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APPLICATIONS OF THE METHOD OF BARRIERS I. SOME BOUNDARY-VALUE PROBLEMS

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Abstract. Two-point boundary-value problems are investigated by the method of barriers for ordinary differential equations of the second and the third order.

1. In studying boundary-value problems for ordinary differential equations one often comes across situations where the most popular methods of obtaining a priori estimates including the maximum principle, the method of successive approximations, etc., turn out to be inapplicable because the problem under investigation possesses singularities. In that case the proofs have to be accomplished by methods employing conditions for the existence of solutions that are close to the necessary and sufficient ones. Among these methods is, in particular, the method of barrier functions used to investigate boundary-value problems for ordinary differential equations.

Let us consider the Dirichlet boundary-value problem for an ordinary differential equation of the second order

$$x'' = f(t,x,x'), \quad t \in (a,b), \tag{1}$$

$$x(a) = A, \quad x(b) = B. \tag{2}$$

Definition 1. Functions $\alpha(t) \in C^2(a,b)$ and $\beta(t) \in C^2(a,b)$ such that the inequalities

$$\alpha(t) \le \beta(t), \quad t \in (a,b), \tag{3}$$

$$\alpha'' \ge f(t,\alpha,\alpha'), \quad \beta'' \le f(t,\beta,\beta'), \quad t \in (a,b), \tag{4}$$

$$\alpha(a) \le A \le \beta(a), \quad \alpha(b) \le B \le \beta(b). \tag{5}$$

are valid are called the lower $\alpha(t)$ and upper $\beta(t)$ barrier functions (barriers) for problem (1), (2).

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A relationship between the existence of a solution of problem (1), (2) and the existence of the lower and upper barriers was established by M. Nagumo in [1].

Theorem 1 (M. Nagumo). *Let the following conditions be fulfilled:*
1) $f(t, x, y) \in C([a, b] \times R^2)$;
2) *there exists a positive function* $\varphi(z) \in C[0, \infty)$ *such that the relations*

$$\int^{\infty} \frac{sds}{\varphi(s)} = \infty, \quad |f(t, x, y)| \le \varphi(|y|) \tag{6}$$

hold for $\forall (t, x, y) \in [a, b] \times R^2$;
3) *there exist the lower* $\alpha(t)$ *and upper* $\beta(t)$ *barrier functions.*

Then problem (1), (2) *has a solution; moreover, for this solution we have the estimate*

$$\alpha(t) \le x(t) \le \beta(t), \quad t \in [a, b].$$

In what follows the function $\varphi(z)$ from the formulation of Theorem 1 will be called the Nagumo function.

The idea of using differential inequalities to prove the existence of solutions of initial and boundary-value problems originates from the method of a priori estimates developed by Bernshtein [2] and Chaplygin [3]. In particular, Bernshtein showed that if conditions (6) are not fulfilled, then problem (1), (2) has no solution in the general case. The method of barriers was further developed in [4]–[6] and other papers.

It was shown in [7] that Theorem 1 remains valid if the barrier functions are continuous, twice piecewise-continuously differentiable, and satisfy inequalities (4) at points of the existence of second derivatives. At points at which the condition of continuity of the first derivatives of barrier functions is not fulfilled, it is assumed that there exist the right and left derivatives of these functions and the inequalities

$$\alpha'(t - 0) \le \alpha'(t + 0), \quad \beta'(t - 0) \ge \beta'(t + 0) \tag{7}$$

hold.

In [8] an analogue of Nagumo's theorem is proved for some boundary-value problems connected with differential equations of the third order, whereas in [9] and [10] the method of barriers is used to investigate solutions of boundary-value problems for equations of the nth order and for systems of equations.

Note that Nagumo's theorem guarantees the existence but not the uniqueness of solutions of the original problem. The theorem does not exclude the possibility of the existence of several solutions bounded by given barriers, as well as of the existence of other pairs of barrier functions and, accordingly, of other solutions.

Note also that the existence of barriers is not only sufficient but also necessary for the existence of a solution of the boundary-value problem, since by virtue of the fact that inequalities (3)–(5) are not rigorous, the solution itself is, in particular, the lower and upper barriers. In this connection it is clear that the question of constructing (or even of proving the existence of) barriers for a concrete boundary-value problem remains open and, generally speaking, is as difficult as proving the existence of a solution of this problem.

The latter statement obviously holds when for the considered boundary-value problem only solutions of this problem may themselves play the role of barrier functions, i.e., when inequalities (3)–(5) become equalities. It appears that such a situation occurs for a whole class of boundary-value problems.

In what follows barrier functions that are not solutions of the considered problem will be called nontrivial barriers.

Definition 2. An equation

$$Lx \equiv (p(t)x')' + q(t)x = f(t), \quad t \in (a,b), \tag{8}$$

is called an equation without conjugate points if in the interval $[a,b]$ the corresponding homogeneous equation has no nontrivial solutions vanishing more that at one point of this interval.

Theorem 2. *For the boundary-value problem connected with equation* (8) *and the boundary conditions*

$$x(a) = x(b) = 0, \tag{9}$$

where $p(t)$, $p'(t)$, $q(t)$,$f(t) \in C(a,b)$, $p(t) > 0$, *nontrivial barriers exist if and only if equation* (8) *is an equation without conjugate points.*

Proof. Let $x = x_0(t)$ be a solution of problem (8), (9) and $\beta_1(t)$ be a nontrivial upper barrier such that $x_0(t) \le \beta_1(t)$. Then the function $\beta(t) = \beta_1(t) - x_0(t)$ is a nontrivial upper barrier for the homogeneous problem corresponding to problem (8), (9).

Let the equation $Lx = 0$ have a nontrivial solution $x = x_1(t)$ vanishing at least at two points $t = t_1$, $t = t_2$ of the interval $[a,b]$; it can be assumed without loss of generality that $x_1(t) > 0$ for $t \in (t_1,t_2)$.

We shall first consider the interval $[t_1,t_2]$, assuming that in this interval there are no other conjugate points of equation (8). Let in the interval $[t_1,t_2]$ the upper barrier function $\beta(t)$ not coincide with the solution of the equation $Lx = 0$ vanishing at the points $t = t_1$, $t = t_2$. Then the function $\beta(t)$ is nonnegative for $t \in [t_1, t_2]$ and at least one of the following inequalities holds:

$$\beta(t_1) > 0, \quad \beta(t_2) > 0, \quad \left[(p(t)\beta')' + q(t)\beta(t)\right]\big|_{t=t_0} < 0,$$

where t_0 is some point of $[t_1, t_2]$. Therefore

$$(p(t)\beta')' + q(t)\beta = \varphi(t), \quad t \in [t_1, t_2],$$
$$\beta(t_1) \geq 0, \quad \beta(t_2) \geq 0,$$

where $\varphi(t)$ is a nonpositive function.

If $x = x_2(t)$ is a solution of the equation $Lx = 0$ in the interval $[a, b]$, is linearly independent of the solution $x = x_1(t)$, and is such that $x_2(t_1) = 1$, then the function $\beta(t)$ has the form

$$\beta(t) = c_1 x_1(t) + c_2 x_2(t) + \int_{t_1}^{t} \frac{\varphi(\tau)}{p(\tau)w(\tau)} [x_1(\tau)x_2(t) - x_1(t)x_2(\tau)]d\tau,$$

where $w(t)$ is the Wronsky determinant of the system of functions $x_1(t)$, $x_2(t)$ and c_1, c_2 are some constants. We have

$$\beta(t_1) = c_2 \geq 0,$$
$$\beta(t_2) - c_2 x_2(t_2) = x_2(t_2) \int_{t_1}^{t_2} \frac{\varphi(\tau)}{p(\tau)w(\tau)} x_1(\tau)d\tau.$$

Since $x_2(t_2) < 0$, $w(\tau) < 0$, then the left-hand side of the latter equality is nonnegative, while the right-hand side is nonpositive. Therefore $c_2 = 0$, $\beta(t_2) = 0$, $\varphi(t) \equiv 0$ for $t \in [t_1, t_2]$.

If $c_1 \neq 0$, then $\beta'(t_2 - 0) < 0$; at the same time $\beta'(t_2 + 0) \geq 0$, whereas $\beta(t) \geq 0$ for $t > t_2$. Thus condition (7) can be fulfilled at the point $t = t_2$ only if $c_1 = 0$, and therefore $\beta(t) \equiv 0$ for $t \in [t_1, t_2]$. Repeating, if necessary, our reasoning, we can show that if $t = t_k$ is the greatest zero of the function $x_1(t)$ in the interval $[a, b]$, then $\beta(t) \equiv 0$ for $t \in [t_1, t_k]$.

If $t_k < b$, then $\beta'(t_k + 0) = 0$ and the function $\beta(t)$ satisfies the condition

$$L\beta \equiv (p(t)\beta')' + q\beta = \varphi(t), \quad \varphi(t) \leq 0$$

for $t > t_k$.

Multiplying both sides of the latter equality by the function $x_1(t)$ and integrating from t_k to t, we obtain

$$\psi(t) = (\beta(t)/x_1(t))' = [p(t)x_1^2(t)]^{-1} \int_{t_k}^{t} \varphi(\tau)x_1(\tau)d\tau.$$

Let $x_1(t) > 0$ for $t > t_k$, $\varphi(t) \not\equiv 0$. We introduce the notation

$$t_0 = \inf_{t>t_k} \{t|\varphi(t) < 0\}.$$

Due to the above reasoning $\beta(t_0) = 0$, $\beta'(t_0) = 0$. The function $\beta(t)/x_1(t)$ is positive and decreasing for $t > t_0$. Therefore it has a positive limit value (finite or infinite) for $t \to t_0 + 0$. But this is impossible, since if $t_0 > t_k$, then $\beta(t_0)/x_1(t_0) = 0$ and if $t_0 = t_k$, then

$$\lim_{t\to t_k+0} \beta(t)/x_1(t) = \lim_{t\to t_k+0} \beta'(t)/x_1'(t) = 0.$$

The case $x_1(t) < 0$ for $t > t_k$ is treated similarly. Thus $\beta(t) \equiv 0$ for $t \in [t_1, b]$. The same proof applies to the relation $\beta(t) \equiv 0$ for $t \in [a, t_1]$ when $t_1 > a$.

Let us now prove the statement of the theorem when equation (8) has no conjugate points in the interval $[a, b]$. It is obvious that in that case the solution $x = x(t)$ of problem (8), (9) exists and is unique. We shall construct two linearly independent solutions $x = x_1(t)$ and $x = x_2(t)$ of the corresponding homogeneous equation in a manner such that they are positive for $t \in (a, b)$, $x_1(a) = 0$, $x_2(b) = 0$. Let $\varphi(t)$ be an arbitrary continuous function in the interval $[a, b]$, $\varphi(t) \le 0$ for $t \in [a, b]$. Then a general solution of the equation $Lz = \varphi(t)$ is written as

$$z(t) = c_1x_1(t) + c_2x_2(t) + \int_a^t \frac{\varphi(\tau)}{p(\tau)w(\tau)} x_1(\tau)x_2(t)d\tau +$$
$$+ \int_t^b \frac{\varphi(\tau)}{p(\tau)w(\tau)} x_1(t)x_2(\tau)d\tau.$$

Since $w(t) < 0$ for $t \in [a, b]$, the function $z(t)$ is nonnegative for nonnegative values of the constants c_1, c_2. If $|c_1| + |c_2| + |\varphi(t)| \not\equiv 0$, then the functions $\alpha(t) = x(t) - z(t)$, $\beta(t) = x(t) + z(t)$ are nontrivial lower and upper barriers for the said solution of problem (8), (9). □

Corollary. *If equation* (8) *has no conjugate points in the interval* $[a, b]$, *then there exists a twice continuously differentiable function* $v(t) \ge 1$ *such that the change of the desired function* $x(t) = v(t)u(t)$ *leads to the equation*

$$L_1u \equiv (p_1(t)u')' + q_1(t)u = f_1(t),$$

where $p_1(t)$, $p_1'(t)$, $q_1(t)$, $f_1(t)$ *are continuous functions,* $q_1(t) \le -1$ *for* $t \in [a, b]$.

Indeed, if $p_1(t) = p(t)v(t)$, then

$$q_1(t) = p(t)v'' + p'(t)v' + q(t)v.$$

Since (8) is the equation without conjugate points, by an appropriate choice of the function $\varphi(t)$ and constants c_1, c_2 one may choose a nontrivial upper barrier to play the part of the function $v(t)$.

The latter statement implies, in particular, that if the original equation is the equation without conjugate points in the considered interval, then by a linear change of the desired function this differential equation can be reduced to a differential equation to which we can apply the maximum principle (see, for example, [11]).

An **example** illustrating Theorem 2 is the problem

$$x'' + k^2 x = 0, \quad t \in (0,1),$$
$$x(0) = x(1) = 0.$$

If $k^2 < \pi^2$, then this equation is the one without conjugate points, and functions $\beta(t) = -\alpha(t) = \sin m(t+\delta)$ can be taken as nontrivial barriers for some values of the constants m, δ, $k \le m \le \pi$, $0 \le \delta \le \delta_0$ provided that the constant δ_0 is sufficiently small. If, however, $k^2 \ge \pi^2$, then the absence of nontrivial barriers for the considered problem is substantiated in [4].

In the case of boundary-value problems for nonlinear differential equations one might expect the statement of Theorem 2 on the existence or nonexistence of nontrivial barriers to be valid if the corresponding equation in variations is or is not the equation without conjugate points. But concrete examples show that an analogue of Theorem 2 does not hold for nonlinear equations.

Indeed, as shown in [6], the boundary-value problem

$$Lx \equiv x'' + \lambda(x - x^3) = 0, \quad t \in (0,1),$$
$$x(0,\lambda) = x(1,\lambda) = 0,$$

has at least several solutions for each sufficiently large value of the parameter λ; among them are, in particular, solutions

$$x = x_1(t,\lambda), \quad x = x_2(t,\lambda), \quad 0 < x_1(t,\lambda) < 1, \quad -1 < x_2(t,\lambda) < 0,$$

such that the uniform relations

$$\lim_{\lambda\to\infty} x_1(t,\lambda) = 1, \quad \lim_{\lambda\to\infty} x_2(t,\lambda) = -1$$

hold in each interval $0 < a < t < b < 1$.

It is easy to see that for $c \ge 1$ the functions $\beta(t) \equiv c = const$, $\alpha(t) \equiv -c$ are nontrivial barriers for the solutions $x_1(t,\lambda)$, $x_2(t,\lambda)$, $x_3(t,\lambda) \equiv 0$, whereas for $c_1 > 1$ the functions $\beta_1(t) = c_1 x_1(t,\lambda)$, $\alpha_1(t) = c_1^{-1} x_1(t,\lambda)$

are nontrivial upper and lower barriers only for the solution $x = x_1(t, \lambda)$. However, the equation in variations for a trivial solution of this problem

$$\delta'' + \lambda\delta = 0,$$

is evidently the equation with conjugate points for $\lambda > \pi^2$.

2. The method of barrier functions is also used to prove the existence of solutions of boundary-value problems connected with differential equations of third and higher orders. In that case, however, the right-hand sides of the equations must be subjected to additional restrictions, limiting substantially the applicability domain of the proven theorems. At the same time, these restrictions are, by all means, not always needed. For example, for boundary-value problems of the form

$$x''' = f(t, x, x', x''), \quad t \in (a, b), \tag{10}$$

$$x(a) = A_0, \quad x'(a) = A_1, \quad x'(b) = B_1, \tag{11}$$

considered in [8–10] and other papers the corresponding generalization of Nagumo's theorem is proved when a sufficiently rigorous assumption is made about a monotone nondecrease of the function $f(t, x, y, u)$ with respect to the second argument. However, the requirement that the function $f(t, x, y, u)$ be monotone can be rejected at the expense of a certain additional condition imposed on the barrier functions (which appears unessential for quite a number of specific problems). To be more exact, we have

Theorem 3. *Let a function $f(t, x, y, u) \in C([a, b] \times R^3)$ be continuously differentiable with respect to the variables x, y, and let u satisfy condition* (6) *with respect to the variable u. Let there exist upper $\alpha(t)$ and lower $\beta(t)$ barrier functions such that the following relations hold:*

$$\alpha(t) \le \beta(t), \ \ \alpha'(t) \le \beta'(t), \ \ t \in (a, b), \tag{12}$$

$$\alpha''' \ge f(t, \alpha, \alpha', \alpha''), \ \ \beta''' \le f(t, \beta, \beta', \beta''), \quad t \in (a, b), \tag{13}$$

$$\alpha(a) = \beta(a) = A_0, \quad \alpha'(a) \le A_1 \le \beta'(a), \quad \alpha'(b) \le B_1 \le \beta'(b). \tag{14}$$

Then the boundary-value problem (10), (11) *has at least one solution satisfying the inequalities*

$$\alpha(t) \le x(t) \le \beta(t), \quad \alpha'(t) \le x'(t) \le \beta'(t), \quad t \in [a, b].$$

The *proof* of this theorem practically repeats that of Theorem 7 from [8], and the corresponding inequalities ensuing (see [8]) from the monotonicity of the function $f(t, x, y, u)$ with respect to the second argument will be valid in the absence of monotonicity too if the first of relations (14) is fulfilled. A complete proof of Theorem 3 based on Shauder's theorem on a fixed point of a completely continuous operator can be found in [12].

A statement similar to that of Theorem 3 can also be proved when the barrier functions are continuous, together with the first derivatives, in the interval $[a, b]$ and possess piecewise continuous derivatives of the second and the third order in the interval (a, b). In that case, besides conditions (12)–(14), additional restrictions

$$\alpha''(t+0) \geq \alpha''(t-0), \quad \beta''(t+0) \leq \beta''(t-0), \quad t \in (a, b). \tag{15}$$

are imposed on the functions $\alpha(t)$, $\beta(t)$.

Moreover, Theorems 2, 3 can be generalized to problems with boundary conditions of other types. Finally, all the above-proven statements are generalized to the boundary-value problems for differential equations of form (1), (10) whose right-hand sides may have discontinuities of the first kind with respect to the variable t in a finite number of points of the interval (a, b); such proofs can be found in [12].

Remark. Theorem 3 turns out to be rather helpful in substantiating asymptotic representations of boundary-value problems connected with singularly perturbed differential equations of the third order. In problems of this kind, barrier functions satisfying relations (14) are constructed in a sufficiently natural manner, while the requirement that the function $f(t, x, y, u)$ be nondecreasing with respect to the second argument excludes from consideration many interesting problems of applied nature, including problems with the so-called internal transition layers.

3. Barrier functions and the properties of the existence theorems based on these functions appear also to be very useful in investigating differential equations of the form

$$p(t)x' + h(t, x) = 0, \quad t \in (a, b), \tag{16}$$

$p(t) \in C^k[a, b]$, $h(t, x) \in C^k([a, b] \times R)$ if it is assumed that the function $p(t)$ vanishes in a finite number of points or subintervals of the interval (a, b). The main questions arising during the investigation of equations of form (16) are those of the existence of solutions in the entire interval (a, b) and of their smoothness, depending on the properties of the functions $p(t)$, $h(t, x)$ and the correctness of the formulation of initial and boundary-value problems for such equations.

In [13] these questions are studied for the case of linear dependence of the function $h(t, x)$ on the variable x. The statements below generalize the results obtained in [13].

Lemma 1. *Let the following conditions be satisfied:*

1) $p(t) \in C^k[a, b]$, $h(t, x) \in C^k([a, b] \times R)$, $p(b) = 0$;

2) $h(b, 0) = 0$, $h'_x(t, x) \geq h_0 > 0$ *for* $(t, x) \in [b - \delta, b] \times R$ *where* $\delta > 0$ *is some constant,* $\delta \leq b - a$.

In that case if $p(t) < 0$ for $t \in [b-\delta, b)$, then equation (16) has, in the interval $[b-\delta, b]$, a unique solution where $x = x(t) \in C^k[b-\delta, b]$ and $x(b) = 0$; if, however, $p(t) > 0$ for $t \in [b-\delta, b]$, then for any constant A there exists, in the interval $[b-\delta, b)$, a solution $x = x(t, A)$ of equation (16),

$$x(b-\delta, A) = A, \quad x(b, A) = 0, \quad x(t, A) \in C^m[b-\delta, b]$$

where $m = \min(k, n)$ and the constant n is defined as the maximum possible integer nonnegative number satisfying the inequality

$$np'(b) + h'_x(b, 0) > 0. \tag{17}$$

Proof. Rewrite equation (16) as

$$p(t)x' + x \int_0^1 h'_x(t, \theta x)d\theta = -h(t, 0)$$

and construct a sequence of functions $\{x_r(t)\}$, $x_0(t) \equiv 0$, where the functions $x_r(t)$ are defined as solutions of the equations

$$p(t)x'_r + x_r \int_0^1 h_x(t, \theta x_{r-1})d\theta = -h(t, 0), \quad r \geq 1. \tag{18}$$

Assume that $p(t) < 0$ for $t \in [b-\delta, b)$. According to [13], there exists, in the interval $[b-\delta, b]$, a unique solution of equation (18) which is k times differentiable and vanishes for $t = b$. We write this solution in the form

$$\begin{aligned} x_r(t) = -h(t, 0) \Big/ \int_0^1 h'_x(t, \theta x_{r-1}(t))d\theta + \\ + \int_b^t \Big[h(\tau, 0) \Big/ \int_0^1 h'_x(\tau, \theta x_{r-1}(\tau))d\theta\Big] \times \\ \times \exp\Big\{\int_t^\tau \Big[\int_0^1 h'_x(s, \theta x_{r-1}(s))d\theta\Big] \Big/ p(s)ds\Big\}d\tau. \end{aligned}$$

It is easy to ascertain that the functions $x_r(t)$ are uniformly bounded in the interval $[b-\delta, b]$ and have uniformly bounded derivatives up to order k. Applying Arcela's theorem and relation (16), we begin by proving the first statement of the lemma. The second statement is proved similarly. □

Lemma 2. *Let the following conditions be fulfilled:*

1) $p(t) \in C^k[a,b],$, $h(t,x) \in C^k([a,b] \times R)$, $p(a) = 0$;

2) $h(a,0) = 0$, $h'_x(t,x) \geq h_0 > 0$ *for* $(t,x) \in [a, a+\delta] \times R$, *where* $\delta > 0$ *is some constant,* $\delta \leq b - a$.

In that case if $p(t) > 0$ *for* $t \in [a, a+\delta]$, *then equation* (16) *has, in the interval* $[a, a+\delta]$, *a unique solution where* $x = x(t) \in C^m[a, a+\delta]$, $x(a) = 0$; *if, however,* $p(t) < 0$ *for* $t \in (a, a+\delta]$, *then for any constant* B *there exists, in the interval* $t \in (a, a+\delta]$, *a solution* $x = x(t,B)$ *of equation* (16),

$$x(a+\delta, B) = B, \quad x(a,B) = 0, \quad x(t,B) \in C^m[a, a+\delta]$$

where the constant m *is defined by inequality* (17) *with the coordinate* b *replaced by the coordinate* a.

The *proof* of this lemma repeats that of Lemma 1.

If the condition $h'_x(t,x) \geq h_0 > 0$ (or the condition $h'_x(t,x) \leq -h_0 < 0$) is fulfilled for all $(t,x) \in [a,b] \times R$ with the function $p(t)$ vanishing only at one of the ends of this interval, then in Lemmas 1, 2 one can take the constant δ equal to $b - a$ and obtain the solution of equation (16) in the entire interval $[a,b]$. However, if the function $h'_x(t,x)$ vanishes outside the neighborhood of the point at which the function $p(t)$ vanishes, then the existence of solutions in the entire interval $[a,b]$ is guaranteed by the existence of barrier functions.

Theorem 4. *Let the conditions of Lemma* 1 *be satisfied. Let, moreover, there exist differentiable barrier functions* $\alpha(t)$, $\beta(t)$ *such that the inequalities*

$$\alpha(t) \leq \beta(t), \quad p(t)\alpha' + h(t,\alpha) \leq 0, \quad p(t)\beta' + h(t,\beta) \geq 0 \tag{19}$$

are satisfied for $t \in [a, b-\delta]$. *In that case if* $p(t) < 0$ *for* $t \in [a,b)$ *and the functions* $\alpha(t)$, $\beta(t)$ *satisfy the condition* $\alpha(b-\delta) \leq x(b-\delta) \leq \beta(b-\delta)$, *then for* $t \in [a,b]$ *there exists a unique solution of equation* (16); *this solution belongs to the space* $C^k[a,b]$. *If, however,* $p(t) > 0$ *for* $t \in [a,b)$, *then for any constant* A, $\alpha(a) \leq A \leq \beta(a)$, *there exists, for* $t \in [a,b]$, *a unique solution of equation* (16) *satisfying the condition* $y(a) = A$; *this solution belongs to the space* $C^m[a,b]$, *where the constant* m *is defined by Lemma* 1.

Theorem 5. *Let the conditions of Lemma* 2 *be satisfied. Let, moreover, there exist differential barrier functions* $\alpha(t)$, $\beta(t)$ *such that inequalities* (18) *are satisfied for* $t \in [a+\delta, b]$. *In that case if* $p(t) > 0$ *for* $t \in [a,b]$ *and the functions* $\alpha(t)$, $\beta(t)$ *satisfy the condition* $\alpha(a+\delta) \leq x(a+\delta) \leq \beta(a+\delta)$, *then for* $t \in [a,b]$ *there exists a unique solution of equation* (16); *this solution belongs to the space* $C^k[a,b]$. *If, however,* $p(t) < 0$ *for* $t \in (a,b]$, *then for any constant* B, $\alpha(b) \leq B \leq \beta(b)$, *there exists, for* $t \in [a,b]$, *a unique solution of equation* (16) *satisfying the condition* $y(b) = B$; *this solution belongs to the space* $C^m[a,b]$, *where the constant* m *is defined by Lemma* 2.

Theorems 4, 5 are proved by the methods of successive approximations. The statements of these theorems follow from the fact that for $t \in [a, b - \delta]$ (accordingly, for $t \in [a + \delta, b]$) the successive approximations of $x_r(t)$ cannot come out of the domain $D^-\{(t,x)|t \in [a, b-\delta], \alpha(t) \le x \le \beta(t)\}$ (accordingly, out of the domain $D^+\{(t,x)|t \in [a+\delta, b], \alpha(t) \le x \le \beta(t)\}$).

The theorems in question enable us to describe the properties of solutions of equation (16) in some interval provided that the function $p(t)$ becomes zero at the internal point $t = c$ of the interval $[a, b]$ and has no other zeros in this interval; in that case the properties of solutions of equation (16) are described by

Theorem 6. *If $t = c$, $c \in (a, b)$ is the only zero of the function $p(t)$ in the interval $[a, b]$ and Theorems* 4 *and* 5*, in which points b are, respectively, replaced by the point c, are fulfilled in the intervals $[a, c]$ and $[c, b]$, respectively, then the following statements are valid:*

– if $p(t)(t - c) > 0$ for $t \ne c$, then there exists a unique solution of equation (16) *defined on the interval $[a, b]$;*

– if $p(t)(t - c) < 0$ for $t \ne c$, then for any constants A, B there exists a solution of equation (16) *satisfying the conditions $x(a) = A$, $x(b) = B$;*

– if $p(t) < 0$ for $t \ne c$, then for any constant A there exists a solution of equation (16) *satisfying the condition $x(b) = A$;*

– if $p(t) > 0$ for $t \ne c$, then for any constant A there exists a solution of equation (16) *satisfying the condition $x(a) = A$.*

All these solutions are defined uniquely and belong to the space $C^m[a, b]$ where the number m is defined by means of inequality (17)*, with the coordinate b replaced by the coordinate c.*

Statements analogous to Theorem 6 can be formulated also for the case in which in the interval $[a, b]$ there is a finite number of points and subintervals at which the function $p(t)$ vanishes. One can easily generalize these statements also to the case where the functions $p(t)$, $h(t, x)$ have a finite number of points of first kind discontinuities with respect to the variable t.

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