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ON THE UNIVERSAL C^*-ALGEBRA GENERATED BY PARTIAL ISOMETRY

T. KANDELAKI

Abstract. A universal C^*-algebra is constructed which is generated by a partial isometry. Using grading on this algebra we construct an analog of Cuntz algebras which gives a homotopical interpretation of KK-groups. It is proved that this algebra is homotopy equivalent up to stabilization by 2×2 matrices to $M_2(C)$. Therefore those algebras are KK-isomorphic.

We recall the definition of partial isometry.

Definition 1. Let H_1 and H_2 be Hilbert spaces and $v : H_1 \longrightarrow H_2$ be a bounded linear map. v is called a partial isometry if v^*v is a projection.

We only want to emphasize the following important facts. The projection $p = v^*v$ is called the support projection for v. Standard equations involving partial isometries are the following:

$$v = vv^*v = vp = qv, \quad v^* = v^*vv^* = pv^* = v^*q.$$

It is known that the above equations can be taken to define a partial isometry.

First of all, our aim is to construct an involutive algebra generated by one symbol together with the above relations and prove the existence of a maximal C^*-norm on it. For the general construction and examples see [1].

Let $U(v)$ be the universal involutive complex algebra generated by the symbols v, v^*. Let $J(v)$ be the two-sided $*$-ideal generated by elements of the following form:

(a) $\{v^*\} - \{v\}^*$, (b) $\{v\} - \{vv^*v\}$,

where $\{v\}$ and $\{v^*\}$ denote the elements of $U(v)$ which correspond to v and v^* respectively. Then $\mathbf{U(v)}$ will denote the factor algebra $U(v)/J(v)$, which is a complex $*$-algebra and has the universal property that if B is a

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$*$-algebra generated by a partial isometry ν, then there exists a canonical $*$-homomorphism $\kappa : \mathbf{U(v)} \to B$ such that $\kappa(v) = \nu$.

We say that an algebra seminorm p on a complex involutive algebra A is a C^*-seminorm if $p(xx^*) = p(x)^2$. If $\|x\| = \sup\{p(x) \mid p$ is a C^*-seminorm on $A\}$ is finite for every $x \in A$, then $\|\cdot\|$ defines the largest possible C^*-seminorm on A, and we can form the completion $C^*(A) = \overline{A/N^{\|\cdot\|}}$, $N = \{x \in A \mid \|x\| = 0\}$. $C^*(A)$ has the universal property that any $*$-homomorphism from A into a C^*-algebra factors uniquely through the map $A \to C^*(A)$ (which is not necessarily injective) [1].

Proposition 2. *There exists a C^*-seminorm $\|\cdot\|$ on $\mathbf{U(v)}$ (which is the supremum of all C^*-seminorms on the same algebra). Let $C^*(v) = C^*(\mathbf{U(v)})$ and $\kappa : \mathbf{U(v)} \to C^*(v)$ be the canonical $*$-homomorphism; then $C^*(v)$ has the following universal property: if $\varphi : U(v) \to B$ is a $*$-homomorphism into a C^*-algebra B, then there exists a unique $*$-homomorphism $\psi : C^*(v) \to B$ such that the diagram*

$$\begin{tikzcd} U(v) \arrow[r, "\kappa"] \arrow[dr, "\varphi"'] & C^*(v) \arrow[d, "\psi"] \\ & B \end{tikzcd}$$

is commutative.

Proof. Let θ be any C^*-seminorm on $\mathbf{U(v)}$; then $\theta(v)^2 = \theta(v^*v) \le 1$ because v^*v is a projection. If $e \in \mathbf{U(v)}$, then $e = \sum\limits_i \lambda_i f_{i_1} \cdots f_{i_{n_i}}$, where f_{i_k} is v or v^*. Thus $\theta(e) \le \sum\limits_i |\lambda_i| \theta(f_{i_1}) \cdots \theta(f_{i_{n_1}}) \le \sum\limits_i |\lambda_i|$. So there exists $\pi(e) = \sup_\theta \|\theta(e)\|$, where θ runs over all C^*-seminorms on $\mathbf{U(v)}$. It is easy to check that π is a C^*-seminorm and $C^*(v) = C^*(\mathbf{U(v)})$ has the above universal property. $\square$

The algebra $C^*(v)$ has a Z_2-grading which is induced by the automorphism defined by $v \to -v$; this graded algebra will be denoted by $G^*(v)$. It follows from the definition that $\deg v = 1$. If B is a Z_2-graded C^*-algebra and v is a partial isometry with $\deg v = 1$, then there exists a unique graded $*$-homomorphism $\psi : G^*(v) \to B$ such that $\psi(v) = v$.

Now we need a definition of KK-groups in the style given in [2].

Definition 3. A superquasimorphism (sqm) from A to B is a triple $\Phi = (\phi, G, \mu)$, where ϕ is a graded homomorphism from A to a Z_2-graded algebra D with a graded (invariant) ideal $J \lhd D$ and $G \in D$ is an element of degree 1, such that

$$(G - G^*)\phi(x) \in J, \quad (1 - G^2)\phi(x) \in J, \quad [\phi(x), G] \in J$$

for $x \in A$, $\mu : A \to B$ is a homomorphism. This will be writtten as $A \xrightarrow{\phi} D \overset{G}{\triangleright} J \xrightarrow{\mu} B$ or shortly $\Phi : A - - \triangleright B$.

A mapping between two sqm's Φ_1, Φ_2 is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\phi_1} & D_1 & \overset{G_1}{\triangleright} & J_1 & \xrightarrow{\mu_1} & B \\ || & & \downarrow & & \downarrow & & || \\ A & \xrightarrow{\phi_2} & D_2 & \overset{G_2}{\triangleright} & J_2 & \xrightarrow{\mu_2} & B \end{array} ;$$

$D_1 \to D_2$ maps G_1 to G_2. We will say that Φ_1 is contained in Φ_2 if the vertical homomorphisms are injective and that Φ_2 is a quotient of Φ_1 if they are surjective.

If $f : A' \to A$, $g : B \to B'$ are homomorphisms and $\Phi : A - - \triangleright B$ is a sqm then the composition $g \circ \Phi \circ f$ gives a sqm from A' to B'.

Let $q_t : B[0;1] \to B$ be the evolution at time t. A sqm Π from A to $B[0;1]$ will be called a homotopy from Φ to Ψ, if $\Phi = q_0 \circ \Pi$, $\Psi = q_1 \circ \Pi$. Φ and Ψ will be called equivalent if there is a chain $\{\Phi_{i,} i = 0, \ldots, n\}$ consisting of finitely many sqm $\Phi_{i,}$ such that $\Phi_0 = \Phi, \Phi_n = \Psi$ and, for each i, Φ_i and Φ_{i+1} are connected either by a mapping or by a homotopy [2].

Following [2], let us denote by $KK(A,B)$ the set of equivalence classes of sqm's from A to $\overset{\wedge}{\mathcal{K}} \otimes B$, where $\overset{\wedge}{\mathcal{K}} = \hat{M}_2 (\mathcal{K})$ with the standard even grading on $\hat{M}_2$, and $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space. Choose a fixed (standard) isomorphism $\overset{\wedge}{\mathcal{K}} \otimes \hat{K} = \hat{K}$ such that the identity $id_{\overset{\wedge}{\mathcal{K}}}$ is homotopic to the embedding j_0 mapping $x \in \hat{K}$ to $e \otimes x$ with e a minimal projection of degree 0 in the left upper corner.

On this set there is a (commutative) addition defined by

$$[\Phi_1] \oplus [\Phi_2] = [\Phi_1 \oplus \Phi_2], \quad \Phi_1 \oplus \Phi_2 : A - - \triangleright M_2(\overset{\wedge}{\mathcal{K}} \otimes B) \simeq \overset{\wedge}{\mathcal{K}} \otimes B.$$

The result will not depend up to an equivalence on the chosen isomorphism $M_2(\overset{\wedge}{\mathcal{K}} \otimes B) \simeq \hat{K} \otimes B$.

It follows from the definition that every sqm is contained in a sqm with unital D. Let $\Phi = A \xrightarrow{\phi} D \overset{G}{\triangleright} J \xrightarrow{\mu} B$ be a sqm, where A and D are separable and D is unital. Let

$D' = \{d \in D \mid [\phi(x), d] \in J, \text{for all } x \in A\}$ and

$J' = \{d \in D \mid d \cdot \phi(x) \in J, \phi(x) \cdot d \in J, \text{ for all } x \in A\}$.

It is easy to check that D' is a unital C^*-algebra and J' is a closed two-sided ideal in D'. Note that $G \in D'$, $G - G^* \in J'$, and $1 - G^2 \in J'$. Thus G gives G', the unitary element in the factor algebra D'/J'. It is known that every element of a factor C^*-algebra can be lifted to the C^*-algebra preserving the norm [3], [4]. Thus G' can be lifted to D' preserving the norm, i.e., there exists an element F such that $F - G \in J'$ and $||F|| = 1$. The matrix

$$U_F = \begin{pmatrix} F & -(1 - FF^*)^{1/2} \\ (1 - F^*F)^{1/2} & F^* \end{pmatrix}$$

is a unitary element in the $M_2(D')$ and $M_2(D)$. Thus we have the following sqm's

$$\Phi_k = A \overset{i_0\phi}{\to} M_2(D) \overset{U_k}{\rhd} M_2(J) \overset{\mu}{\to} M_2(B), \quad k = 0, 1,$$

where $U_0 = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, $U_1 = U_F$, i_0 is an inclusion in the upper left corner. It is easy to see that $\Phi_0 \sim \Phi_1$ because

(a) $(1 - F^*F)^{1/2}\phi(x)$ and $(1 - FF^*)^{1/2}\phi(x)$ are in J, for all $x \in A$;

(b) U_F is homotopic to $U_d = \begin{pmatrix} F & 0 \\ 0 & F^* \end{pmatrix}$.

Therefore we have

Lemma 4. *Every sqm $A \overset{\phi}{\to} D \overset{G}{\rhd} J \overset{\mu}{\to} B$, with unital D, is equivalent up to 2×2 matrices to sqm $A \overset{i_0\phi}{\to} M_2(D) \overset{U}{\rhd} M_2(J) \overset{\mu}{\to} M_2(B)$, where U is a unitary element in $M_2(D)$.*

Corollary 5. *Every sqm is equivalent up to 2×2 matrices to a sqm with G a partial isometry.*

This corollary can be made more precise if A and D are separable. It is known that if $A \to B$ is an epimorphism of separable C^*-algebras then any unitary element of B can be lifted to a partial isometry of A [4]. Thus G' can be lifted to a parial isometry $\Upsilon \in D'$ because D' is separable. Note that $(G - \Upsilon)\phi(x) \in J, [\phi(x), \Upsilon] \in J$. Elementary calculation shows that $\Phi' = A \overset{\phi}{\to} D \overset{\Upsilon}{\rhd} J \overset{\mu}{\to} B$ is a superquasimorphism, which is equivalent to Φ by the homotopy $\Pi = A \overset{\phi_c}{\to} D[0;1] \overset{H}{\rhd} J[0;1] \overset{\rho}{\to} B[0;1]$, where $\phi_c(x)(t) = \phi(x), H(t) = G - t(G - \Upsilon)$ and $\rho(f)(t) = \mu(f(t))$. Thus we get

Lemma 6. *Every sqm $A \overset{\phi}{\to} D \overset{G}{\rhd} J \overset{\mu}{\to} B$ with separable A is equivalent to the sqm $A \to D_1 \overset{\Upsilon}{\rhd} J_1 \to B$, where Υ is a partial isometry.*

Therefore we have two variants of the definition of KK-groups:

(1) Consider the definition of sqm's with partial isometry as G. Then we get the same KK-groups. In this case we have the following universal sqm:

Let $A * C^*(v)$ be the sum of C^*-algebras in the category of C^*-algebras and $*$-homomorphisms (it is exactly the free product of C^*-algebras). Let $v(A)$ be the closed two-sided ideal in $A * C^*(v)$ generated by the elements $(v - v^*) \cdot a$, $(1 - v^2) \cdot a$, $[a, v]$. Then we have the sqm $A \overset{i_0}{\to} A * C^*(v) \overset{v}{\rhd} v(A)$

which is universal in the following sense: if $A \xrightarrow{\phi} D \overset{G}{\triangleright} J \xrightarrow{\mu} B$ is a given sqm, where G is a partial isometry, then there exist a unique $*$-homomorphism $v(A) \xrightarrow{v(\mu)} B$ and a mapping

$$\begin{array}{ccccccc} A & \xrightarrow{i_0} & A * C^*(v) & \overset{v}{\triangleright} & v(A) & \xrightarrow{v(\mu)} & B \\ \| & & \downarrow \phi_g & & \downarrow \phi_g & & \| \\ A & \xrightarrow{\phi} & D & \overset{G}{\triangleright} & J & \xrightarrow{\mu} & B \end{array}$$

where $\phi_g \mid A = \phi$ and $\phi_g \mid C^*(A)(v) = G$. Therefore we have

Theorem 7. $KK(A,B) = [v(A), \overset{\wedge}{\mathcal{K}} \otimes B]$, *where* $[v(A), \overset{\wedge}{\mathcal{K}} \otimes B]$ *denotes the set of homotopy classes of* (*graded*) *homomorphisms.*

Theorems of such a type come from [5]; see also [2].

(2) Consider a sqm with unital algebras D and unitary element $G \in D$. In this case we have the following universal sqm:

Let $\hat{C}(S^1)$ be the standard C^*-algebra of the circle with grading induced by the automorphism $z \to -z$ and let A^+ be obtained from A by adjoining the unit. Let $A^+ \bullet \hat{C}(S^1)$ be the sum in the category of unital C^*-algebras and unital homomorphisms (this is the factor algebra of the free product by the ideal generated by the element $1_A - 1_{S^1}$) and define $z(A)$ as the closed ideal generated by the elements $[a,z]$, $\quad (z - z^*)a$, $\quad a \in A$; then we have the universal sqm $A \xrightarrow{i_A} A^+ \bullet \hat{C}(S^1) \overset{z}{\triangleright} z(A)$ with the following property: if Φ is a sqm with unital D and unitary element G, then there exists a unique homomorphism $z(\mu) : z(A) \to B$ such that the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{i_A} & A^+ \bullet \overset{\wedge}{C^*}(S^1) & \overset{z}{\triangleright} & z(A) & \xrightarrow{z(\mu)} & B \\ \| & & \downarrow \phi_z & & \downarrow & & \| \\ A & \xrightarrow{\phi} & D & \overset{G}{\triangleright} & J & \xrightarrow{\mu} & B \end{array}$$

is commutative, where $\phi_z = \phi^+ \bullet g_z$ and $g_z : \overset{\wedge}{C^*}(S^1) \to D$ is the homomorphism defined by the formula $g_z(z) = G$ (because G is a unitary element in D). We have the following analog of Theorem 7:

Theorem 8. $KK(A,B) = [z(A), \overset{\wedge}{\mathcal{K}} \otimes B]$.

We give below some homotopic and KK-theoretic properties of C^*-algebras $C^*(v)$ and $G^*(v)$.

Remark. Let S be the operator on the Hilbert space $l_2(N)$ given on the basis by $e_n \to e_{n+1}$ (it is called the unilateral shift). The Toeplitz algebra $\mathbf{T}$ is the unital separable C^*-subalgebra of $l_2(N)$ generated by S. The algebra $\mathbf{T}$ is the universal C^*-algebra generated by an isometry [1]. The algebra

$\mathbf{T}$ has the grading defined by the map $S \longmapsto -S$. We denote this graded Toeplitz algebra by $\widehat{\mathbf{T}}$. The C^*-algebras $C^*(v)$ and $G^*(v)$ are sufficiently "large" algebras, because the universal property implies that the canonical $*$-homomorphisms $\nu : C^* \to \mathbf{T}$ and $\upsilon : G^*(A) \to \widehat{\mathbf{T}}$ are $*$-epimorphisms.

We will need the folowing $*$-homomorphisms: $i_0 : M_2(C) \to M_2(M_2(C))$ and $j_0 : C^*(v) \to M_2(C^*(v))$ which are $*$-inclusions into the upper left corner.

We recall that two $*$-homomorphisms $f_0, f_1 : A \to B$ of C^*-algebras are homotopic if there is a path $\{\varphi_t\}$ of $*$-homomorphisms $\varphi_t : A \to B$ such that $t \longmapsto \varphi_t(a)$ is a norm continuous map from $[0;1]$ to B for fixed $a \in A$ and such that $\varphi_0 = f_0$, $\varphi_1 = f_1$.

Now we are ready to prove

Theorem 9. *Let $C^*(v)$ be the universal C^*-algebra of a partial isometry. There exists a $*$-homomorphism $\varphi_v : M_2(C) \to M_2(C^*(v))$ such that $\varphi_v\mu$ is homotopic to j_0, and such that $\mu\varphi_v$ is homotopic to i_0 (here μ stands for both the evolution map $M_2(C^*(v)) \to M_2(M_2(C))$ and the one $C^*(v) \to M_2(C)$). That is, $C^*(v)$ and $M_2(C)$ are homotopy equivalent up to stabilization by 2×2 matrices.*

Proof. Let $\varphi_v : M_2(C) \to M_2(C^*(v))$ be defined by the formula

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}.$$

There is homotopy $h(t) = \begin{pmatrix} \cos t \cdot v & \sin t \cdot v \\ 0 & 0 \end{pmatrix}$ from $\begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$, where $t \in [0;1]$. Note that $h(t)$ is a partial isometry in $M_2(C^*(v)$, for any $t \in [0;1]$:

$$h(t) \cdot h(t)^* \begin{pmatrix} \cos t \cdot v & \sin t \cdot v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos t \cdot v^* & 0 \\ \sin t \cdot v^* & 0 \end{pmatrix} = \begin{pmatrix} vv^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus we have, by the universal property of $C^*(v)$, a path of $*$-homomorphisms: $H_t : C^*(v) \to M_2(C^*(v))$ which is defined by the formula

$$H_t(v) = \begin{pmatrix} \cos t \cdot v & \sin t \cdot v \\ 0 & 0 \end{pmatrix}$$

such that $H_0 = j_0$ and $H_1 = \varphi_0\mu$. We have to prove now that $\mu\varphi_0$ is homotopic to i_0. Note that $\mu\varphi_0$ is defined by the formula

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The elements

$$v' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v'' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are unitary equivalent with the unitary element

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

i.e., $u^*v'u = v''$. The element u is homotopic to 1 in the group of unitary elements, by the homotopy

$$u_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{\pi}{2}t & 0 & \sin\frac{\pi}{2}t \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\frac{\pi}{2}t & 0 & \cos\frac{\pi}{2}t \end{pmatrix},$$

$t \in [0;1]$. Thus the path of $*$-homomorphisms $H'_t : M_2(C) \to M_2(M_2(C)$ defined by the formula $H'_t(x) = u_t^* i_0 u_t$ defines the homotopy from i_0 to $\mu\varphi_v$. $\square$

As an application of the theorem we now have to prove

Corollary 10. *The canonical $*$-homomorphism $\mu : C^*(v) \to M_2(C)$ induces an invertible element in $KK(C^*(v); M_2(C))$.*

Proof. It follows from the definition of KK-groups that i_0 and j_0 induce identity elements of rings $KK(C^*(v), C^*(v))$ and $KK(M_2(C), M_2(C))$. Homomorphisms μ and φ_v give elements of the groups $KK(C^*(v), M_2(C))$ and $KK(M_2(C); C^*(v))$, respectively. From the theorem it immediately follows that $[\varphi_v] \cdot [\mu] = [1_{C^*(v)}]$ and $[\mu] \cdot [\varphi_v] = [1_{M_2(C)}]$.

Thus C^*-algebras $C^*(v)$ and $M_2(C)$ are KK-isomorphic.

Let A be a Z_2-graded C^*-algebra. On $M_2(A)$ we have two essential Z_2-gradings given by $*$-automorphisms of period 2:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\gamma_0} \begin{pmatrix} \gamma(a) & \gamma(b) \\ \gamma(c) & \gamma(d) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\gamma_1} \begin{pmatrix} \gamma(a) & -\gamma(b) \\ -\gamma(c) & \gamma(d) \end{pmatrix}$$

where $\gamma : A \to A$ is the $*$-automorphism inducing the given Z_2-grading on A. Let $M_2(A)$ be the algebra of 2×2 matrices with an odd grading and $\hat{M}_2\,(C)$ be the same algebra with an even grading. The partial isometry $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has degree 1 in the odd grading and so we have the canonical

Z_2-graded $*$-homomorphism $\omega : G^*(v) \to \overset{\wedge}{M_2}(C)$ defined by the formula $\omega(v) = e_{12}$, which follows from the universal property of $G^*(v)$. It follows from the above definitions of Z_2-gradings that the canonical $*$-inclutions

$$i_0 : \overset{\wedge}{M_2}(C) \to M_2(\overset{\wedge}{M_2}(C)) \quad \text{and} \quad j_0 : G^*(v) \to M_2(G^*(v))$$

are graded homomorphisms. $\square$

We have the following graded analog of Theorem 9.

Theorem 11. *Let $G^*(v)$ be the universal Z_2-graded algebra of a partial isometry of degree one. There exists a graded $*$-homomorphism $\varphi_v : \overset{\wedge}{M_2}(C) \to M_2(G^*(v))$ such that $\omega\varphi_v$ is homotopic to i_0 and $\varphi_v\omega$ is homotopic to j_0. That is, $G^*(v)$ and $\overset{\wedge}{M_2}(C)$ are homotopy equivalent up to stabilization by odd graded 2×2 matrices.*

Proof. It exactly coincides with the proof of Theorem 9.

Corollary 12. *The canonical $*$-homomorphism ω induces an invertible element of $KK(G^*(v), \overset{\wedge}{M_2}(C))$.*

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Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia