_2)(t)$ $(i=1,2)$ are continuously differentiable on the interval $]a,b[$ for any $v=(v_1,v_2)\in\mathbb{B}_\rho$ and

$$\begin{aligned} \frac{d}{dt}\,W_1(v_1,v_2)(t) &= h_0\Big(t,v_1,\frac{v_2}{\sigma}\Big)\,\delta^{-1}(t)\,W_2(v_1,v_2)(t), \\ \frac{d}{dt}\,W_2(v_1,v_2)(t) &= -\Big(h_1\Big(t,v_1,\frac{v_2}{\sigma}\Big)\,W_1^{-\lambda}(v_1,v_2)(t) + h_2(t,v_1,v_2)\Big)\delta(t) \\ &\quad + \frac{\delta'(t)}{\delta(t)}\,W_2(v_1,v_2)(t). \end{aligned}$$

From this, taking into account estimates (21) and (24), we obtain

$$\Big|\frac{d}{dt}\,W_i(v_1,v_2)\Big| \le \eta_i(t) \quad (i=1,2) \quad \text{for} \quad a<t<b, \tag{25}$$

where $\eta_i :]a,b[\to[0,+\infty[$ are continuous functions defined by the equalities

$$\eta_1(t) = \rho\,\frac{h(t)}{\delta(t)}, \quad \eta_2(t) = \delta(t)\,\varphi(t) + \rho\,\frac{h(t)}{\delta_1(t)}\int_a^b h(\tau)\,d\tau.$$

Analogously, from estimates (22) and (23) we obtain

$$\big|W_1(v_1,v_2)(t)\big| + \big|W_2(v_1,v_2)(t)\big| \le \varepsilon_0(t) \quad \text{for} \quad a\le t\le b, \tag{26}$$

where the continuous function $\varepsilon_0 : [a,b]\to[0,+\infty[$ is defined by the equality

$$\begin{aligned} \varepsilon_0(t) = 2\,l^2\Big(\int_a^b h(\tau)\,d\tau\Big)^{-1}\Big[\int_t^b h(\tau)\,d\tau\int_a^t\Big(\int_a^s h(\tau)\,d\tau\Big)\varphi(\tau)\,d\tau \\ + \int_a^t h(\tau)\,d\tau\int_t^b\Big(\int_s^b h(\tau)\,d\tau\Big)\varphi(\tau)\,d\tau\Big], \end{aligned}$$

and therefore

$$\varepsilon_0(a) = 0, \quad \varepsilon_0(b) = 0. \tag{27}$$

From (25)–(27) we conclude that the set of functions $W(\mathbb{B}_\rho)$ is equicontinuous and uniformly bounded, i.e., the continuous operator W transforms the bounded convex set $\mathbb{B}_\rho$ into its compact subset and, by virtue of the Shauder's principle of fixed point, equation (19) has at least one solution. □

Theorem 2 is proved in an analogous manner with the only difference that we use Lemma 2.

Acknowledgement

The work was supported by a fellowship from the INTAS YS 2001-2/80 as for a visiting professor at the University of Udine, Italy.

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(Received 1.05.200)

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