GEORGIAN MATHEMATICAL JOURNAL: Vol. 2, No. 2, 1995, 151-164

SEQUENTIAL CONVERGENCE IN TOPOLOGICAL VECTOR SPACES

A. K. KATSARAS AND V. BENEKAS

Abstract. For a given linear topology τ, on a vector space E, the finest linear topology having the same τ convergent sequences, and the finest linear topology on E having the same τ precompact sets, are investigated. Also, the sequentially bornological spaces and the sequentially barreled spaces are introduced and some of their properties are studied.

Introduction

For a locally convex space E, Webb constructed in [1] the finest locally convex topology on E having the same convergent sequences as the initial topology and the finest locally convex topology having the same precompact sets as the initial topology.

In this paper, using the notion of a string introduced in [2], we construct the finest linear topology which has the same convergent sequences as the initial linear topology and the finest linear topology having the same precompact sets. Some of the properties of these topologies are studied. We also give the notion of a sequentially bornological space. A space E is S-bornological if every bounded linear map from E to an arbitrary topological vector space is sequentially continuous. This is equivalent to the following: Every bounded pseudo-seminorm on E is sequentially continuous. The concept of a sequentially barreled space is also given and some of the properties of the S-bornological spaces and the S-barreled spaces are investigated.

1. Preliminaries

All vector spaces considered in this paper will be over the field of real numbers or the field of complex numbers. Following [2], we will call a string, in a vector space E, any sequence $U = (V_n)$ of balanced absorbing sets such

1991 *Mathematics Subject Classification.* 46 A.
Key words and phrases. String, sequential space, sequentially barreled space, sequentially bornological space, sequential convergence.

1072-947X/95/0300-0151$07.50/0

that $V_{n+1} + V_{n+1} \subset V_n$ for all n. A string $U = (V_n)$, in a topological vector space E, is called topological if every V_n is a neighborhood of zero. The string U is called closed (resp. bornivorous) if every V_n is closed (resp. bornivorous). Every string $U = (V_n)$ defines a linear topology $\tau + U$ having as a base at zero the sequence $\{V_n : n \in N\}$. The set $N_U \bigcap\limits_n V_n$ is a subspace of E. We will denote by E_U the quotient space E/N_U. The space E_U with the quotient topology is a metrizable topological vector space. By $\widehat{E}_U$ we will denote the completion of E_U, and so $\widehat{E}_U$ is a Frechet space and is a complete metrizable topological vector space. Any family $\mathcal{F}$ of strings in E defines a linear topology, namely the supremum of the topologies $\tau_{_U}$, $U \in \mathcal{F}$. A family $\mathcal{F}$ of topological strings, in a topological vector space E, is called a base for the topological strings if for any topological string $W = (W_n)$ there exists $U = (V_n)$ in $\mathcal{F}$ with $U \leq W$, where $U \leq W$ means that $V_n \subset W_n$ for all n. The family of all closed topological strings is a base for the topological strings.

Definition 1.1 ([3,6.1]). A pseudo-seminorm, on a vector space E over $\mathbb{K}$, is a function $p : E \to \mathbb{R}$ with the following properties:

1) $p(x + y) \leq p(x) + p(y)$, for all x, y in E.
2) $p(\lambda x) \leq p(x)$ for all $x \in E$ when $|\lambda| \leq 1$.
3) If $\lambda_n \to 0$, then $\mathbb{K}$ for all $p(\lambda_n x) \to 0$ in $x \in E$.
4) If $p(x_n) \to 0$, then $p(\lambda_n x) \to 0$ for all λ in $\mathbb{K}$.

If $p(x) > 0$ when $x \neq 0$, then p is called a pseudo-norm.

Lemma 1.2. *If p is any pseudo-seminorm on E, then $d_p : E \times E \to \mathbb{R}$, $d_p(x, y) = p(x - y)$, is a pseudo-metric, and the corresponding topology τ_p is linear. If p is a pseudo-norm, then d_p is a metric.*

Lemma 1.3. *If p is a pseudo-seminorm on E and $V_n = \{x : p(x) \leq 2^{-n}\}$, then $U = (V_n)$ is a string and $\tau_{_U} = \tau_p$.*

Lemma 1.4 ([3,6.1]). *If $U = (V_n)$ is a string in E, then there exists a pseudo-seminorm $p_{_U}$ on E such that*

$$\left\{x \in E \ : \ p_{_U}(x) \leq 2^{-n-1}\right\} \subset V_n \subset \left\{x \ : \ p_{_U}(x) \leq 2^{-n}\right\}$$

and so $\tau_{_U} = \tau_{p_{_U}}$.

Any linear topology on E is generated by the family of all continuous pseudo-seminorms on E.

Lemma 1.5. *Let (p_n) be a sequence of pseudo-seminorms on E and define p on E by $p(x) = \sum\limits_{n=1}^{\infty} \dfrac{1}{2^n} \cdot \dfrac{p_n(x)}{1 + p_n(x)}$. Then p is a pseudo-seminorm on E and τ_p coincides with the topology generated by the sequence (p_n) of pseudo-seminorms.*

Lemma 1.6. *A topological vector space E is pseudo-metrizable iff its topology is generated by a countable family of pseudo-seminorms and this is true iff the topology of E is generated by a single pseudo-seminorm.*

2. Sequential Spaces

Definition 2.1. Let V be a subset of a topological vector space E. Then V is called a sequential neighborhood (S-neighborhood) of zero if every null sequence in E lies eventually in V. The set V is called sequentially closed (S-closed) if every x in E, which is a limit of a sequence in V, belongs to V.

Definition 2.2. A string $U = (V_n)$, in a topological vector space E, is called S-topological if every V_n is an S-neighborhood of zero.

Lemma 2.3. *Let $U = (V_n)$ be a string in a topological vector space E and let p be a pseudo-seminorm on E such that $\tau_p = \tau_U$. Then, U is S-topological iff p is sequentially continuous.*

Proof. Let U be S-topological and let (x_n) be a sequence in E converging to some x. Given $\varepsilon > 0$, the set $W = \{x : p(x) \leq \varepsilon\}$ is a τ_p-neighborhood of zero and so there is m such that $V_m \subset W$. Since $x_n - x \to 0$ in E and V_m is an S-neighborhood, there exists n_0 such that $x_n - x \in V_m$ when $n \geq n_0$, and so $p(x_n - x) \leq \varepsilon$ if $n \geq n_0$. The converse is proved analogously. $\square$

Definition 2.4. A topological vector space E is said to be a sequential space if every S-topological string in E is topological.

Proposition 2.5. *Let $U = (V_n)$ be a string in a topological vector space and let $\widehat{E}_U$ be the completion of E_U. If $J = J_U : E \to E_U$ is the canonical mapping, then the following are equivalent:*

(1) *U is S-topological in E.*
(2) *J is sequentially continuous.*
(3) *J is sequentially continuous as a map from E to $\widehat{E}_U$.*

Proof. It is clear that (1)⇒(2)⇒(3).

(3)⇒(1). For each subset A of E_U, let $\overline{A}$ denote its closure in $\widehat{E}_U$. Then, the family $\{\overline{J(V_n)} : n \in N\}$ is a base at zero in $\widehat{E}_U$. Since the closure of $J(V_{n+2})$ in E_U is contained in $J(V_{n+2}) + J(V_{n+2}) \subset J(V_{n+1})$, we have

$$J^{-1}(\overline{V_{n+2}}) = J^{-1}(J(\overline{V_{n+2}}) \cap E_U) \subset J^{-1}(J(V_{n+1})) =$$
$$= V_{n+1} + N_U \subset V_{n+1} + V_{n+1} \subset V_N,$$

which shows that V_n is an S-neighborhood of zero in E since J is sequentially continuous. $\square$

Proposition 2.6. *For a topological vector space E, the following are equivalent:*

(1) *E is sequential.*

(2) *Every sequentially continuous linear map, from E to an arbitrary topological vector space, is continuous.*

(3) *Every sequentially continuous linear map, from E to an arbitrary Frechet space, is continuous.*

(4) *Every sequentially continuous pseudo-seminorm on E is continuous.*

Proof. (1)⇒(2). Let $f : E \to F$ be a linear and sequentially continuous and let V be a neighborhood of zero in F. There exists a topological string $U = (V_n)$ in F with $V_1 = V$. Since f is sequentially continuous, $f^{-1}(U)$ is S-topological and hence topological by (1). Hence $f^{-1}(V)$ is a neighborhood of zero in E.

(3)⇒(1). Let $U = (W_n)$ be an S-topological string. By Proposition 2.5, the canonical mapping $J : E \to \widehat{E}_U$ is sequentially continuous and hence by (3) continuous. As in the proof of the implication (3)⇒(1) in the preceding Proposition, it follows that each V_n is a neighborhood of zero in E.

(1)⇒(4). Let p be a sequentially continuous pseudo-seminorm and set $V_n = \{x : p(x) \leq 2^{-n}\}$. Then $U = (V_n)$ is an S-topological string and hence topological, which implies that p is continuous.

(4)⇒(1). Let $U = (W_n)$ be an S-topological string and let S be as in Lemma 1.4. By Lemma 2.3, p_U is sequentially continuous and hence by (4) continuous, which implies that (W_n) is topological. □

Proposition 2.7. *Let $\{E_\alpha : \alpha \in I\}$ be a family of topological vector spaces, E a vector space and, for each α, $f_\alpha : E_\alpha \to E$ a linear map. If τ is the finest linear topology on E for which each f_α is continuous and if each E_α is sequentially, then (E, τ) is sequential.*

Proof. If U is an S-topological string in (E, τ), then each $f_\alpha^{-1}(U)$ is S-topological in E_α and so it is topological in E, which implies that U is topological in (E, τ). □

Corollary 2.8. *Quotient space and direct sums of sequential spaces are sequential.*

The family $\mathcal{F}_S$ of all S-topological strings in (E, τ) is direct. We will denote by τ^S the linear topology generated by the family $\mathcal{F}_S$.

Proposition 2.9. (1) *τ^S is the coarsest sequential topology finer than τ.*

(2) *τ^S is the finest linear topology on E having the same with τ convergent sequences.*

(3) *The topologies τ and τ^S have the same bounded sets.*

(4) *If τ_1 is a linear topology such that every τ-null sequence is also τ_1-null, then $\tau_1 \leq \tau^S$.*

(5) *A linear map f, from (E,τ) to an arbitrary topological vector space F, is sequentially continuous iff it is τ^S-continuous.*

(6) *τ^S is generated by the family of all sequentially continuous pseudo-seminorms on (E,τ).*

Proof. Clearly $\tau \leq \tau^S$. It follows directly from the definition that every τ^S-null sequence is also τ^S-null and so τ and τ^S have the same convergent sequences and the same bounded sets. If τ_1 is a linear topology on E such that every τ-null sequence is also τ_1-null, then every τ_1-topological string is sequentially τ-topological, which implies that $\tau_1 \leq \tau^S$. This proves (2),(3), and (4).

(1) Since every τ^S-sequential string is also τ-sequential (by (2)), the topology τ^S is sequential and it is clearly the coarser sequential topology finer than τ.

(5) If the linear map $f : (E,\tau) \to F$ is sequentially continuous, then for each topological string U in F, $f^{-1}(U)$ is an S-topological string in (E,τ) and so $f^{-1}(U)$ is τ^S-topological, which proves that f is τ^S-continuous. The converse is clear.

(6) It follows from Lemma 2.3. □

Proposition 2.10. *If the linear map $f : (E,\tau) \to (F,\tau_1)$ is continuous, then f is (τ^S, τ_1^S)-continuous.*

Proof. It follows from the fact that, for each S-topological string in F, $f^{-1}(U)$ is S-topological in E. □

Corollary 2.11. *If $(E,\tau) = \prod_{\alpha \in I} (E_\alpha, \tau_\alpha)$, then $\tau^S \geq \prod_{\alpha \in I} \tau_\alpha^S$.*

Proposition 2.12. *If $(E,\tau) = \prod_{k=1}^{n} (E_k, \tau_n)$, then $\tau^S = \prod_{k=1}^{n} \tau_k^S$.*

Proof. Let $\tau_0 = \prod_{k=1}^{n} \tau_k^S$. By the preceding corollary, we have $\tau_0 \leq \tau^S$. On the other hand, let $U(V_n)$ be an S-topological string in (E,τ). For each k, $1 \leq k \leq n$, let $j_k : E_k \to E$ be the canonical mapping. Set $V_k^m = j_k^{-1}(V_{m+n+1})$. Each V_k^m is an S-neighborhood of zero in (E_k, τ_k) and so $U_k = (V_k^m)_{m=1}^{\infty}$ is an S-topological string in E_k. The set

$$W_m = \prod_{k=1}^{n} V_k^m$$

is a τ_0-neighborhood of zero in E and $W_m \subset V_m$, which implies that U is τ_0-topological. Thus $\tau^S \leq \tau_0$, and the result follows. □

Proposition 2.13. *Let* $(E,\tau) = \bigoplus_{\alpha\in I}(E_\alpha,\tau_\alpha)$*, where each* (E_α,τ_α) *is Hausdorff. Then* $\tau^S = \bigoplus_{\alpha\in I}\tau_\alpha^S$*.*

Proof. Let $\sigma = \bigoplus_{\alpha\in I}\tau_\alpha^S$. If $f_\alpha : E_\alpha \to E$ is the canonical mapping, then f_α is (τ_α^S,τ^S)-continuous and so $\tau^S \le \sigma$. On the other hand, let $U = (V_n)$ be an S-topological string in (E,τ) and let $(x^{(n)})$ be a null sequence in (E,τ). By [2, 5.10], there exists a finite subset $I_0 = \{\alpha_1,\dots,\alpha_m\}$ of I such that $x_\alpha^{(n)} = 0$ for all n and all $\alpha \notin I_0$. Let

$$g : (E,\tau) \to \bigoplus_{k=1}^m (E_{\alpha_k},\tau_{\alpha_k}) = \prod_{k=1}^m (E_{\alpha_k},\tau_{\alpha_k})$$

be a canonical projection. If $\tau_0 = \prod_{k=1}^m \tau_{\alpha_k}$, then g is (τ,τ_0)-continuous and so g is (τ^S,τ_0^S)-continuous. The canonical embedding

$$h : \bigoplus_{k=1}^m (E_{\alpha_k},\tau_{\alpha_k}^S) \to (E,\sigma)$$

is continuous. But

$$\bigoplus_{k=1}^n \tau_{\alpha_k}^S = \prod_{k=1}^n \tau_{\alpha_k}^S = \tau_0^S.$$

Thus, h is (τ_0^S,σ)-continuous and so

$$h\circ g : (E,\tau^S) \to (E,\sigma)$$

is continuous and hence $x^{(n)} = h\circ g(x^{(n)}) \to 0$ in (E,σ). This proves that the identity map from (E,τ^S) to (E,σ) is sequentially continuous and therefore continuous. This implies that $\sigma \le \tau^S$ and the result follows. □

Proposition 2.14. *Let* F *be a topologically complemented subspace, of a topological vector space* (E,τ)*, and let* $\tau\big|F$ *be the topology induced by* τ *on* F*. Then* $\tau^S\big|F = (\tau\big|F)^S$*.*

Proof. Since the inclusion map from F to E is $((\tau\big|F)^S,\tau^S)$-continuous, we have that $\tau^S\big|F \le (\tau\big|F)^S$. To prove the inverse inequality, let G be a topological complement of E and let π_1,π_2 be the projections of E onto F and G, respectively. Then π_1,π_2 are continuous. Let now $\tau_1 = \tau\big|F$ and let $U = (V_n)$ be an S-topological string in (F,τ_1). Let $W_n = V_n + G$. Each W_n is an S-neighborhood of zero in (E,τ). In fact, let (x_n) be a τ-null sequence. Since $\pi_1(x_m)\to 0$ in F, there exists m_0 such that $\pi_1(x_m)\in V_n$ if $m \ge m_0$ and so $x_0 \in W_n$ if $m \ge n_0$. Now, $W = (W_n)$ is an S-topological string in (E,τ). Since $W_n \cap F = V_n$, it follows that U is $(\tau^S\big|F)$-topological, which proves that $\tau_1^S \le \tau^S\big|F$. □

3. The Topology τ^p

Definition 3.1. A string $U = (V_n)$ in a topological vector space (E, τ) is called precompactly-topological (pr-topological) if for each n and each τ-precompact subset A of E there exists a finite subset S_n of A such that $A \subset S_n + V_n$.

Lemma 3.2. *If $U = (V_n)$ and $W = (W_n)$ are pr-topological strings in E and λ a non-zero scalar, then $\lambda U = (\lambda V_n)$ and $U \cap W = (V_n \cap W_n)$ are also pr-topological.*

Proof. It is easy to see that λU is pr-topological. For the $U \cap W$, let A be a non-empty τ-precompact set and let $k \in N$. There are $x_1, \dots, x_n$, $y_1, \dots, y_m$ in A such that

$$A \subset \bigcup_{i=1}^{n} \big(x_i + V_{k+1}\big), \quad A \subset \bigcup_{j=1}^{n} \big(y_j + W_{k+1}\big).$$

Let

$$D = \big\{(i,j) : A \cap (x_i + V_{k+1}) \cap (y_j + W_{k+1}) \neq \varnothing\big\}.$$

Clearly,

$$A \subset \bigcup_{(i,j) \in D} \big(x_i + V_{k+1}\big) \cap \big(y_j + W_{k+1}\big).$$

For each $\alpha = (i,j) \in D$, choose $z_\alpha \in A \cap (x_i + V_{k+1}) \cap (y_j + W_{k+1})$. If $z \in (x_i + V_{k+1}) \cap (y_j + W_{k+1})$, then

$$z - x_i \in V_{k+1}, \ \ z_\alpha - x_i \in V_{k+1}, \ \ z - y_j \in W_{k+1}, \ \ z_\alpha - y_j \in W_{k+1}.$$

Thus

$$z - z_\alpha = (z - x_i) - (z_\alpha - x_i) \in V_{k+1} + V_{k+1} \subset V_k$$

and similarly $z - z_\alpha \in W_k$, i.e., $z - z_\alpha \in V_k \cap W_k$. Thus $A \subset \{z_\alpha : \alpha \in D\} + W_k \cap V_k$. $\square$

Notation 3.3. For a linear topology τ on E, we will denote by τ^p the linear topology generated by the family of all pr-topological strings in (E, τ).

We omit the proof of the following easily established proposition.

Proposition 3.4. 1) *τ^p is the finest linear topology on E having the same τ precompact sets.*

2) *$\tau \leq \tau^p$ and if τ_1 is a linear topology on E such that every τ-precompact set is also τ_1-precompact, then $\tau_1 \leq \tau^p$.*

Proposition 3.5. *A linear map $f : (E, \tau) \to F$ is τ^p-continuous iff it maps τ-precompact sets in E into precompact sets in F.*

Proof. The condition is clearly necessary since every τ-precompact set is also τ^p-precompact and images of precompact sets, under continuous linear mappings, are precompact. For the sufficiency, let $U = (V_n)$ be a topological string in F. If A is a τ-precompact set, then $f(A)$ is precompact in F and so, given n, there exists a finite subset S_n of A such that $f(A) \subset f(S_n) + V_n$, which implies that $A \subset S_n + f^{-1}(V_n)$. Thus $f^{-1}(U)$ is pr-topological in (E, τ), which proves that f is τ^p-continuous. $\square$

Proposition 3.6. *If the linear map $f : (E, \tau) \to (F, \tau_1)$ is continuous, then f is (τ^p, τ_1^p)-continuous.*

Proof. It follows from the fact that for every pr-topological string in (F, τ_1), $f^{-1}(U)$ is pr-topological in (E, τ). $\square$

Corollary 3.7. *If $(E, \tau) = \prod_{\alpha \in I} (E_\alpha, \tau_\alpha)$, then $\tau^p \geq \prod_{\alpha \in I} \tau_\alpha^p$.*

For the proof of the following Proposition we use an argument analogous to that of Proposition 2.12.

Proposition 3.8. *If $(E, \tau) = \prod_{k=1}^{n} (E_k, \tau_k)$, then $\tau^p = \prod_{k=1}^{n} \tau_k^p$.*

Proposition 3.9. *Let $(E, \tau) = \bigoplus_{\alpha \in I} (E_\alpha, \tau_\alpha)$, where each τ_α is Hausdorff. Then $\tau^p = \bigoplus_{\alpha \in I} \tau_\alpha^p$.*

Proof. Let $\sigma = \bigoplus_{\alpha \in I} \tau_\alpha^p$. If $f_\alpha : E_\alpha \to E$ is the canonical embedding, then f_α is (τ_α^p, τ^p)-continuous and so $\tau^p \leq \sigma$. On the other hand, let $U = (V_n)$ be a σ-topological string. We need to show that U is pr-topological in (E, τ). To this end, let A be a τ-precompact set. By [2, 5.10], there exists a finite subset $I' = \{\alpha_1, \ldots, \alpha_n\}$ of I such that $x_\alpha = 0$ for all $x \in A$ and all $\alpha \notin I'$. If $\pi_k : E \to E_{\alpha_k}$ is the canonical projection, then $A_k = \pi_k(A)$ is precompact in E_{α_k}. Given n, there exists a finite subset S of A such that

$$A_k \subset \pi_k(S) + f_{\alpha_k}^{-1}(V_{n+m-1})$$

for $k = 1, \ldots, m$. Given $x \in A$, there are $x_k \in A_k$, $k = 1, \ldots, m$, such that $x = \sum_{k=1}^{m} f_{\alpha_k}(x_k)$. Since

$$\underbrace{V_{n+m-1} + V_{n+m-1} + \cdots + V_{n+m-1}}_{m-times} \subset V_n$$

and since $f_{\alpha_k}(x_k) \in f_\alpha(S_k) + V_{m+n-1}$, $S_k = \pi_k(S)$, it follows that $x \in T + V_n$, $T = \sum_{k=1}^{m} f_{\alpha_k}(S_k)$. Thus, $A \subset T + V_n$, which shows that U is pr-topological in (E, τ). This clearly completes the proof. $\square$

Proposition 3.10. *Let F be a topologically complemented subspace of a topological vector space (E, τ). Then $\tau^p \big| F = (\tau \big| F)^p$.*

Proof. For the proof we use an argument analogous to that of Proposition 2.14. □

Proposition 3.11. *Let $\{(E_\alpha, \tau_\alpha) : \alpha \in I\}$ be a family of topological vector spaces, E a vector space, and $f_\alpha : E_\alpha \to E$ a linear map, for each $\alpha \in I$. If τ is the finest linear topology on E for which each f_α is continuous and if $\tau_\alpha^p = \tau^p$ for all $\alpha \in I$, then $\tau^p = \tau$.*

Proof. If U is a pr-topological string in (E, τ), then $f_\alpha^{-1}(U)$ is pr-topological in (E_α, τ_α) and so $f^{-1}(U)$ is τ_α-topological, which implies that U is τ-topological. □

Corollary 3.12. *Let F be a subspace of (E, τ) and let τ_0 be the quotient topology on $E \big| F$. If $\tau^p = \tau$, then $\tau_0^p = \tau_0$.*

Lemma 3.13. *Let τ_1, τ_2 be two linear topologies on E. If every τ_1-precompact set is also τ_2-precompact, then every τ_1-bounded set is τ_2-bounded.*

Proof. Assume the set A is τ_1-bounded but not τ_2-bounded. Then there exists a balanced τ_2-neighborhood V of zero in E and a sequence (x_n) in A such that $x_n \notin n^2 V$. If $y_n = n^{-1} x_n$, then $y_n \xrightarrow{\tau_1} 0$ (since A is τ_1-bounded), and so the set $B = \{y_n : n \in N\}$ is τ_1-precompact. By our hypothesis, B is τ_2-precompact and hence B is τ_2-bounded, which is not true since B is not absorbed by V. □

Corollary 3.14. 1) *τ and τ^p have the same bounded sets.*
2) *If (E, τ) is a bornological space, then $\tau = \tau^S = \tau^p$.*

Recall that a topological vector space E is called countably barreled if every string in E which is the intersection of a countable number of closed topological strings is topological. Also, E is called countably quasi-barreled if every bornivorous string in E which is the intersection of a countable number of closed topological strings is topological.

Notation 3.15. If τ is a linear topology on E, we will denote by τ^β the linear topology generated by the family of all τ-closed strings. By $\tau^{\beta *}$ we will denote the linear topology generated by the family of all closed bornivorous strings in (E, τ).

Proposition 3.16. *Let (E, τ) be a topological vector space.*
1) *If (E, τ) is countably quasi-barreled, then $\tau^{\beta *} \leq \tau^p$.*
2) *If (E, τ) is countably barreled, then $\tau^\beta \leq \tau^p$.*

Proof. It follows from [2, 18.5] since every τ-precompact set is $\tau^{\beta *}$ (resp. τ^{β})-precompact when (E,τ) is countably quasi-barreled (resp. countably barreled). $\square$

4. Sequentially Bornological Spaces

Definition 4.1. A topological vector space E is called sequentially bornological (S-bornological) if every bornivorous string is S-topological.

Since τ and τ^S have the same bounded sets, (E,τ) is S-bornological iff (E,τ^S) is bornological.

Definition 4.2. A pseudo-seminorm p, on a topological vector space E, is called bounded if for every bounded sequence (x_n) in E and every null sequence of scalars (λ_n) we have $p(\lambda_n x_n) \to 0$.

Note. If p is a seminorm, then p is bounded iff it is bounded on bounded sets.

Definition 4.3. A linear map f between two topological vector spaces E, F is called bounded if it maps bounded sets in E into bounded sets in F.

Proposition 4.4. *For a topological vector space (E,τ) the following properties are equivalent:*

(1) *(E,τ) is S-bornological.*

(2) *Every bounded linear map, from (E,τ) to an arbitrary topological vector space, is sequentially continuous.*

(3) *Every bounded linear map, from (E,τ) to an arbitrary Frechet space, is sequentially continuous.*

(4) *Every bounded pseudo-seminorm on E is sequentially continuous.*

Proof. (1)$\Rightarrow$(2). Let $f : (E,\tau) \to F$ be linear and bounded. If U is a topological string in F, then $f^{-1}(U)$ is a bornivorous string in (E,τ), and hence $f^{-1}(U)$ is S-topological. This proves that f is τ^S-continuous and hence f is sequentially τ-continuous.

(3)$\Rightarrow$(1). Let $U = (V_n)$ be a bornivorous string in E and consider the Frechet space $F = \widehat{E}_U$. Let $j : E \to F$ be the canonical mapping. If A is bounded in (E,τ), then for each n there exists a scalar λ such that $A \subset \lambda V_n$ and so $j(A) \subset \lambda j(V_n)$, which proves that $J(A)$ is bounded in F. Thus, j is bounded and hence j is sequentially continuous by our hypothesis, which implies that U is S-topological by Proposition 2.5.

(1)$\Rightarrow$(4). Let p be a bounded pseudo-seminorm on (E,τ) and set $W_m = \{x : p(x) \le 2^{-m}\}$. Every W_m is bornivorous. In fact, assume that W_m does not absorb a τ-bounded set A. Then, there exists a sequence (x_n) in A with $x_n \notin 2^n W_m$. Now (x_n) is a bounded sequence and $p(2^{-n}x_n) > 2^{-m}$ for all

n, which is a contradiction. Thus, $U = (W_n)$ is a bornivorous string and hence S-topological by (1). Let now m be given and let (x_n) be a τ-null sequence. Since W_m is an S-neighborhood of zero, there exists n_0 such that $x_n \in W_m$ and so $p(x_n) \le 2^{-}$ if $n \ge n_0$.

(4)⇒(1). Let $W = (V_n)$ be a bornivorous string in (E, τ) and let $p = p_W$ be as in Lemma 1.4. Then, p is bounded. In fact, let (x_n) be a bounded sequence and let $\lambda_n \to 0$. Given k, there exists a scalar α such that $(x_n) \subset \alpha V_k$. Now

$$\lambda_n x_n \in \lambda_n \, \alpha V_k \subset V_k$$

for large n, which implies that $p(\lambda_n x_n) \le 2^{-k}$ eventually. By (4), p is sequentially continuous and hence W is S-topological by Lemma 2.3. □

Proposition 4.5. *A countable product* $(E, \tau) = \prod_{n=1}^{\infty} (E_k, \tau_k)$ *of S-bornological spaces is S-bornological.*

Proof. Each τ_n^S is bornological and so $\tau_0 = \prod_{n=1}^{\infty} \tau_n^S$ is bornological by [2, 11.6]. If τ^b is the bornological topology associated with τ, then $\tau^S \le \tau^b$, since τ^S has the same τ bounded sets. Since $\tau \le \tau_0 \le \tau_S$ (by Corollary 2.11), we have that $\tau_0 = \tau^b = \tau^S$ and so τ^S is bornological. □

Proposition 4.6. *Let F be a topologically complemented subspace of a topological vector space (E, τ). If (E, τ) is S-bornological, then F is S-bornological.*

Proof. Let $\tau_0 = \tau \big| F$. By Proposition 2.14, we have $\tau_0^S = \tau^S \big| F$. Thus, it suffices to assume that (E, τ) is bornological and show that τ_0 is bornological. If $U = (V_n)$ is a bornivorous string in (F, τ_0) and if G is a topological complement of F, then (W_n), where $W_n = V_n + G$ is a bornivorous string in (E, τ) and $W_n \cap F = V_n$, which shows that U is τ_0-topological. □

Proposition 4.7. *Let* $(E, \tau) = \bigoplus_{\alpha \in I} (E_\alpha, \tau_\alpha)$*, where each τ_α is Hausdorff. If each (E_α, τ_α) is S-bornological, then (E, τ) is S-bornological.*

Proof. By Proposition 2.13, we have $\tau^S = \bigoplus_{\alpha \in I} \tau_\alpha^S$. Since the direct sum of bornological spaces is bornological ([2, 11.5]), the result follows. □

5. Sequentially Barreled Spaces

Definition 5.1. A topological vector space E is called sequentially-barreled (S-barreled) if every closed string in E is S-topological.

Proposition 5.2. *A topological vector space E is S-barreled iff every lower-semicontinuous pseudo-seminorm on E is sequentially continuous.*

Proof. Assume that E is S-barreled and let p be a lower-semicontinuous pseudo-seminorm on E. If $V_n = \{x : p(x) \leq 2^{-n}\}$, then $U = (V_n)$ is a closed string and hence S-topological, which implies that p is sequentially continuous (by Lemma 2.3). Conversely, suppose that every lower-semicontinuous pseudo-seminorm on E is sequentially continuous and let $W = (W_n)$ be a closed string in E. Let $p = p_W$ be as in Lemma 1.4. Let $\{V_\gamma : \gamma \in \Gamma\}$ be a base at zero in E consisting of balanced sets and make Γ into a direct set by defining $\gamma_1 \geq \gamma_2$ if $V_{\gamma_1} \subset V_{\gamma_2}$. Define p on E by

$$p(x) = \sup_{\gamma \in \Gamma} \inf_{y \in V_\gamma} p_W(x + y).$$

Then, p is lower-semicontinuous (it is the biggest lower-semicontinuous function f with $f \leq p$) and it is easy to show that $p(x+y) \leq p(x) + p(y)$ and that $p(\lambda x) \leq p(x)$ when $|\lambda| \leq 1$. Moreover

$$\{x : p(x) < 2^{-n-1}\} \subset W_n. \tag{$*$}$$

In fact, let $p(x) < 2^{-n-1}$. For each $\gamma \in \Gamma$, there exists y_γ in V_γ with $p_W(x + y_\gamma) < 2^{-n-1}$ and so $x + y_\gamma \in W_n$. Since the net $(y_\gamma)_{\gamma \in \Gamma}$ converges to zero and W_n is closed, we have that $x \in W_n$. It now follows easily that p is a pseudo-seminorm which, by our hypothesis, is sequentially continuous. Now $(*)$ implies that W is S-topological and the result follows. □

Proposition 5.3. *A topological vector space (E, τ) is S-barreled iff any linear topology τ_1 on E, having a base at zero consisting of τ-closed sets, is weaker than τ^S.*

Proof. Suppose that (E, τ) is S-barreled and let τ_1 have the stated properties. Then, there is a base $\mathcal{F}$ for the τ_1-topological strings, consisting of τ-closed strings. Since (E, τ) is S-barreled, every member of $\mathcal{F}$ is S-topological in (E, τ) and hence $\tau_1 \leq \tau^S$. Conversely, assume that (E, τ) has the same property stated in the proposition. The family $\mathcal{F}_c$ of all τ-closed strings is directed and it generates a linear topology τ_2. This topology has a base at zero consisting of τ-closed sets and hence $\tau_2 \leq \tau^S$ by our hypothesis. It follows that every τ-closed string is S-topological in (E, τ) and so (E, τ) is S-barreled. □

Definition 5.4. A family $\mathcal{F}$ of linear maps, from a topological vector space E to another F, is called sequentially equicontinuous if for each neighborhood V of zero in F the set $\bigcap_{f \in \mathcal{F}} f^{-1}(V)$ is an S-neighborhood of zero in E.

Proposition 5.5. *For a topological vector space E, the following properties are equivalent:*

(1) *E is S-barreled.*

(2) *Every pointwise bounded family $\mathcal{F}$ of continuous linear maps, from E into another topological vector space F, is S-equicontinuous.*

(3) *Every pointwise bounded family of continuous linear maps, from E into an arbitrary Frechet space, is S-equicontinuous.*

Proof. (1)$\Rightarrow$(2). Let V be a neighborhood of zero in F. Choose a closed topological string $U = (V_n)$ in F with $V_1 \subset V$. Let $W_n = \bigcap_{f\in\mathcal{F}} f^{-1}(V_n)$. Then (W_n) is a closed string in E and hence S-topological by (1). Thus $\bigcap_{f\in\mathcal{F}} f^{-1}(V)$ is an S-neighborhood because it contains W_1.

(3)$\Rightarrow$(1). For the proof we use an argument analogous to the one used in [2, 7.2]. $\square$

Corollary 5.6. *Let E be an S-barreled space, F an arbitrary topological vector space, and $(f_\alpha)_{\alpha\in I}$ a pointwise bounded net of continuous linear maps from E to F. If the limit* $\lim f_\alpha(x) = f(x)$ *exists for each $x \in E$, then f is sequentially continuous.*

Corollary 5.7. *Every pointwise limit of continuous linear maps, from an S-barreled space into an arbitrary topological vector space, is sequentially continuous.*

Proposition 5.8. *Let E be a Hausdorff S-barreled space and let F be a subspace of finite codimension. Then, F is S-barreled.*

Proof. We may consider the case in which the codimension of F is one. Let $U = (V_n)$ be a closed string in F. There are two possible cases.

Case I. Each V_n is a proper subset of its closure $\overline{V}_n$ in E. Then, each $\overline{V}_n$ is absorbing in $\overline{E}$. In fact, let $x_0 \in \overline{V}_{n+1}\backslash V_{n+1}$ and let $x \in E$. Then $x = y + \lambda x_0$ with $y \in F$. Since V_{n+1} is absorbing in F, there exists a scalar μ, $|\mu| \geq |\lambda|$, with $y \in \mu V_{n+1}$, and so

$$x \in \mu V_{n+1} + \lambda \overline{V}_{n+1} \subset \mu \overline{V}_n.$$

Now, $(\overline{V}_n)$ is a closed string in (E,τ) and hence each $\overline{V}_n$ is an S-neighborhood in (E,τ). Moreover, $\overline{V}_n \cap F = V_n$. It is clear now that U is S-topological in F.

Case II. There exists an n such that V_n is closed in E. We will show that each V_m is an S-neighborhood in F. Without loss of generality, we may assume that V_1 is closed in F. Then, each V_m is closed in E. Choose $x_0 \in E$ with $x_0 \notin F$ and let $W = \{\lambda x_0 : |\lambda| \leq 1\}$. The set W is compact in E and hence each $W_n = V_n + 2^{-n}W$ is closed in E. Clearly (W_n) is a closed string in E and $W_n \cap F = V_n$. Since each W_n is an S-neighborhood of zero in E, it follows that V_n is an S-neighborhood of zero in F. $\square$

Definition 5.9. A linear map f, between two topological vector spaces E and F, is called sequentially open (S-open) if $f(V)$ is an S-neighborhood of zero in F for each S-neighborhood V of zero in E.

Proposition 5.10. *Let E, F be topological vector spaces and let $f : E \to F$ be linear, continuous and sequentially open. If E is S-barreled, then F is S-barreled.*

Proof. It is clear that f is onto. Let $U = (V_n)$ be a closed string in F. Then, $f^{-1}(U)$ is a closed string in E and so $f^{-1}(U)$ is S-topological. Since $f(f^{-1}(V_n)) = V_n$ and f is sequentially open, it follows that U is S-topological in F. $\square$

References

1. J. H. Webb, Sequential convergence in locally convex spaces. *Proc. Camb. Phil. Soc.* **64**(1968), 341-464.
2. N. Adasch, B. Ernst, and D. Kein, Topological vector spaces. The theory without convexity conditions. *Springer-Verlag, Berlin, Heidelberg, New York,* 1978.
3. H. H. Schaefer, Topological Vector Spaces. *Springer-Verlag, New York, Heidelberg, Berlin,* 1971.
4. R. M. Dudley, On sequential convergence. *Trans. Amer. Math. Soc.* **112**(1964), 483-507.
5. L. M. Sānchez Ruiz, On the closed graph theorem between topological vector spaces and Frechet spaces. *Math. Japon.* **36**(1991), 271-275.
6. L. M. Sānchez Ruiz, On the Banach–Steinhaus theorem between topological vector spacers and locally convex spaces. *Math. Japon.* **36**(1991), No. 1, 143-145.
7. L. M. Sānchez Ruiz, Two maximal classes of topological vector spaces. *Math. Japon.* **36**(1991), No. 4, 643-646.
8. L. M. Sānchez Ruiz, On locally bounded linear mappings between topological vector spaces and Frechet spaces. *Math. Japon.* **36** (1991), No. 2, 277-282.
9. E. F. Snipes, S-barreled topological vector spaces. *Canad. Math. Bull.* **21(2)**(1978).
10. R. F. Snipes, C-sequential and S-bornological vector spaces. *Math. Ann.* **202**(1973), 273-283.

(Received 29.11.1993)

Authors' address:
Department of Mathematics
University if Ioannina
Ioannina, Greece