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WEIGHT INEQUALITIES FOR SINGULAR INTEGRALS DEFINED ON SPACES OF HOMOGENEOUS AND NONHOMOGENEOUS TYPE

D. E. EDMUNDS, V. KOKILASHVILI, AND A. MESKHI

Dedicated to Professor Hans Triebel on his 65th birthday

Abstract. Optimal sufficient conditions are found in weighted Lorentz spaces for weight functions which provide the boundedness of the Calderón–Zygmund singular integral operator defined on spaces of homogeneous and nonhomogeneous type.

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Introduction

In the present paper optimal sufficient conditions are found for the weight functions which provide the boundedness of Calderón–Zygmund singular integral operator defined on spaces of homogeneous as well as nonhomogeneous type in weighted Lorentz spaces. In the nonhomogeneous case the results are new even in Lebesgue spaces.

Two-weight strong type inequalities for singular integrals in Lebesgue spaces on spaces of homogeneous type (SHT) were established in [5] (see also [6], Chapter 9, and [14]), while a similar problem for singular integrals defined on homogeneous groups was considered in [16] in the case of Lorentz spaces.

Two-weight inequalities for Hilbert transforms with monotonic weights were studied in [17]. Analogous problems for singular integrals in Euclidean spaces were considered in [8] and generalized in [7] for singular integrals on Heisenberg groups. Optimal conditions for a pair of weights ensuring the validity of two-weight inequalities for Calderón–Zygmund singular integrals were obtained in [3] (and also in [19] for the case of Lorentz spaces). The latter result was generalized to the setting of homogeneous groups in [13], [15].

Finally, we would like to mention that singular integrals on spaces of nonhomogeneous type (measure spaces with a metric) were studied in [18].

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1. Preliminaries

In this section we give the notion of SHT and defined Lorentz space. The definition of a singular integral and several well-known results concerning the Hardy-type operators on SHT are also given.

Definition 1.1. A space of homogeneous type (SHT) (X, d, μ) is a topological space X endowed with a complete measure μ such that: (a) the space of compactly supported continuous functions is dense in $L^1(X,\mu)$, and (b) there exists a non-negative real-valued function (quasimetric) $d : X \times X \to \mathbb{R}^1$ satisfying:

(i) $d(x, x) = 0$ for arbitrary $x \in X$;

(ii) $d(x, y) > 0$ for arbitrary $x, y \in X$, $x \neq y$;

(iii) there exists a positive constant a_0 such that the inequality

$$d(x, y) \leq a_0 d(y, x)$$

holds for all $x, y \in X$;

(iv) there exists a positive constant a_1 such that the inequality

$$d(x, y) \leq a_1(d(x, z) + d(z, y))$$

holds for arbitrary $x, y, z \in X$;

(v) for every neighborhood V of any point $x \in X$ there exists a number $r > 0$ such that the ball

$$B(x, r) = \Big\{y \in X : \ \ d(x, y) < r\Big\}$$

with center in x and radius r is contained in V;

(vi) the balls $B(x, r)$ are measurable for all $x \in X$, $r > 0$ and, moreover, $0 < \mu B(x, r) < \infty$;

(vii) there exists a positive constant b such that the inequality (doubling condition)

$$\mu B(x, 2r) \leq b\mu B(x, r)$$

is true for all $x \in X$ and for all positive r.

It will be supposed that for some $x_0 \in X$

$$\mu\{x_0\} = \mu\{x \in X : \ \ d(x_0, x) = a\} = 0,$$

where

$$a \equiv \sup \Big\{d(x_0, x) : \ \ x \in X\Big\}.$$

Note that the condition $a = \infty$ is equivalent to the condition $\mu(X) = \infty$, and if $a = \infty$, then $\mu\{x \in X : \ d(x_0, x) = a\} = 0$.

For the definitions, various examples and properties of SHT, we refer to [2], [23] (also to [6]).

Definition 1.2. By a spaces of nonhomogeneous type we mean a measure space with a quasimetric (X, d, μ) satisfying conditions (i)–(v) of Definition 1.1, i.e., the doubling condition is not assumed and may fail.

In the sequel it will be assumed that there exists a point $x_0 \in X$ such that

$$B(x_0, R)\backslash B(x_0, r) \neq \varnothing \tag{1}$$

for all r and R provided $0 < r < R < a$.

In the remaining part of this section (X, d, μ) is assumed to be a space of nonhomogeneous type.

Definition 1.3. An almost everywhere positive, locally integrable function $w : X \to \mathbb{R}^1$ is called a weight.

For weight functions w, we denote by $L^{pq}_w(X)$ the Lorentz space with weight w which represents the class of all μ-measurable functions $f : X \to \mathbb{R}^1$ for which

$$\|f\|_{L^{pq}_w(X)} = \bigg(q \int\limits_0^{+\infty} \bigg(\int\limits_{\{x\in X\colon |f(x)|>\lambda\}} w(x)\, d\mu \bigg)^{q/p} \lambda^{q-1}\, d\lambda \bigg)^{1/q} < \infty$$

when $1 \leq p \leq \infty$, $1 \leq q < \infty$, and

$$\|f\|_{L^{p\infty}_w(X)} = \sup_{\lambda>0} \lambda \bigg(\int\limits_{\{x\in X\colon |f(x)|>\lambda\}} w(x)\, d\mu \bigg)^{1/p} < \infty$$

when $1 < p < \infty$.

It is clear that if $p = q$, then $L^{pq}_w(X) = L^p_w(X)$ is a Lebesgue space. If $w \equiv 1$, then we put

$$L^p_w(X) \equiv L^p(X).$$

The following lemma holds.

Lemma A. *Let $E \subset X$ be a μ-measurable set, $1 < p, q < \infty$ and suppose that w is a weight function on X. Further, let f, f_1 and f_2 be μ-measurable functions on X. Then:*

(i)

$$\|\chi_E(\cdot)\|_{L^{pq}_w(X)} = \bigg(\int\limits_E w(x)\, d\mu \bigg)^{1/p};$$

(ii)

$$\|f\|_{L^{pq_1}_w(X)} \leq \|f\|_{L^{pq_2}_w(X)}$$

for fixed p and $q_1 \leq q_2$;

(iii) *there exists a positive constant c such that*

$$\|f_1 f_2\|_{L^{pq}_w(X)} \leq c\|f_1\|_{L^{p_1q_1}_w(X)} \|f_2\|_{L^{p_2q_2}_w(X)},$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

The first part of this Lemma is proved in [22], Chpater V, Section 3 and the rest of the proof is given in [10].

Lemma B ([1], [19]). *Let $\{E_k\}$ be a sequence of μ-measurable subsets of X, such that $\sum\limits_k \chi_{E_k} \leq c_0 \chi_{\cup E_k}$. Then*

(i) *there exists a positive constant $k_1 = k_1(c_0)$, such that for arbitrary f, the following inequality holds:*

$$\sum_k \|f(\cdot)\chi_{E_k(\cdot)}\|^{\lambda}_{L^{rs}_w(X)} \le k_1 \|f(\cdot)\chi_{\cup E_k(\cdot)}\|^{\lambda}_{L^{rs}_w(X)},$$

where $\max\{r,s\} \le \lambda$*;*

(ii) *there exists a positive constant $k_2 = k_2(c_0)$, such that for every f, the following inequality holds:*

$$\|f(\cdot)\chi_{\cup E_k(\cdot)}\|^{\gamma}_{L^{pq}_w(X)} \le k_2 \sum_k \|f(\cdot)\chi_{E_k(\cdot)}\|^{\gamma}_{L^{pq}_w(X)},$$

where $0 < \gamma \le \min\{p,q\}$.

Definition 1.4. If $1 < p < \infty$, then $A_p(X)$ is the set of all weights w such that

$$\sup \Big(\frac{1}{\mu(B)} \int\limits_B w(x)\,d\mu\Big)\Big(\frac{1}{\mu(B)} \int\limits_B w^{1-p'}(x)\,d\mu\Big)^{p-1} < \infty, \quad p' = \frac{p}{p-1},$$

where the supremum is taken over all balls $B \subset X$.

We recall that $a \equiv \sup\{d(x_0,x):\ x \in X\}$ for some $x_0 \in X$ with $\mu\{x_0\} = 0$.

Lemma C ([5]). *Let $a = \infty$, $1 < p < \infty$, $\rho \in A_p(X)$ and $0 < c_1 \le c_2 < c_3 < \infty$. Then there exists a positive constant c such that the inequality*

$$\int\limits_{B(x_0,c_3t)\setminus B(x_0,c_2t)} \rho(x)\,d\mu \le c \int\limits_{B(x_0,c_1t)} \rho(x)\,d\mu$$

holds for arbtrary $t > 0$.

In the sequel we shall need the following results concerning the Hardy-type operators on the measure space (X,μ)

$$\mathcal{H}f(t) = \int\limits_{\{x:\, d(x_0,x)<t\}} f(x)\,d\mu(x), \quad t \in (0,a),$$

and

$$\widetilde{\mathcal{H}}f(t) = \int\limits_{\{x:\, d(x_0,x)>t\}} f(x)\,d\mu(x), \quad t \in (0,a).$$

The following result is from [5].

Proposition A. *Let $1 < p \le q < \infty$, $\mu\{x_0\} = 0$. Then the inequality*

$$\Big(\int\limits_0^a v(t)|\mathcal{H}f(t)|^q\,dt\Big)^{1/q} \le c\Big(\int\limits_X |f(x)|^p w(x)\,d\mu\Big)^{1/p}, \tag{1.1}$$

holds with some $c > 0$ independent of f, $f \in L^p_w(X)$, if and only if

$$D = \sup_{0<t<a} \Big(\int\limits_t^a v(\tau)\,d\tau\Big)^{1/q}\Big(\int\limits_{\{x:\, d(x_0,x)<t\}} w^{1-p'}(x)\,d\mu\Big)^{1/p'} < \infty.$$

Moreover, if c is the best constant in (1.2)*, then $D \le c \le 4D$.*

Proposition B ([5]). *Let $1 < p \le q < \infty$, $\mu\{x:\ d(x_0,x) = a\} = 0$. Then the inequality*

$$\Big(\int\limits_0^a v(t)|\widetilde{\mathcal{H}}f(t)|^q\,dt\Big)^{1/q} \le c\Big(\int\limits_X |f(x)|^p w(x)\,d\mu\Big)^{1/p} \tag{1.2}$$

holds with a constant c independent of f, $f \in L^p_w(X)$, if and only if

$$\widetilde{D} = \sup_{0<t<a}\Big(\int\limits_0^t v(\tau)\,d\tau\Big)^{1/q}\Big(\int\limits_{\{x:\ d(x_0,x)>t\}} w^{1-p'}(x)\,d\mu\Big)^{1/p'} < \infty.$$

Moreover, if c is the best constant in (1.3)*, then $\widetilde{D} \le c \le 4\widetilde{D}_1$.*

Let f be a μ-measurable function defined on X and suppose that

$$H_{\varphi\psi}f(x) = \varphi(x)\int\limits_{B(x_0,d(x_0,x))} f(y)\psi(y)w(y)\,d\mu(y)$$

is a Hardy-type operator on X, where φ and ψ are positive μ-measurable functions on X.

The following proposition is true (see Theorem 2.4 of [4] for $a = \infty$. The case $a < \infty$ can be obtained similarly).

Proposition C. *Let $r = s = 1$ or $r \in (0,\infty]$ and $s \in [1,\infty]$, $p = q = 1$ or $p \in (1,\infty)$, $q \in [1,\infty]$, $\max\{r,s\} \le \min\{p,q\}$. The operator $H_{\varphi\psi}$ is bounded from $L^{rs}_w(X)$ to $L^{pq}_v(X)$ if and only if*

$$A \equiv \sup_{0<t<a}\Big\|\varphi(\cdot)\chi_{\{d(x_0,y)>t\}}(\cdot)\Big\|_{L^{pq}_v(X)}\Big\|\psi(\cdot)\chi_{\{d(x_0,y)\le t\}}(\cdot)\Big\|_{L^{r's'}_w(X)} < \infty$$

with $\|H_{\varphi\psi}\| \approx A$.

Let

$$H'_{\varphi\psi}f(x) = \varphi(x)\int\limits_{\{y:\ d(x_0,y)>d(x_0,x)\}} f(y)\psi(y)\,d\mu(y).$$

By the dual arguments we deduce the following result:

Proposition D. *Let r, s, p and q satisfy the conditions of Proposition* C*. The operator $H'_{\varphi,\psi}$ is bounded from $L^{rs}_w(X)$ to $L^{pq}_v(X)$ if and only if:*

$$A' = \sup_{t>0}\Big\|\varphi(\cdot)\chi_{\{d(x_0,y)\le t\}}(\cdot)\Big\|_{L^{pq}_v(X)}\Big\|\psi(\cdot)\chi_{\{d(x_0,y)>t\}}(\cdot)\Big\|_{L^{r's'}_w(X)} < \infty$$

with $\|H'_{\varphi\psi}\| \approx A'$.

2. Singular integrals on SHT

In this section the existence of singular integrals is investigated and optimal sufficient conditions for the boundedness of singular integral operators in weighted Lorentz spaces defined on SHT (X, d, μ) are established.

Let $k : X \times X \backslash \{(x,x) : \ x \in X\} \to \mathbb{R}^1$ be a measurable function satisfying the condition

$$|k(x,y)| \le \frac{c}{\mu B(x, d(x,y))},$$

for all x, $y \in X$, $x \ne y$, and

$$|k(x_1,y) - k(x_2,y)| + |k(y,x_1) - k(y,x_2)| \le c\omega\Big(\frac{d(x_2,x_1)}{d(x_2,y)}\Big)\frac{1}{\mu(B(x_2, d(x_2,y)))},$$

for all x_1, x_2, $y \in X$ provided that $d(x_2,y) > bd(x_1,x_2)$, where ω is a positive non-decreasing function on $(0,\infty)$ satisfying the Δ_2-condition (i.e., $\omega(2t) \le c\omega(t)$ for all $t \in (0,\infty)$) and the Dini condition

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty.$$

We shall also suppose that for some p_0, $1 < p_0 < \infty$, and all $f \in L^{p_0}(X)$, the limit

$$Kf(x) = \lim_{\varepsilon \to 0+} \int_{X \backslash B(x,\varepsilon)} k(x,y) f(y)\, d\mu$$

exists a.e. on X and that the operator$K$ is bounded in $L^p_w(X)$.

For the definition of singular integrals and for other related remarks see, e.g., [21], Chapter I, pp. 29–36, also [6], p. 295, and [5].

Theorem A ([9]). *Let $1 < p < \infty$. If $w \in A_p(X)$, then the operator K is bounded in $L^p_w(X)$.*

Theorem B ([12], p. 207). *Let $1 < p,q < \infty$. If $w \in A_p(X)$, then the operator K is bounded in $L^{pq}_w(X)$. If the Hilbert transform H,*

$$Hf(x) = p.v. \int_{-\infty}^{+\infty} f(t)(x-t)^{-1}\, dt,$$

is bounded in $L^{pq}_w(\mathbb{R})$, then $w \in A_p(\mathbb{R})$.

The following Lemma holds.

Lemma 2.1. *Let $1 < s \le p < \infty$, $\rho \in A_p(X)$. Suppose that for weight functions w and w_1 the following conditions hold:*

(1) there exist a positive increasing function σ defined on $(0, 4a_1a)$ and a positive constant b such that

$$\sigma(d(x_0,x))\rho(x) \le bw(x)w_1(x)$$

for almost all $x \in X$;

(2) *for all* t, $0 < t < a$, *we have*

$$\Big\| \frac{1}{w(\cdot)w_1(\cdot)}\, \chi_{\{d(x_0,y)\le t\}}(\cdot)\Big\|_{L_w^{p's'}(X)} < \infty.$$

Then $Kg(x)$ *exists a.e. on* X *for any* g *satisfying the condition*

$$\|g(\cdot)w_1(\cdot)\|_{L_w^{ps}(X)} < \infty.$$

Proof. Let us fix some positive α satisfying $0 < \alpha < \frac{\alpha}{\alpha_1}$ and put

$$S_\alpha = \Big\{x \in X : \quad d(x_0, x) \ge \frac{\alpha}{2}\Big\}.$$

Assume that $\|g(\cdot)w_1(\cdot)\|_{L_w^{ps}(X)} < \infty$. Introduce g in the following way:

$$g(x) = g_1(x) + g_2(x),$$

where $g_1(x) = g(x) \cdot \chi_{S_a}(x)$, $g_2(x) = g(x) - g_1(x)$.

For g_1 we have

$$\int_X |g_1(x)|^p \rho(x)\, d\mu = \frac{\sigma(\alpha/2)}{\sigma(\alpha/2)} \int_{S_a} |g(x)|^p \rho(x)\, d\mu$$

$$\le \frac{1}{\sigma(\alpha/2)} \int_{S_a} |g(x)|^p \rho(x)\sigma(2a_1 d(x_0,x))\, d\mu \le \frac{b}{\sigma(\alpha/2)} \int_{S_a} |g(x)|^p w_1^p(X) w(x)\, d\mu$$

$$= \frac{b_1}{\sigma(\alpha/2)}\, \|g(\cdot)w_1(\cdot)\|^p_{L_w^p(X)} \le \frac{b}{\sigma(\alpha/2)}\, \|g(\cdot)w_1(\cdot)\|^p_{L_w^{ps}(X)}.$$

(In the latter inequality we have used Lemma B, part (ii).) Hence $g_1 \in L^p_\rho(X)$ and, according to Theorem A, $Kg \in L^p_\rho(X)$ and consequently $Kg(x)$ exists a.e. on X.

Now, let $d(x_0, x) > \alpha a_1$, and let $d(x_0, y) < \alpha/2$. We have

$$d(x_0, x) \le a_1(d(x_0, y) + d(y, x)) \le a_1(d(x_0, y) + a_0 d(x, y)).$$

Hence

$$d(x, y) \ge \frac{d(x_0, x)}{a_1 a_0} - \frac{1}{a_0}\, d(x_0, y) \ge \frac{\alpha}{a_0} - \frac{\alpha}{2a_0} = \frac{\alpha}{2a_0}.$$

Moreover,

$$B(x_0, d(x, y)) \subset B\Big(x, a_1(1 + a_1(1 + a_0^2))d(x, y)\Big)$$

and consequently we obtain the inequality

$$\mu B(x_0, \alpha/2) \le c_1 \mu B(x, d(x, y)),$$

where c_1 is independent of x. So, for Kg_2, using the latter estimate and the Hölder's inequality, we have

$$|Kg_2(x)| = \Big| \int_X g(y)k(x,y)\,d\mu \Big| \le c_2 \int_{B(x_0,\alpha/2)} \frac{|g(y)|}{\mu B(x,d(x,y))}\,d\mu$$
$$\le \frac{c_3}{\mu B(x_0,\frac{\alpha}{2c_0})} \int_{B(x_0,\alpha/2)} |g(y)|\,d\mu$$
$$\le \frac{c_3}{\mu B(x_0,\frac{\alpha}{2c_0})} \int_{B(x_0,\alpha/2)} |g(y)| \frac{1}{w(y)w_1(y)} w_1(y)w(y)\,d\mu$$
$$\le \frac{c_3}{\mu B(x_0,\frac{\alpha}{2c_0})} \Big\| \chi_{B(x_0,\frac{\alpha}{2})}(\cdot) \frac{1}{w(\cdot)w_1(\cdot)} \Big\|_{L_w^{p's'}(X)} \|g(\cdot)w_1(\cdot)\|_{L_w^{ps}(X)} < \infty.$$

Thus $Kg_2(x)$ converges absolutely for all x with $d(x_0,x) > \alpha a_1$. Since we can choose α arbitrarily small and $\mu\{x_0\} = 0$, we conclude that $Kg(x)$ exists μ- a.e. on X. □

From Lemma 2.1 it is easy to derive

Lemma 2.2. *Let $1 < s \le p < \infty$. If u and u_1 are positive increasing functions on $(0, 4a_1a)$ and*

$$\Big\| \frac{1}{u(d(x_0,\cdot))u_1(d(x_0,\cdot))} \chi_{\{d(x_0,y)\le t\}}(\cdot) \Big\|_{L^{p's'}_{u(d(x_0,\cdot))}(X)} < \infty$$

for all t satisfying $0 < t < a$, then for arbitrary g satisfying the condition

$$\|\varphi(\cdot)u_1(d(x_0,\cdot))\|_{L^{ps}_{u(d(x_0,\cdot))}(X)} < \infty,$$

$K\varphi(x)$ exists a.e. on X.

The following lemma is proved in the same way as Lemma 2.1.

Lemma 2.3. *Let $a = \infty$, $1 < s \le p < \infty$, $\rho \in A_p(X)$. If the weight functions w and w_1 satisfy the conditions:*

(1) *there exists a decreasing positive function σ on $(0,\infty)$ such that for almost all $x \in X$ we have*

$$\sigma(d(x_0,\cdot))\rho(x) \le bw(x)w_1^p(x),$$

where the positive constant b does not depend on $x \in X$;

(2) *if for every $t > 0$*

$$\Big\| \frac{\mu B(d(x_0,\cdot))^{-1}}{w(\cdot)w_1(\cdot)} \chi_{\{d(x_0,y)>t\}}(\cdot) \Big\|_{L_w^{p's'}(X)} < \infty,$$

then $Kg(x)$ exists a.e. on X for arbitrary g provided that $\|g(\cdot)w_1(\cdot)\|_{L_w^{ps}(X)} < \infty$.

From Lemma 2.3 the next lemma follows easily:

Lemma 2.4. *Let $a = \infty$, $1 < s \le p < \infty$. Assume that u and u_1 are positive decreasing functions on $(0, \infty)$ and the condition*

$$\Big\| \frac{\mu B(d(x_0, \cdot))^{-1}}{u(d(x_0, \cdot))u_1(d(x_0, \cdot))} \chi_{\{d(x_0,y)>t\}}(\cdot) \Big\|_{L^{p's'}_{u(d(x_0,\cdot)}(X)} < \infty$$

holds for any $t > 0$. Then $Kg(x)$ exists a.e. on X for arbitrary g satisfying the condition $g(\cdot)u_1(d(x_0, \cdot)) \in L^{ps}_{u(d(x_0,\cdot))}(X)$.

Now we pass to the weight inequalities for the operator K.

Theorem 2.1. *Let $1 < s \le p \le q < \infty$, let w be a weight function on X, suppose that σ is a positive increasing continuous function on $(0, 4a_1a)$, $\rho \in A_p(X)$ and $v(x) = \sigma(d(x_0, x))\rho(x)$. Suppose the following conditions are satisfied:*

(a) *there exists a positive constant b such that the inequality*

$$\sigma(2a_1 d(x_0, x))\rho(x) \le bw(x)$$

holds for almost every $x \in X$;

(b)

$$B \equiv \sup_{0<t<a} \Big\| \mu B(x_0, d(x_0, \cdot))^{-1} \chi_{\{d(x_0,y)>t\}} \Big\|_{L^{pq}_v(X)} \Big\| \frac{1}{w(\cdot)} \chi_{\{d(x_0,y)\le t\}}(\cdot) \Big\|_{L^{p's'}_w(X)} < \infty.$$

Then there exists a positive constant c such that

$$\|Kf(\cdot)\|_{L^{pq}_v(X)} \le c\|f(\cdot)\|_{L^{ps}_w(X)} \tag{2.1}$$

for all f, $f \in L^{ps}_w(X)$.

Proof. First, let us assume that σ is of the kind

$$\sigma(t) = \sigma(0+) + \int_0^t \varphi(\tau)\, d\tau, \quad \varphi \ge 0.$$

Then we have

$$\|Kf(\cdot)\|_{L^{pq}_v(X)} \le c_1 \Big(q \int_0^\infty \lambda^{q-1} \Big(\int_{\{x\in X:\, |Kf(x)|>\lambda\}} \rho(x)\sigma(0+)\, d\mu \Big)^{q/p} d\lambda \Big)^{1/q}$$

$$+ c_1 \Big(q \int_0^\infty \lambda^{q-1} \Big(\int_{\{x\in X:\, |Kf(x)|>\lambda\}} \rho(x) \Big(\int_0^{d(x_0,x)} \varphi(t)\, dt \Big) d\mu \Big)^{q/p} d\lambda \Big)^{1/q} \equiv I_1 + I_2.$$

In the case, where $\sigma(0+) = 0$, we have $I_1 = 0$; otherwise, by Theorem B and Lemma A (part (ii)), we have

$$I_1 = c_1\sigma(0+)^{1/p} \|Kf(\cdot)\|_{L^{pq}_\rho(X)} \le c_2\sigma^{1/p}(0+)\|f(\cdot)\|_{L^{pq}_\rho(X)}$$
$$\le c_2\sigma^{1/p}(0+)\|f(\cdot)\|_{L^{ps}_\rho(X)} \le c_2\|f(\cdot)\|_{L^{ps}_w(X)}.$$

Now, let us estimate I_2. Set

$$f_{1t}(x)=f(x)\chi_{\{d(x_0,x)>\frac{t}{2a_1}\}},\quad f_{2t}(x)=f(x)-f_{1t}(x).$$

We have

$$I_2=c_1\Big(q\int_0^\infty\lambda^{q-1}\Big(\int_0^a\varphi(t)\Big(\int_{\{x:\,d(x_0,x)>t,|Kf(x)|>\lambda\}}\rho(x)\,d\mu\Big)\,dt\Big)^{q/p}d\lambda\Big)^{1/q}$$

$$\leq c_3\Big(q\int_0^\infty\lambda^{q-1}\Big(\int_0^a\varphi(t)\Big(\int_{\{x:\,d(x_0,x)>t\}}\chi_{\{x:\,|Kf_{1t}(x)|>\frac{\lambda}{2}\}}\rho(x)\,d\mu\Big)\,dt\Big)^{q/p}d\lambda\Big)^{1/q}$$

$$+c_3\Big(q\int_0^\infty\lambda^{q-1}\Big(\int_0^a\varphi(t)\Big(\int_{\{x:\,d(x_0,x)>t\}}\chi_{\{x:\,|Kf_{2t}(x)|>\frac{\lambda}{2}\}}\rho(x)\,d\mu\Big)\,dt\Big)^{q/p}d\lambda\Big)^{1/q}\equiv I_{21}+I_{22}.$$

Applying the Minkowki's inequality twice ($\frac{q}{p}\geq 1$, $\frac{p}{s}\geq 1$) and using Theorem B and Lemma A (part (ii)), we obtain

$$I_{21}\leq c_4\Big(\int_0^a\varphi(t)\Big(\int_0^\infty\lambda^{q-1}\Big(\int_{\{x\in X:\,|Kf_{1t}(x)|>\frac{\lambda}{2}\}}\rho(x)\,d\mu\Big)^{q/p}d\lambda\Big)^{p/q}dt\Big)^{1/p}$$

$$\leq c_5\Big(\int_0^a\varphi(t)\|f_{1t}(\cdot)\|^p_{L^{pq}_\rho(X)}\,dt\Big)^{1/p}\leq c_5\Big(\int_0^a\varphi(t)\|f_{1t}(\cdot)\|^p_{L^{ps}_\rho(X)}\,dt\Big)^{1/p}$$

$$\leq c_5\Big(\int_0^\infty\lambda^{s-1}\Big(\int_{\{x\in X:\,|f(x)|>\lambda\}}\rho(x)\Big(\int_0^{2a_1d(x_0,x)}\varphi(t)\,dt\Big)\,d\mu\Big)^{s/p}d\lambda\Big)^{1/s}$$

$$\leq c_5\|f(\cdot)\|_{L^{ps}_w(X)}\leq c_6\|f(\cdot)\|_{L^{ps}_w(X)}.$$

Next we shall estimate I_{22}. Note that if $d(x_0,x)>t$ and $d(x_0,y)<\frac{t}{2a_1}$, then

$$d(x_0,x)\leq a_1(d(x_0,y)+d(y,x))\leq a_1(d(x_0,y)+a_0d(x,y))$$
$$\leq a_1\Big(\frac{t}{2a_1}+a_0d(x,y)\Big)\leq a_1\Big(\frac{d(x_0,x)}{2a_1}+a_0d(x,y)\Big).$$

Hence

$$\frac{d(x_0,x)}{2a_1a_0}\leq d(x,y),$$

and we also have

$$\mu(B(x,d(x_0,x)))\leq b\mu\Big(B\Big(x,\frac{d(x_0,x)}{2a_1a_0}\Big)\Big)\leq b\mu(B(x,d(x,y))).$$

Moreover, it is easy to show that

$$\mu B(x_0,d(x_0,x))\leq b_1\mu B(x,d(x_0,x)).$$

Hence

$$\mu B(x_0,d(x_0,x))\leq b_2\mu B(x,d(x,y)).$$

Taking into account this inequality and using Proposition C we see that

$$I_{22} \le c_7 \Big\| \frac{1}{\mu B(x_0, d(x_0,\cdot))} \int\limits_{\{d(x_0,y)<d(x_0,\cdot)\}} |f(y)|\,dy \Big\|_{L^{pq}_v(X)} \le c_9 \|f(\cdot)\|_{L^{ps}_w(X)}.$$

Now, let σ be a positive continuous, but not absolutely continuous, increasing function on $(0, 4a_1a)$; then there exists a sequence of absolutely continuous functions σ_n such that $\sigma_n(t) \le \sigma(t)$ and $\lim_{n\to\infty} \sigma_n(t) = \sigma(t)$ for arbitrary $t \in (0, 4a_1a)$. For these functions, we can take $\sigma_n(t) = \sigma(0+) + n \int_0^t [\sigma(\tau) - \sigma(\tau - \frac{1}{n})]\,d\tau$.

Denote $v_n(x) = \rho(x)\sigma_n(d(x_0,x))$; then $B_n < B$, where

$$B_n \equiv \sup_{t>0} \big\| \chi_{\{d(x_0,y)>t\}}(\cdot)\mu(B(x_0,d(x_0,\cdot)))^{-1} \big\|_{L^{pq}_{v_n}(X)} \Big\| \frac{1}{w(\cdot)}\,\chi_{\{d(x_0,y)\le t\}}(\cdot) \Big\|_{L^{p's'}_w(X)}.$$

By virtue of what has been proved, if $B < \infty$, then the following inequality holds:

$$\|Kf(\cdot)\|_{L^{pq}_{v_n}(X)} \le c\|f(\cdot)\|_{L^{ps}_w(X)},$$

where the constant $c > 0$ does not depend on n.

By passing to the limit as $n \to \infty$, we obtain inequality (2.1). □

Using the representation $\sigma(t) = \sigma(+\infty) + \int_t^\infty \psi(\tau)\,d\tau$, where $\sigma(+\infty) = \lim_{t\to\infty} \sigma(t)$ and $\psi \ge 0$ on $(0,\infty)$ and Proposition D, we obtain the following result.

Theorem 2.2. *Let $a = \infty$, $1 < s \le p \le q < \infty$, and suppose that σ is a positive decreasing continuous function on $(0,\infty)$. Assume that $\rho \in A_p(X)$ and $v(X) = \sigma(d(x_0,x))\rho(x)$. Suppose the following conditions are satisfied:*

(a) *there exists a positive constant b such that the inequality*

$$\rho(x)\sigma\Big(\frac{d(x_0,x)}{2a_1}\Big) \le bw(x)$$

is true for almost every $x \in X$;

(b)

$$B' = \sup_{t>0} \|\chi_{\{d(x_0,y)\le t\}}(\cdot)\|_{L^{pq}_v(X)} \Big\| \frac{(\mu B(x_0,d(x_0,x)))^{-1}}{w(\cdot)}\,\chi_{\{d(x_0,y)>t\}}(\cdot) \Big\|_{L^{p's'}_w(X)} < \infty.$$

Then inequality (2.1) *holds.*

Now let us consider the particular cases of Theorems 2.1 and 2.2.

Theorem 2.3. *Let $a = \infty$, $1 < p \le q < \infty$, suppose σ_1 and σ_2 are positive, increasing functions on $(0,\infty)$, let σ_1 be a continuous function and suppose that $\rho \in A_p(X)$. Put $v(x) = \sigma_2(d(x_0,x))\rho(x)$, $w(x) = \sigma_1(d(x_0,x))\rho(x)$. If*

$$\sup_{t>0} \big\| (\mu B(x_0,d(x_0,\cdot)))^{-1}\chi_{\{d(x_0,y)>t\}}(\cdot) \big\|_{L^{pq}_v(X)} \Big\| \frac{1}{w(\cdot)}\,\chi_{\{d(x_0,y)\le t\}}(\cdot) \Big\|_{L^{p'}_w(X)} < \infty,$$

then there exists a positive constant c such that

$$\|Kf(\cdot)\|_{L^{pq}_v(X)} \le c\|f(\cdot)\|_{L^p_w(X)}$$

for all $f \in L^p_w(X)$.

Proof. By Theorem 2.1 it is sufficient to show that there exists a positive constant b such that $\sigma_2(2a_1t) \le b\sigma_1(t)$ for all $t \in (0,\infty)$.

From the doubling condition (see (vii) in Definition 1.1) and the condition (1.1) it follows that the measure μ satisfies the reverse doubling condition at the point x_0 (see, e.g., [20] and [23], Lemma 20). In other words, there exist constants $\eta_1 > 1$ and $\eta_2 > 1$ such that

$$\mu(B(x_0,\eta_1 r)) \ge \eta_2\mu(B(x_0,r))$$

for all $r > 0$.

Applying the Hölder's inequality and using Lemma C, the reverse doubling condition and the fact that $\rho^{1-p'} \in A_{p'}(X)$, we obtain

$$\begin{aligned}
\frac{\sigma_2(2a_1t)}{\sigma_1(t)} &\le \Big(\mu(B(x_0,2a_1\eta_1t)\backslash B(x_0,2a_1t))\Big)^{-p}\Big(\int\limits_{B(x_0,2a_1\eta_1t)\backslash B(x_0,2a_1t)}\rho(x)\,d\mu\Big)\\
&\quad\times\Big(\int\limits_{B(x_0,2a_1\eta_1t)\backslash B(x_0,2a_1t)}\rho^{1-p'}(x)\,d\mu\Big)^{p-1}\frac{\sigma_2(2a_1t)}{\sigma_1(t)}\\
&\le \Big(\Big(1-\frac{1}{\eta_2}\Big)\mu B(x_0,2a_1\eta t)\Big)^{-p}\Big(\int\limits_{B(x_0,2a_1\eta_1t)\backslash B(x_0,2a_1t)}\rho(x)\,d\mu\Big)\\
&\quad\times\Big(\int\limits_{B(x_0,t)}\rho^{1-p'}(x)\,d\mu\Big)^{p-1}\frac{\sigma_2(2a_1t)}{\sigma_1(t)}\\
&\le c_1(\mu B(x_0,2a_1\eta t))^{-p}\Big\|\chi_{(B(x_0,2a_1\eta_1t)\backslash B(x_0,2a_1t))}(\cdot)\Big\|^p_{L^{pq}_v(X)}\Big\|\frac{1}{w(\cdot)}\,\chi_{B(x_0,t)}(\cdot)\Big\|^p_{L^{p'}_w(X)}\\
&\le c_2\Big\|\Big(\mu B(x_0,d(x_0,\cdot))\Big)^{-1}\chi_{\{d(x_0,y)>t\}}(\cdot)\Big\|^p_{L^{pq}_v(X)}\Big\|\frac{1}{w(\cdot)}\,\chi_{B(x_0,t)}(\cdot)\Big\|^p_{L^{p'}_w(X)} \le c. \qquad \square
\end{aligned}$$

An analogous theorem dealing with singular integrals on homogeneous groups was proved in [16], and for singular integrals on SHT in the Lebesgue space in [5].

Theorem 2.4. *Let $a = \infty$, $1 < s \le p \le q < \infty$; suppose σ_1, σ_2, u_1 and u_2 are weight functions defined on X. Let $\rho \in A_p(X)$, and suppose $v = \sigma_2\rho$, $w = \sigma_1\rho$. Assume that the following conditions are fulfilled:*

(1) *there exists a positive constant b such that for all $t > 0$*

$$\sup_{F_t}\sigma_2^{1/p}(x)\sup_{F_t}u_2(x) \le b\inf_{F_t}\sigma_1^{1/p}(x)\inf_{F_t}u_1(x)$$

holds, where $F_t = \{x \in X:\ \frac{t}{a_1} \le d(x_0,x) < 8a_1t\}$;

(2)

$$\sup_{t>0}\Big\|u_2(\cdot)\Big(\mu(B(x_0,d(x_0,\cdot)))\Big)^{-1}\chi_{\{d(x_0,t)>t\}}(\cdot)\Big\|_{L_v^{pq}(X)}$$
$$\times\Big\|\frac{1}{u_1(\cdot)w(\cdot)}\,\chi_{\{d(x_0,y)\le t\}}(\cdot)\Big\|_{L_w^{p's'}(X)}<\infty;$$

(3)

$$\sup_{t>0}\Big\|u_2(\cdot)\chi_{\{d(x_0,t)\le t\}}(\cdot)\Big\|_{L_v^{pq}(X)}\Big\|\frac{(\mu B(x_0,d(x_0,\cdot)))^{-1}}{u_1(\cdot)w(\cdot)}\,\chi_{\{d(x_0,y)>t\}}(\cdot)\Big\|_{L_w^{p's'}(X)}<\infty.$$

Then

$$\|u_2(\cdot)Kf(\cdot)\|_{L_v^{pq}(X)}\le c\|u_1(\cdot)f(\cdot)\|_{L_w^{ps}(X)}, \tag{2.2}$$

where the positive constant c does not depend on f.

Proof. Let

$$E_k\equiv B(x_0,2^{k+1})\backslash B(x_0,2^k),\quad G_{k,1}\equiv B(x_0,2^{k-1}/a_1),$$
$$G_{k,2}\equiv B(x_0,a_1 2^{k+2})\backslash B(x_0,2^{k-1}/a_1),\quad G_{k,3}\equiv X\backslash B(x_0,a_1 2^{k+2}).$$

We obtain

$$\|u_2(\cdot)Kf(\cdot)\|^p_{L_v^{pq}(X)}\le c_1\Big\|\sum_{k\in Z}u_2(\cdot)K(f\chi_{G_{k,1}})(\cdot)\chi_{E_k}(\cdot)\Big\|^p_{L_v^{pq}(X)}$$
$$+c_1\Big\|\sum_{k\in Z}u_2(\cdot)K(f\chi_{G_{k,2}})(\cdot)\chi_{E_k}(\cdot)\Big\|^p_{L_v^{pq}(X)}$$
$$+c_1\Big\|\sum_{k\in Z}u_2(\cdot)K(f\chi_{G_{k,3}})(\cdot)\chi_{E_k}(\cdot)\Big\|^p_{L_v^{pq}(X)}\equiv c_1(S_1^p+S_2^p+S_3^p).$$

Now we estimate S_1. Note that

$$d(x_0,y)<\frac{2^{k-1}}{a_1}\le\frac{d(x_0,x)}{2a_1}$$

when $x\in E_k$, and $y\in G_{k,1}$. From the latter inequality we have

$$\mu B(x,d(x_0,x))\le b_1\mu B(x,d(x,y)).$$

Indeed,

$$d(x_0,x)\le a_1(d(x_0,y)+d(y,x))\le a_1\Big(\frac{d(x_0,x)}{2a_1}+a_0d(x,y)\Big).$$

Hence

$$\frac{1}{2a_1a_0}\,d(x_0,x)\le d(x,y).$$

Correspondingly,

$$\mu B(x,d(x_0,x))\le b_2\mu B\Big(x,\frac{d(x_0,x)}{2a_1a_0}\Big)\le b_2\mu B(x,d(x,y)).$$

It is easy to see that

$$\mu B(x_0, d(x_0,x)) \le b_3 \mu B(x, d(x_0,x))$$

and, finally, we obtain

$$\mu B(x_0, d(x_0,x)) \le b_4 \mu B(x, d(x,y)).$$

By considering the latter inequality we have

$$\begin{aligned}\left|K(f\chi_{G_{k1}})(x)\right| &\le b_5 \int\limits_X \frac{|f(y)|\chi_{G_{k1}}(y)}{\mu B(x,d(x,y))}\,d\mu \\ &\le \frac{b_6}{\mu B(x_0,d(x_0,x))} \int\limits_{B(x_0,d(x_0,x))} |f(y)|\,d\mu,\end{aligned}$$

when $x \in E_k$, and by Proposition C we obtain

$$\begin{aligned}S_1^p &\le c_2 \Big\| u_2(\cdot)\Big(\mu B(x_0,d(x_0,\cdot))\Big)^{-1} \int\limits_{B(x_0,d(x_0,x))} |f(y)|\,d\mu \Big\|^p_{L^{pq}_v(X)} \\ &\le c_3 \|u_1(\cdot)f(\cdot)\|^p_{L^{ps}_w(X)}.\end{aligned}$$

Now we shall estimate S_3^p. It is easy to check that if $x \in E_k$ and $y \in G_{k,3}$, then $d(x_0,y) \le d(x,y)$ and

$$\mu B(x_0, d(x_0,y)) \le b_7 \mu B(x, d(x,y)).$$

By virtue of Proposition D we obtain

$$S_3^p \le c_4 \Big\| u_2(\cdot) \int\limits_{\{d(x_0,y)>d(x_0,x)\}} \frac{|f(y)|}{\mu B(x_0,d(x_0,y))}\,d\mu \Big\|^p_{L^{pq}_v(X)} \le c_5 \|u_1(\cdot)f(\cdot)\|^p_{L^{pq}_w(X)}.$$

Now let us estimate S_2^p. By Lemma B (part (ii)),

$$S_2^p \le \sum_{k\in u} \Big\| u_2(\cdot)K(f\chi_{G_{k,2}})(\cdot)\chi_{E_k}(\cdot)\Big\|^p_{L^{pq}_v(X)} \equiv \sum_{k\in u} S^p_{k,2}.$$

We shall use the following notation:

$$u_{2,k} \equiv \sup_{x\in E_k} u_2(x), \quad \sigma_{2,k} \equiv \sup_{x\in E_k} \sigma_2(x), \quad u_{1,k} \equiv \inf_{x\in G_{k,2}} u_1(x), \quad \sigma_{1,k} \equiv \inf_{x\in G_{k,2}} \sigma_1(x).$$

By Theorem C and Lemma A we have

$$\begin{aligned}S_{k,2} &\le u_{2,k}\sigma_{2,k}^{1/p} \Big\| K(f\chi_{G_{k,2}})(\cdot)\Big\|_{L^{pq}_\rho(X)} \le c_6 u_{2,k}\sigma_{2,k}^{1/p} \Big\| f(\cdot)\chi_{G_{k,2}}(\cdot)\Big\|_{L^{pq}_\rho(X)} \\ &\le c_6 u_{2,k}\sigma_{2,k}^{1/p} \Big\| f(\cdot)\chi_{G_{k,2}}(\cdot)\Big\|_{L^{ps}_\rho(X)} \le c_7 u_{1,k}\sigma_{1,k}^{1/p} \Big\| f(\cdot)\chi_{G_{k,2}}(\cdot)\Big\|^p_{L^{ps}_\rho(X)} \\ &\le c_8 \Big\| u_1(\cdot)f(\cdot)\chi_{G_{k,2}}(\cdot)\Big\|_{L^{ps}_w(X)}.\end{aligned}$$

By Lemma B (part (i)) we finally obtain

$$S_2^p \le c_9 \|u_1(\cdot)f(\cdot)\|_{L^{ps}_w(X)}. \qquad \square$$

Remark 2.1. It is easy to check that Theorem 2.4 is still valid if we replace condition (1) by the condition

(1′)

$$\sup_{E_x} \sigma_2(y) \sup_{E_x} u_2^p(y) \le \bar{b}\sigma_1(x)u_1^p(x),$$

where $\bar{b} > 0$ does not depend on $x \in X$ and where

$$E_x = \Big\{y: \ \frac{d(x_0,x)}{4a_1} \le d(x_0,y) < 4a_1 d(x_0,x)\Big\}.$$

Indeed, we have

$$\begin{aligned}
S_{k,2} &\le \sigma_{2,k}^{1/p} u_{2,k} \|T(f\chi_{G_{k,2}})(\cdot)\|_{L^{pq}_\rho(X)} \le \sigma_{2,k}^{1/p} u_{2,k} \|T(f\chi_{G_{k,2}})(\cdot)\|_{L^{p}_\rho(X)} \\
&\le b_1 \sigma_{2,k}^{1/p} u_{2,k} \|f(\cdot)\chi_{G_{k,2}}(\cdot)\|_{L^p_\rho(X)} \\
&= b_1 \int\limits_{G_{k,2}} \Big(\sup_{2^k \le d(x_0,y) < 2^{k+1}} \sigma_2(y)\Big)\Big(\sup_{2^k \le d(x_0,y) < 2^{k+1}} u_2^p(y)\Big) |f(x)|^p \rho(x)\, d\mu\Big)^{1/p} \\
&\le b_1 \Big(\int\limits_{G_{k,2}} \Big(\sup_{E_x} \sigma_2(y)\Big)\Big(\sup_{E_x} u_2^p(y)\Big) |f(x)|^p \rho(x)\, d\mu\Big)^{1/p} \\
&\le b_2 \Big(\int\limits_{G_{k,2}} \sigma_1(x) u_1^p(x) \rho(x) |f(x)|^p \, d\mu\Big)^{1/p} \\
&= b_2 \Big\| f(\cdot) u_1(\cdot) \chi_{G_{k,2}}(\cdot)\Big\|_{L^p_w(X)} \le b_2 \Big\| f(\cdot) u_1(\cdot) \chi_{G_{k,2}}(\cdot)\Big\|_{L^{sp}_w(X)}.
\end{aligned}$$

In [19] it is proved that conditions (2) and (3) of Theorem 2.4 are also necessary for equality (2.2) to be fulfilled when K is the Hilbert transform.

Theorem 2.5. *Let $\mu(X) = \infty$, $1 < s \le p \le q$ and suppose φ_1, φ_2, and v are positive increasing functions on $(0,\infty)$.*

If the condition

$$\begin{aligned}
\sup_{t>0} B(t) \equiv \sup_{t>0} \Big\| \varphi_2(d(x_0,\cdot))\Big(\mu B(x_0, d(x_0,x))\Big)^{-1} \chi_{\{d(x_0,y)>t\}}(\cdot)\Big\|_{L^{pq}_{v(d(x_0,\cdot))}(X)} \\
\times \Big\| \frac{1}{\varphi_1(d(x_0,\cdot))} \chi_{\{d(x_0,y)\le t\}}(\cdot)\Big\|_{L^{p's'}(X)} < \infty
\end{aligned}$$

is fulfilled, then the following weighted inequality holds:

$$\Big\| Kf(\cdot)\varphi_2(d(x_0,\cdot))\Big\|_{L^{pq}_{v(d(x_0,\cdot))}(X)} \le c \Big\| f(\cdot)\varphi_1(d(x_0,\cdot))\Big\|_{L^{ps}(X)}. \tag{2.3}$$

Proof. First, let us prove the inequality

$$\varphi_2(8a_1 t) v^{1/p}(8a_1 t) \le b_1 \varphi_1\Big(\frac{t}{a_1}\Big),$$

where the positive constant b_1 does not depend on $t>0$. Indeed, by Lemma A (part (i)) and by the reverse doubling condition for μ we have

$$
\begin{aligned}
c \geq B(t) &\geq \Big\| \varphi_2(d(x_0,\cdot))(\mu B(x_0,d(x_0,\cdot))^{-1}\chi_{\{t<d(x_0,y)<\eta_1 t\}}(\cdot)\Big\|_{L^{ps}_{v(d(x_0,\cdot))}(X)} \\
&\quad\times \Big\| \frac{1}{\varphi_1(d(x_0,\cdot))}\,\chi_{\{d(x_0,y)\leq \frac{t}{8a_1^2}\}}(\cdot)\Big\|_{L^{p's'}(X)} \\
&\geq c_1(\mu B(x_0,\eta_1 t))^{-1}\varphi_2(t)\bigg(\int\limits_{\{t<d(x_0,y)<\eta_1 t\}} v(d(x_0,y))\,d\mu\bigg)^{1/p} \\
&\quad\times \varphi_1^{-1}\Big(\frac{t}{8a_1^2}\Big)\Big(\mu B\Big(x_0,\frac{t}{8a_1^2}\Big)\Big)^{1/p} \\
&\geq c_2\varphi_2(t)v^{1/p}(t)\mu B(x_0,t)^{-1}\mu B(x_0,t)^{1/p'}\varphi_1^{-1}\Big(\frac{t}{8a_1^2}\Big)\mu B(x_0,t)^{1/p'} \\
&= c_2\frac{\varphi_2(t)v^{1/p}(t)}{\varphi_1(\frac{t}{8a_1^2})}\,.
\end{aligned}
$$

Now, we are to show that the following condition is fulfilled:

$$
\begin{aligned}
\sup_{t>0} B_1(t) \equiv \sup_{t>0} &\Big\|\varphi_2(d(x_0,\cdot))\,\chi_{\{d(x_0,y)\leq t\}}(\cdot)\Big\|_{L^{pq}_v(X)} \\
&\times \Big\|\frac{\mu B(x_0,d(x_0,\cdot))^{-1}}{\varphi_1(d(x_0,\cdot))}\,\chi_{\{d(x_0,y)>t\}}(\cdot)\Big\|_{L^{p's'}(X)} < \infty.
\end{aligned}
$$

Indeed, by the monotonic property of the functions φ_1, φ_2, and v and by Lemma A (part (i)), we obtain:

$$
\begin{aligned}
B_1(t) \leq c_3\varphi_2(t)(\mu B(x_0,t))^{1/p}v^{1/p}(t)\varphi_1^{-1}\Big(\frac{t}{8a_1^2}\Big) \\
\times \Big\|(\mu B(x_0,d(x_0,\cdot))^{-1}\chi_{\{d(x_0,y)>t\}}(\cdot)\Big\|_{L^{p's'}(X)}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\Big\|(\mu B(x_0,d(x_0,\cdot))^{-1}\chi_{\{d(x_0,y)>t\}}(\cdot)\Big\|_{L^{p's'}(X)} \\
&= \bigg(s'\int\limits_0^\infty \lambda^{s'-1}\Big(\mu\Big\{x:\ \big(\mu B(x_0,d(x_0,x))^{-1}>\lambda\Big\}\cap\{d(x_0,x)\geq t\}\Big)\Big)^{s'/p'}\,d\lambda\bigg)^{1/s'} \\
&= \bigg(s'\int\limits_0^{(\mu B(x_0,t))^{-1}} \lambda^{s'-1}\Big(\mu\Big\{x:\ \mu B(x_0,d(x_0,x))<\lambda^{-1}\Big\}\Big)^{s'/p'}\,d\lambda\bigg)^{1/s'} \\
&\leq c_4\bigg(\int\limits_0^{(\mu B(x_0,t))^{-1}} \lambda^{s'-1}\lambda^{-s'/p'}\,d\lambda\bigg)^{1/s'} = c_5(\mu B(x_0,t))^{-1/p}.
\end{aligned}
$$

Here we have used the inequality

$$\mu\Big\{x: \ \ \mu B(x_0, d(x_0,x)) < \lambda^{-1}\Big\} \le b\lambda^{-1},$$

where the positive constant b is from the doubling condition for μ. Thus we obtain

$$B_1(t) \le c_6 \varphi_2(t) v^{1/p}(t) \varphi_1^{-1}\bigg(\frac{t}{8a_1^2}\bigg) \le c_7$$

for arbitrary $t > 0$.

By Theorem 2.4 we conclude that inequality (2.3) holds. □

Remark 2.2. All results of this section remain valid if we omit the continuity of σ, but require that $\mu B(x_0, r)$ be continuous with respect to r, where x_0 is the same fixed point as above. This follows from the fact that in this case the sequence $\sigma_n(d(x_0,x))$, where σ_n are absolutely continuous functions (see the proof of Theorem 2.1.), converges to $\sigma(d(x_0,x))$ a.e. on X. On the other hand, the continuity of $\mu B(x_0,r)$ with respect to r is equivalent to the condition $\mu\{x: \ d(x_0,x) = r\} = 0$ for all $r > 0$.

3. Singular Integrals on Spaces of Nonhomogeneous Type

In this section we present weighted inequalities for Calderón–Zygmund singular integrals defined on spaces of nonhomogeneous type.

Let (X, d, μ) be a spaces of nonhomogeneous type with metric d and measure μ (i.e., $a_0 = 1$, $a_1 = 1$, we have equality instead of the inequality in (iii) and (vii) need not be valid in Definition 1.1) satisfying the condition

$$\mu\overline{B}(x,r) \le r^{\alpha}, \quad x \in X, \ \ r > 0,$$

for some $\alpha > 0$, where $\overline{B}(x,r) \equiv \{y: \ d(x,y) \le r\}$.

Let the kernel k satisfy the following conditions:

(1)

$$|k(x,y)| \le c_1 d(x,y)^{-\alpha},$$

for all x, $y \in X$, $x \ne y$;

(2) there exist $c_2 > 0$ and $\varepsilon \in (0,1]$ such that

$$\max\Big\{|k(x,y) - k(x_1,y)|, |k(y,x) - k(y,x_1)|\Big\} \le c_2 \,\frac{d(x,x_1)^{\varepsilon}}{d(x,y)^{\alpha+\varepsilon}}$$

whenever $d(x,x_1) \le 2^{-1} d(x,y)$, $x \ne y$. Assume also that the integral operator

$$Tf(x) = \lim_{\varepsilon \to 0} \int\limits_{X \setminus B(x_0,\varepsilon)} f(y) k(x,y)\, d\mu$$

is bounded in $L^2(X)$.

The following theorem is valid.

Theorem C ([18]). *The operator T is bounded in $L^p(X)$ for $1 < p < \infty$ and is of weak type $(1,1)$.*

Recall that $a \equiv \sup\{d(x_0, x) : \ x \in X\}$, where x_0 is a given point of X and $B(x_0, R) \backslash B(x_0, r) \neq \varnothing$ whenever $0 < r < R < a$.

We remark that the condition $\mu\{x\} = 0$ is automatically satisfied for all $x \in X$.

Lemma 3.1. *Let $1 < p < \infty$, and let w be a weight function on X. Assume that the following two conditions are satisfied:*

(i) *there exists a positive increasing function v on $(0, 4a)$ such that for almost all $x \in X$ the inequality*

$$v(2d(x_0, x)) \leq b_1 w(x)$$

holds, where the positive constant b_1 does not depend on x;

(ii)

$$I(t) \equiv \int\limits_{B(x_0,t)} w^{1-p'}(x)\, d\mu < \infty$$

for all $t > 0$. Then $T\varphi(x)$ exists μ-a.e. for any $\varphi \in L^p_w(X)$.

Proof. Let $0 < \alpha < a$ and let us denote

$$S_\beta = \Big\{x \in X : \ \ d(x_0, x) \geq \beta/2\Big\}.$$

Suppose $\varphi \in L^p_w(X)$ and represent φ as follows:

$$\varphi(x) = \varphi_1(x) + \varphi_2(x),$$

where $\varphi_1 = \varphi \chi_{S_\beta}$ and $\varphi_2 = \varphi - \varphi_1$. Due to condition (i) we see that

$$\int\limits_X |\varphi_1(x)|^p\, d\mu = \frac{v(\frac{\beta}{2})}{v(\frac{\beta}{2})} \int\limits_{S_\beta} |\varphi(x)|^p\, d\mu$$

$$\leq \frac{1}{v(\frac{\beta}{2})} \int\limits_{S_\beta} |\varphi(x)|^p v(2d(x_0, x))\, d\mu \leq \frac{c_1}{v(\frac{\beta}{2})} \int\limits_{S_\beta} |\varphi(x)|^p w(x)\, d\mu < \infty$$

for arbitrary β, $0 < \beta < a$. Consequently $T\varphi_1 \in L^p(X)$ and, according to Theorem C, $T\varphi_1(x)$ exists μ-a.e. on X.

Now let x be such that $d(x_0, x) > \beta$. If $y \in X$ and $d(x_0, y) < \frac{\beta}{2}$, then

$$d(x_0, x) \leq d(x_0, y) + d(x, y) \leq \frac{d(x_0, x)}{2} + d(x, y).$$

Hence $\beta/2 < \frac{d(x_0,x)}{2} \le d(x,y)$ and we obtain

$$|T\varphi_2(x)| = \Big|\int\limits_X \varphi_2(y)k(x,y)\,d\mu\Big| \le c_2 \int\limits_{B(x_0,\frac{\beta}{2})} \frac{|\varphi(y)|}{d(x,y)^\alpha}\,d\mu$$

$$\le c_3 \int\limits_{B(x_0,\frac{\beta}{2})} \frac{|\varphi(y)|}{d(x,y)^\alpha}\,d\mu \le c_4\beta^{-\alpha} \int\limits_{B(x_0,\frac{\beta}{2})} |\varphi(y)|\,d\mu$$

$$\le c_4\beta^{-\alpha}\Big(\int\limits_{B(x_0,\frac{\beta}{2})} |\varphi(y)|^p w(y)\,d\mu\Big)^{1/p}\Big(\int\limits_{B(x_0,\frac{\beta}{2})} w^{1-p'}(y)\,d\mu\Big)^{1/p'} < \infty.$$

Thus $T\varphi(x)$ is absolutely convergent for arbitrary x such that $d(x_0,x) > \beta$. As we can take β arbitrarily small and $\mu\{x_0\} = 0$, we conclude that $T\varphi(x)$ exists μ-a.e. on X. □

Theorem 3.1. *Let $1 < p < \infty$. Assume that v is a positive increasing continuous function on $(0,4a)$. Suppose that w is a weight on X. Let the following two conditions hold:*

(i) *there exists a constant $b_1 > 0$ such that the inequality*

$$v(2d(x_0,x)) \le b_1 w(x)$$

is fulfilled for μ-almost all $x \in X$;

(ii)

$$\sup_{0<t<a}\Big(\int\limits_{X\setminus B(x_0,t)} \frac{v(d(x_0,x))}{d(x_0,x)^{\alpha p}}\,d\mu\Big)^{1/p}\Big(\int\limits_{B(x_0,t)} w^{1-p'}(x)\,d\mu\Big)^{1/p'} < \infty.$$

Then T is bounded from $L^p_w(X)$ to $L^p_{v(d(x_0,\cdot))}(X)$.

Proof. Without loss of generality we can suppose that v has the form

$$v(t) = v(0+) + \int\limits_0^t \phi(\tau)\,d\tau, \quad \phi \ge 0.$$

We have

$$\int\limits_X |Tf(x)|^p v(d(x_0,x))\,d\mu = v(0+)\int\limits_X |Tf(x)|^p\,d\mu$$

$$+\int\limits_X |Tf(x)|^p\Big(\int\limits_0^{d(x_0,x)} \phi(t)\,dt\Big)\,d\mu \equiv I_1 + I_2.$$

If $v(0+) = 0$, then $I_1 = 0$. If $v(0+) \ne 0$, by Theorem C we obtain

$$I_1 \le c_1 v(0+)\int\limits_X |f(x)|^p\,d\mu \le c_1\int\limits_X |f(x)|^p v(d(x_0,x))\,d\mu \le c_2\int\limits_X |f(x)|^p w(x)\,dx.$$

Changing the order of integration in I_2, we have

$$I_2 = \int_0^a \phi(t)\Big(\int_{\{x:\, d(x_0,x)>t\}} |Tf(x)|^p\, d\mu\Big)\, dt$$

$$\leq c_3 \int_0^a \phi(t)\Bigg(\int_{\{x:\, d(x_0,x)>t\}} \Bigg|\int_{\{y:\, d(x_0,y)>\frac{t}{2}\}} f(y)k(x,y)\, d\mu\Bigg|^p d\mu\Bigg)\, dt$$

$$+c_3 \int_0^a \phi(t)\Bigg(\int_{\{x:\, d(x_0,x)>t\}} \Bigg|\int_{\{y:\, d(x_0,y)\leq\frac{t}{2}\}} f(y)k(x,y)\, d\mu\Bigg|^p d\mu\Bigg)\, dt \equiv I_{21} + I_{22}.$$

Using again the boundedness of T in $L^p(X)$ we obtain

$$I_{21} \leq c_4 \int_0^a \phi(t)\Big(\int_{\{y:\, d(x_0,y)>\frac{t}{2}\}} |f(y)|^p\, d\mu\Big)\, dt = c_4 \int_X |f(y)|^p \Big(\int_0^{2d(x_0,y)} \phi(t)\, dt\Big)\, d\mu$$

$$\leq c_4 \int_X |f(y)|^p v(2a_1 d(x_0,y))\, d\mu \leq c_5 \int_X |f(y)|^p w(y)\, d\mu.$$

Now let us estimate I_{22}. When $d(x_0,x) > t$ and $d(x_0,y) \leq \frac{t}{2}$ we have

$$d(x_0,x) \leq d(x_0,y) + d(y,x) = d(x_0,y) + d(x,y)$$

$$\leq \frac{t}{2} + d(x,y) \leq \frac{d(x_0,x)}{2} + d(x,y).$$

Hence

$$\frac{d(x_0,x)}{2} \leq d(x,y).$$

Consequently,

$$I_{22} \leq c_6 \int_0^a \phi(t)\Bigg(\int_{\{x:\, d(x_0,y)>t\}} \Big(\int_{\{y:\, d(x_0,y)\leq\frac{t}{2}\}} \frac{|f(y)|}{d(x,y)^\alpha}\, d\mu(y)\Big)^p d\mu(x)\Bigg)\, dt$$

$$\leq c_7 \int_0^a \phi(t)\Big(\int_{\{x:\, d(x_0,x)>t\}} \frac{1}{d(x_0,x)^{\alpha p}}\, d\mu\Big)\Big(\int_{B(x_0,t)} |f(y)|\, d\mu\Big)^p dt.$$

It is easy to see that for any s, $0 < s < a$, we have

$$\int_s^a \phi(t)\Big(\int_{\{x:\, d(x_0,x)>t\}} \frac{1}{d(x_0,x)^{\alpha p}}\, d\mu\Big)\, dt$$

$$\leq \int_{\{x:\, d(x_0,x)\geq s\}} \frac{1}{d(x_0,x)^{\alpha p}}\Big(\int_s^{d(x_0,x)} \phi(t)\, dt\Big)\, d\mu \leq \int_{\{x:\, d(x_0,x)\geq s\}} \frac{v(d(x_0,x))}{d(x_0,x)^{\alpha p}}\, d\mu.$$

Finally, using Proposition A, we obtain

$$I_{22} \le c_8 \int\limits_X |f(x)|^p w(x)\, d\mu. \qquad \square$$

Lemma 3.2. *Let $a = \infty$, $1 < p < \infty$, and let w be a weight function on X. Suppose the following conditions are fulfilled:*

(i) *there exists a positive decreasing function v on $(0, \infty)$ such that*

$$v\Big(\frac{d(x_0, x)}{2}\Big) \le cw(x) \quad a.e.;$$

(ii) *for all $t > 0$*

$$\int\limits_{X \setminus B(x_0,t)} w^{1-p'}(x)(d(x_0, x))^{-\alpha p'}\, d\mu < \infty.$$

Then $T\varphi(x)$ exists μ-a.e. for arbitrary $\varphi \in L^p_w(X)$.

Proof. Fix arbitrarily $\beta > 0$ and let

$$S_\beta = \Big\{x : \quad d(x_0, x) \ge \beta\Big\}.$$

Represent φ as follows:

$$\varphi(x) = \varphi_1(x) + \varphi_2(x),$$

where $\varphi_1(x) = \varphi(x)\chi_{S_\beta}(x)$ and $\varphi_2(x) = \varphi(x) - \varphi_1(x)$.

For φ_2 we have

$$\int\limits_X |\varphi_2(x)|^p\, d\mu = \frac{v(\beta)}{v(\beta)} \int\limits_{B(x_0,\beta)} |\varphi(x)|^p\, d\mu$$

$$\le \frac{1}{v(\beta)} \int\limits_{B(x_0,\beta)} |\varphi(x)|^p v(d(x_0, x))\, d\mu \le \frac{c_1}{v(\beta)} \int\limits_{B(x,\beta)} |\varphi(x)|^p w(x)\, d\mu < \infty.$$

Consequently, $\varphi_2 \in L^p(X)$ and by Theorem C we have $T\varphi_2 \in L^p(X)$. Hence $T\varphi_2(x)$ exists a.e. on X.

Now let $x \in X$ and let $d(x_0, x) < \beta/2$. If $d(x_0, y) \ge \beta$, then

$$\frac{d(x_0, y)}{2} \ge d(x, y).$$

Using this inequality, we obtain

$$|T\varphi_1(x)| \le c_2 \int\limits_{S_\beta} \frac{|\varphi(y)|}{d(x, y)^\alpha}\, d\mu \le c_3 \int\limits_{S_\beta} \frac{|\varphi(y)|}{d(x_0, y)^\alpha}\, d\mu$$

$$\le c_3 \Big(\int\limits_{S_\beta} |\varphi(y)|^p w(y)\, d\mu\Big)^{1/p} \Big(\int\limits_{S_\beta} w^{1-p'}(y) d(x_0, y)^{-\alpha p'}\, d\mu\Big)^{1/p'} < \infty.$$

As we may take β arbitrarily large, we conclude that $T\varphi(x)$ exists a.e. $\square$

Theorem 3.2. *Let $a=\infty$, $1<p<\infty$ and let v be a positive continuous decreasing function $(0,\infty)$. Suppose that w is a weight function on X and the following conditions are satisfied:*

(i) *for almost all x*

$$v(d(x_0,x)/2)\le cw(x);$$

(ii)

$$\sup_{t>0}\Big(\int\limits_{B(x_0,t)} v(d(x_0,x))\,d\mu\Big)^{1/p}\Big(\int\limits_{X\setminus B(x_0,t)} w^{1-p'}(x)d(x_0,x)^{-\alpha p'}\,d\mu\Big)^{1/p'}<\infty.$$

Then the operator T is bounded from $L^p_w(X)$ to $L^p_{v(d(x_0,\cdot))}(X)$.

Proof. Without loss of generality we can represent v as

$$v(t)=v(+\infty)+\int\limits_t^\infty \phi(\tau)\,d\tau,\quad \phi\ge 0.$$

Further,

$$\int\limits_X |Tf(x)|^p v(d(x_0,x))\,dx$$

$$=v(+\infty)\int\limits_X |Tf(x)|^p\,d\mu+\int\limits_X |Tf(x)|^p\Big(\int\limits_{d(x_0,x)}^\infty \phi(t)dt\Big)\,d\mu\equiv I_1+I_2.$$

If $v(+\infty)=0$, then $I_1=0$. But if $v(+\infty)\ne 0$, then by virtue of the boundedness of T in $L^p(X)$ we have

$$I_1\le c_1v(+\infty)\int\limits_X |f(x)|^p\,d\mu\le c_1\int\limits_X |f(x)|^p v(d(x_0,x))\,d\mu\le c_2\int\limits_X |f(x)|^p w(x)\,d\mu.$$

Now we pass to I_2 :

$$I_2=\int\limits_0^\infty \phi(t)\Big(\int\limits_{B(x_0,t)} |Tf(x)|^p\,d\mu\Big)\,dt\le c_3\int\limits_0^\infty \phi(t)\Big(\int\limits_{B(x_0,t)} |Tf_t^{(1)}(x)|^p\,d\mu\Big)\,dt$$

$$+\,c_3\int\limits_0^\infty \phi(t)\Big(\int\limits_{B(x_0,t)} |Tf_t^{(2)}(x)|^p\,d\mu\Big)\,dt=I_{21}+I_{22},$$

where $f_t^{(1)}=f\chi_{B(x_0,2t)}$ and $f_t^{(2)}=f-f_t^{(1)}$. Again using Theorem C we have

$$I_{21}\le c_4\int\limits_0^\infty \phi(t)\Big(\int\limits_{B(x_0,2t)} |f(y)|^p\,d\mu\Big)\,dt$$

$$=c_4\int\limits_X |f(y)|^p\Big(\int\limits_{\frac{d(x_0,x)}{2}}^\infty \phi(t)\,dt\Big)\,d\mu\le c_5\int\limits_X |f(y)|^p w(y)\,d\mu.$$

It remains to estimate I_{22}. If $x \in B(x_0,t)$ and $y \in X\backslash B(x_0,2t)$, then

$$\frac{d(x_0,y)}{2} \le d(x,y).$$

Consequently,

$$I_{22} \le c_5 \int_0^\infty \phi(t)\Big(\int_{B(x_0,t)} d\mu\Big)\Big(\int_{X\backslash B(x_0,2a_1 t)} |f(y)| d(x_0,y)^{-\alpha}\, d\mu\Big)^p dt.$$

Moreover,

$$\int_0^s \phi(t)\Big(\int_{B(x_0,t)} d\mu\Big)\, dt = \int_{B(x_0,s)} \Big(\int_{d(x_0,x)}^s \phi(t)\, dt\Big)\, d\mu \le \int_{B(x_0,s)} v(d(x_0,x))\, d\mu$$

and due to Proposition B we finally obtain the boundedness of T. □

Now we are going to establish weighted estimates for the operator T in Lorentz spaces defined on spaces of nonhomogeneous type. Theorem C and the interpolation theorem imply

Proposition E. *Let $1 < p, q < \infty$. Then T is bounded in $L^{pq}(X)$.*

The following lemmas are obtained in the same way as in the homogeneous case. Instead of Theorem A we need to use Theorem C.

Lemma 3.3. *Let $1 < s \le p < \infty$. Let the weight functions w and w_1 satisfy the conditions:*

(1) *there exists an increasing function v on $(0, 4a)$ such that the inequality*

$$v(d(x_0,x)) \le b w(x) w_1(x)$$

holds for almost all $x \in X$;

(2) *for every t, $0 < t < a$, the norm*

$$\Big\| \frac{1}{w(\cdot)w_1(\cdot)}\, \chi_{\{d(x_0,y)\le t\}}(\cdot) \Big\|_{L_w^{p's'}(X)}$$

is finite.

Then $Tg(x)$ exists a.e. on X for any g satisfying the condition

$$\|g(\cdot)w_1(\cdot)\|_{L_w^{ps}(X)} < \infty.$$

Lemma 3.4. *Let $1 < s \le p < \infty$. Suppose also that u and u_1 are positive increasing functions on $(0, 4a_1 a)$ and*

$$\Big\| \frac{1}{u(d(x_0,\cdot))u_1(d(x_0,\cdot))}\, \chi_{\overline{B}(x_0,t)}(\cdot) \Big\|_{L^{p's'}_{u(d(x_0,\cdot))}(X)} < \infty$$

for all t satisfying the condition $0 < t < a$. Then for arbitrary φ with

$$\big\|\varphi(\cdot)u_1(d(x_0,\cdot))\big\|_{L^{ps}_{u(d(x_0,\cdot))}(X)} < \infty,$$

$T\varphi(x)$ exists a.e. on X.

The following lemmas are also true.

Lemma 3.5. *Let $a=\infty$, $1<s\le p<\infty$. Suppose that for weights w and w_1 the following conditions are satisfied:*

(1) *there exists a decreasing positive function v on $(0,\infty)$ such that*

$$v(d(x_0,\cdot))\le bw(x)w_1^p(x)$$

for almost all $x\in X$;

(2) *for every $t>0$,*

$$\Big\|\frac{d(x_0,\cdot)^{-\alpha}}{w(\cdot)w_1(\cdot)}\,\chi_{X\setminus\overline{B}(x_0,t)}(\cdot)\Big\|_{L_w^{p's'}(X)}<\infty.$$

Then $Tg(x)$ exists a.e. on X for arbitrary g satisfying $\|g(\cdot)w_1(\cdot)\|_{L_w^{ps}(X)}<\infty$.

From the previous lemmas easily follows

Lemma 3.6. *Let $a=\infty$, $1<s\le p<\infty$. Suppose also that for the decreasing functions u and u_1 on $(0,\infty)$ the following condition is satisfied*

$$\Big\|\frac{d(x_0,\cdot)^{-\alpha}}{u(d(x_0,\cdot))u_1(d(x_0,\cdot))}\,\chi_{X\setminus\overline{B}(x_0,t)}(\cdot)\Big\|_{L^{p's'}_{u(d(x_0,\cdot))}(X)}<\infty,$$

for all $t>0$. Then $Tg(x)$ exists a.e. on X for g satisfying the condition

$$g(\cdot)u_1(d(x_0,\cdot))\in L^{ps}_{u(d(x_0,\cdot))}(X).$$

Using Propositions E and C, we obtain the following result in the same way as Theorem 2.1.

Theorem 3.3. *Let $1<s\le p\le q<\infty$, and let w be a weight function on X. Assume that v is a positive increasing continuous function on $(0,4a)$. Suppose also that the following two conditions are satisfied:*

(1) *there exists a positive constant c such that the inequality*

$$v(2a_1d(x_0,x))\le cw(x)$$

holds for almost every $x\in X$;

(2)

$$\sup_{0<t<a}\big\|(d(x_0,\cdot))^{-\alpha}\chi_{X\setminus\overline{B}(x_0,t)}\big\|_{L_v^{pq}(X)}\Big\|\frac{1}{w(\cdot)}\,\chi_{\overline{B}(x_0,t)}(\cdot)\Big\|_{L_w^{p's'}(X)}<\infty.$$

Then the operator T is bounded from $L_w^{ps}(X)$ to $L^{pq}_{v(d(x_0,\cdot)}(X)$.

Theorem 3.4. *Let $a=\infty$, and let $1<s\le p\le q<\infty$; suppose that v is a positive decreasing continuous function on $(0,\infty)$. Assume also that:*

(i) *there exists a positive constant b such that the inequality*

$$v\Big(\frac{d(x_0,x)}{2a_1}\Big)\le bw(x)$$

is true for a.e. $x\in X$;

(ii)

$$\sup_{t>0} \|\chi_{\overline{B}(x_0,t)}(\cdot)\|_{L^{pq}_v(X)} \left\| \frac{(d(x_0,\cdot))^{-\alpha}}{w(\cdot)} \chi_{X\setminus \overline{B}(x_0,t)}(\cdot) \right\|_{L^{p's'}_w(X)} < \infty.$$

Then T acts boundedly from $L^{ps}_w(X)$ into $L^{pq}_{v(d(x_0,\cdot))}(X)$.

Finally, we formulate the following result:

Theorem 3.5. *Let $a = \infty$, $1 < s \le p \le q < \infty$. Suppose that v, w, u_1 and u_2 are weights on X. Assume that the following conditions are fulfilled:*
(1) *there exists a positive constant b such that for all $t > 0$*

$$\sup_{F_t} v^{1/p}(x) \sup_{F_t} u_2(x) \le b \inf_{F_t} w^{1/p}(x) \inf_{F_t} u_1(x)$$

holds, where $F_t = \{x \in X : \frac{t}{a_1} \le d(x_0, x) < 8a_1 t\}$;
(2)

$$\sup_{t>0} \left\| u_2(\cdot)(d(x_0,\cdot))^{-\alpha} \chi_{X\setminus \overline{B}(x_0,t)}(\cdot) \right\|_{L^{pq}_v(X)} \left\| \frac{1}{u_1(\cdot)w(\cdot)} \chi_{\overline{B}(x_0,t)}(\cdot) \right\|_{L^{p's'}_w(X)} < \infty;$$

(3)

$$\sup_{t>0} \left\| u_2(\cdot)\chi_{\overline{B}(x_0,t)}(\cdot) \right\|_{L^{pq}_v(X)} \left\| (d(x_0,\cdot))^{-\alpha}(u_1(\cdot)w(\cdot))^{-1} \chi_{X\setminus \overline{B}(x_0,t)}(\cdot) \right\|_{L^{p's'}_w(X)} < \infty.$$

Then the following inequality holds:

$$\|u_2(\cdot)Tf(\cdot)\|_{L^{pq}_v(X)} \le c\|u_1(\cdot)f(\cdot)\|_{L^{ps}_w(X)},$$

where the positive constant c does not depend on f.

Remark 3.1. The results of this section remain valid if we do not require the continuity of v, but assume that $\mu B(x_0, r)$ is continuous with respect to r.

4. Examples of weight functions

Let $(X, d\mu)$ be an SHT such that the condition $\mu(B(x, r)) \approx r$ holds (if $a < \infty$, then we assume that this condition is fulfilled for $0 < r \le 1$). It is known (see [5]) that if $\mu(X) < \infty$, then

$$\int_X |Kf(x)|^p (d(x_0, x))^{p-1}\, d\mu \le c \int_X |f(x)|^p (d(x_0, x))^{p-1} \log^p \frac{b}{d(x_0, x)}\, d\mu,$$

where $x_0 \in X$, $1 < p < \infty$, and $b = 8a_1 a e^{p'}$ and the positive constant c does not depend on f.

Example 4.1. Let $1 < p < q < \infty$, and let $a < \infty$; put $v(t) = t^{p-1}$, $w(t) = t^{p-1} \ln^{\gamma} \frac{b}{t}$ for $t \in (0, 4a_1 a)$, where $b = 8a_1 a e^{\frac{\gamma}{p-1}}$, $\gamma = \frac{p}{q} + p - 1$, and a_1 is the positive constant from Definition 1.1. Then from Theorem 2.1 it follows that

$$\|Kf(\cdot)\|_{L^{pq}_{v(d(x_0,\cdot))}(X)} \le c\|f(\cdot)\|_{L^{p}_{w(d(x_0,\cdot))}(X)}, \tag{4.1}$$

where the positive constant c does not depend on f.

Example 4.2. Let $1 < p < q < \infty$, $a = \infty$, $v(t) = t^{p-1}$ when $0 < t \le 1$ and $v(t) = t^{\alpha}$ when $t > 1$, and let $w(t) = t^{p-1} \ln^{\gamma} \frac{2e^{\frac{\gamma}{p-1}}}{t}$ when $t \le 1$, and $w(t) = t^{\beta} \ln^{\beta}(2e^{\frac{\gamma}{p-1}})$ when $t > 1$, where $\gamma = \frac{p}{q} + p - 1$, $0 < \alpha \le \beta < p-1$. Then inequality (4.1) holds.

An appropriate example for the conjugate function

$$\widetilde{f}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t)}{2 \operatorname{tg} \frac{t}{2}} \, dt$$

is presented in [11].

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Authors' Addresses:

D. E. Edmunds
Centre for Mathematical Analysis
and its Applications
University of Sussex
Brighton BN1 9QH
Sussex
United Kingdom
E-mail: D.E.Edmunds@sussex.ac.uk

V. Kokilashvili and A. Meskhi
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia
E-mail: kokil@rmi.acnet.ge
meskhi@rmi.acnet.ge