CONVERGENCE THEOREMS FOR GENERALIZED PROJECTIONS AND MAXIMAL MONOTONE OPERATORS IN BANACH SPACES

TAKANORI IBARAKI, YASUNORI KIMURA,
AND WATARU TAKAHASHI

Received 20 January 2002

We study a sequence of generalized projections in a reflexive, smooth, and strictly convex Banach space. Our result shows that Mosco convergence of their ranges implies their pointwise convergence to the generalized projection onto the limit set. Moreover, using this result, we obtain strong and weak convergence of resolvents for a sequence of maximal monotone operators.

1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H. For an arbitrary point x of H, consider the set $\{z \in C : \|x - z\| = \min_{y\in C} \|x - y\|\}$. It is known that this set is always a singleton. Let P_C be a mapping from H onto C satisfying

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|. \tag{1.1}$$

Such a mapping P_C is called the *metric projection*. The metric projection has the following important property: $x_0 = P_C x$ if and only if $\langle x - x_0, x_0 - y\rangle \geq 0$, for all $y \in C$.

If C is a nonempty closed convex subset of a Banach space E whose norm is Gâteaux differentiable, then the metric projection P_C has the following property: $x_0 = P_C x$ if and only if

$$\langle J(x - x_0), x_0 - y\rangle \geq 0 \quad \forall y \in C, \tag{1.2}$$

where J is a normalized duality mapping from E to E^*. Likewise, if Q_C is a surjective sunny nonexpansive retraction on a smooth Banach space E, then $x_0 = Q_C x$ if and only if

$$\langle x - x_0, J(x_0 - y)\rangle \geq 0 \quad \forall y \in C. \tag{1.3}$$

Abstract and Applied Analysis 2003:10 (2003) 621–629
2000 Mathematics Subject Classification: 41A65, 47H05, 46B20
URL: http://dx.doi.org/10.1155/S1085337503207065

Notice that Q_C is identical with the metric projection if E is a Hilbert space.

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E and suppose that $\{C_n\}$ converges to C_0 in a sense of Mosco [4]. In [7], Tsukada proved that $\{P_{C_n}\}$ converges weakly to P_{C_0} if E is reflexive and strictly convex. Moreover, if E has the Kadec-Klee property, the convergence is in the strong topology. On the other hand, Kimura and Takahashi [3] proved the following. Suppose that each C_n is a sunny nonexpansive retract, E is a reflexive Banach space with a uniformly Gâteaux differentiable norm, and every weakly compact convex subset of E has the fixed-point property for nonexpansive mappings. If the duality mapping J is weakly sequentially continuous, then Q_{C_n} converges strongly to Q_{C_0}.

One of the purposes of this paper is to obtain an analogous result for a generalized projection Π_C which was defined by Alber [1]. A weak convergence theorem is in Section 3 and a strong convergence theorem appears in Section 4.

In Section 5, we discuss sequences of maximal monotone operators. For a single operator A with $A^{-1}0 \neq \emptyset$, it is known that, for every $x^* \in E^*$, $(J+\lambda A)^{-1}x^*$ converges strongly to $\pi^*_{A^{-1}0}x^*$ as $\lambda \to \infty$ when E is smooth and E^* has a Fréchet differentiable norm [5]. The mapping $\pi^*_{A^{-1}0}$ is defined by $\pi^*_{A^{-1}0} = \Pi_{A^{-1}0} \circ J^{-1}$. Using convergence theorems shown in Sections 3 and 4, we obtain a result which replaces a single operator A with a sequence of operators $\{A_n\}$.

2. Preliminaries

Let E be a real Banach space with its dual E^*. We denote by J the normalized duality mapping from E to E^*. If E is smooth, reflexive, and strictly convex, J is a bijection. Let C be a nonempty closed convex subset of E. Define $V : E \times E \to \mathbb{R}$ by

$$V(x, y) = \|x\|^2 - 2\langle J(x), y\rangle + \|y\|^2. \tag{2.1}$$

Suppose that E is smooth, reflexive, and strictly convex. Then, for arbitrarily fixed $x \in E$, there exists a unique point $y_x \in C$ such that

$$V(x, y_x) = \min_{y \in C} V(x, y). \tag{2.2}$$

Following the notation of [1], we let $\Pi_C(x) = y_x$ and call Π_C a *generalized projection* onto C. Notice that if E is a Hilbert space, then Π_C is identical with the metric projection onto C.

The following is a well-known result. See, for example, [1, 5].

Proposition 2.1. *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle J(x) - J(x_0), x_0 - y\rangle \geq 0 \quad \forall y \in C. \tag{2.3}$$

Using a generalized projection Π_C, we define a mapping π_C^* from E^* to E by

$$\pi_C^* = \Pi_C \circ J^{-1}. \tag{2.4}$$

From Proposition 2.1, we obtain that, for $x^* \in E^*$, $x_0 = \pi_C^* x^*$ if and only if

$$\langle x^* - J(x_0), x_0 - y\rangle \geq 0 \quad \forall y \in C. \tag{2.5}$$

Let E be a Banach space and let $C_1, C_2, C_3, \ldots$ be a sequence of weakly closed subsets of E. We denote by $\text{s-Li}_n C_n$ the set of limit points of $\{C_n\}$, that is, $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, we denote by $\text{w-Li}_n C_n$ the set of cluster points of $\{C_n\}$; $y \in \text{w-Li}_n C_n$ if and only if there exists $\{y_{n_i}\}$ such that $\{y_{n_i}\}$ converges weakly to y and that $y_{n_i} \in C_{n_i}$ for all $i \in \mathbb{N}$. Using these definitions, we define the Mosco convergence [4] of $\{C_n\}$. If C_0 satisfies

$$\operatorname*{s-Li}_n C_n = C_0 = \operatorname*{w-Ls}_n C_n, \tag{2.6}$$

we say that $\{C_n\}$ is a Mosco convergent sequence to C_0 and write

$$C_0 = \operatorname*{M-lim}_{n\to\infty} C_n. \tag{2.7}$$

Notice that the inclusion $\text{s-Li}_n C_n \subset \text{w-Ls}_n C_n$ is always true. Therefore, to show the existence of $\text{M-lim}_{n\to\infty} C_n$, it is sufficient to prove $\text{w-Ls}_n C_n \subset \text{s-Li}_n C_n$. For more details, see [2].

3. Weak convergence of a sequence of generalized projections

In this section, we prove a pointwise weak convergence theorem for a sequence of generalized projections. The sequence of ranges of these projections is assumed to converge in the sense of Mosco.

Theorem 3.1. *Let E be a smooth, reflexive, and strictly convex Banach space and C a nonempty closed convex subset of E. Let $C_1, C_2, C_3, \ldots$ be nonempty closed convex subsets of C. If $C_0 = \text{M-lim}_{n\to\infty} C_n$ exists and nonempty, then C_0 is a closed convex subset of C and, for each $x \in C$, $\Pi_{C_n}(x)$ converges weakly to $\Pi_{C_0}(x)$.*

Proof. It is easy to prove that C_0 is closed and convex if C_n is a closed convex subset of C for each $n \in \mathbb{N}$. Fix $x \in C$. For the sake of simplicity, we write x_n instead of $\Pi_{C_n}(x)$ for $n \in \mathbb{N}$. Since $C_0 = \text{M-lim}_{n\to\infty} C_n$, we have, for each $y \in C_0$ there exists $\{y_n\} \subset E$ such that $y_n \to y$ as $n \to \infty$ and that $y_n \in C_n$ for each $n \in \mathbb{N}$. From Proposition 2.1, we have

$$\langle J(x) - J(x_n), x_n - y_n\rangle \geq 0. \tag{3.1}$$

Hence, we obtain

$$\begin{aligned} 0 &\le \langle J(x) - J(x_n), x_n - x\rangle + \langle J(x) - J(x_n), x - y_n\rangle \\ &\le -(\|x\| - \|x_n\|)^2 + (\|x\| + \|x_n\|)\|x - y_n\|, \end{aligned} \tag{3.2}$$

thus

$$(\|x\| - \|x_n\|)^2 \le (\|x\| + \|x_n\|)\|x - y_n\|. \tag{3.3}$$

Assume that $\{x_n\}$ is unbounded. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty}\|x_{n_i}\| = \infty$. From $y_n \to y$ and (3.3), we get a contradiction. Hence $\{x_n\}$ is bounded.

Since $\{x_n\}$ is bounded, there exists a subsequence, again denoted by $\{x_n\}$, such that it converges weakly to $x_0 \in C$. From the definition of C_0, we get $x_0 \in C_0$.

Now, we prove that $\Pi_{C_0}(x) = x_0$. From lower semicontinuity of the norm, we have

$$\begin{aligned} \liminf_{n\to\infty} V(x, x_n) &= \liminf_{n\to\infty}\left(\|x\|^2 - 2\langle J(x), x_n\rangle + \|x_n\|^2\right) \\ &\ge \|x\|^2 - 2\langle J(x), x_0\rangle + \|x_0\|^2 \\ &= V(x, x_0). \end{aligned} \tag{3.4}$$

On the other hand, we get

$$\liminf_{n\to\infty} V(x, x_n) \le \liminf_{n\to\infty} V(x, y_n) = V(x, y), \tag{3.5}$$

that is,

$$V(x, x_0) = \min_{y\in C_0} V(x, y). \tag{3.6}$$

Hence we get $\Pi_{C_0}(x) = x_0$.

According to our consideration above, each sequence $\{x_n\}$ has, in turn, a subsequence which converges weakly to the unique point $\Pi_{C_0}(x)$. Therefore, the sequence $\{x_n\}$ converges weakly to $\Pi_{C_0}(x)$. □

4. Strong convergence of a sequence of generalized projections

A Banach space E is said to have the *Kadec-Klee property* if a sequence $\{x_n\}$ of E satisfying that $\text{w-}\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty}\|x_n\| = \|x_0\|$ converges strongly to x_0. It is known that E^* has a Fréchet differentiable norm if and only if E is reflexive, strictly convex, and has the Kadec-Klee property; see, for example, [6].

Theorem 4.1. *Let E be a smooth Banach space such that E^* has a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E. Let $C_1, C_2, C_3, \ldots$ be nonempty closed convex subsets of C. If $C_0 = \text{M-}\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in C$, $\Pi_{C_n}(x)$ converges strongly to $\Pi_{C_0}(x)$.*

Proof. Fix $x \in C$ arbitrarily. We write $x_n = \Pi_{C_n}(x)$ and $x_0 = \Pi_{C_0}(x)$. By Theorem 3.1, we obtain $\text{w-lim}_{n\to\infty} x_n = x_0$. Since E^* has a Fréchet differentiable norm, E has the Kadec-Klee property. Therefore, it is sufficient to prove that $\|x_n\| \to \|x_0\|$ as $n \to \infty$. Since $x_0 \in C_0$, there exists a sequence $\{y_n\} \subset C$ such that $y_n \to x_0$ as $n \to \infty$ and $y_n \in C_n$ for each $n \in \mathbb{N}$. It follows that

$$\begin{aligned} V(x, x_0) &\leq \liminf_{n\to\infty} V(x, x_n) \leq \limsup_{n\to\infty} V(x, x_n) \\ &\leq \lim_{n\to\infty} V(x, y_n) \leq V(x, x_0). \end{aligned} \tag{4.1}$$

Hence we obtain $V(x, x_0) = \lim_{n\to\infty} V(x, x_n)$. Since $\langle J(x), x_n\rangle$ converges to $\langle J(x), x_0\rangle$, we get

$$\lim_{n\to\infty} ||x_n|| = ||x_0||. \tag{4.2}$$

Using the Kadec-Klee property of E, we obtain that $\{x_n\}$ converges strongly to x_0. □

On the other hand, the following theorem shows that the pointwise strong convergence of $\{\Pi_{C_n}(x)\}$ implies the Mosco convergence of $\{C_n\}$ under certain conditions.

THEOREM 4.2. *Let E be a reflexive and strictly convex Banach space with a Fréchet differentiable norm, and C a nonempty closed convex subset of E. Let $C_0, C_1, C_2, \ldots$ be nonempty closed convex subsets of C. Suppose that*

$$\lim_{n\to\infty} \Pi_{C_n}(x) = \Pi_{C_0}(x) \quad \forall x \in C. \tag{4.3}$$

Then

$$C_0 = \underset{n\to\infty}{\text{M-lim}}\, C_n. \tag{4.4}$$

Proof. For the sake of simplicity, we write Π_n instead of Π_{C_n} for $n \in \mathbb{N} \cup \{0\}$. For an arbitrary $x \in C_0$, we have

$$x = \Pi_0(x) = \lim_{n\to\infty} \Pi_n(x) \tag{4.5}$$

and $\Pi_n(x) \in C_n$ for all $n \in \mathbb{N}$. This means that $x \in \text{s-Li}_n C_n$ and hence we have $C_0 \subset \text{s-Li}_n C_n$. Next, we show that $\text{w-Ls}_n C_n \subset C_0$. For any $z \in \text{w-Ls}_n C_n$, there exists $\{z_i\}$ such that $\{z_i\}$ converges weakly to z as $i \to \infty$ and that $z_i \in C_{n_i}$ for each $i \in \mathbb{N}$. Using Proposition 2.1, we have

$$\langle J(z) - J(\Pi_i(z)), \Pi_i(z) - z_i\rangle \geq 0. \tag{4.6}$$

Since E has a Fréchet differentiable norm, the duality mapping J is strongly continuous. Thus we get

$$\langle J(z) - J(\Pi_0(z)), \Pi_0(z) - z\rangle \geq 0. \tag{4.7}$$

By the strict convexity of E, J is strictly monotone. Hence $z = \Pi_0(z) \in C_0$. This means that $\text{w-Ls}_n C_n \subset C_0$, and consequently, we obtain $C_0 = \text{M-lim}_{n\to\infty} C_n$. $\square$

5. Convergence of resolvents for a sequence of maximal monotone operators

In this section, we consider a set-valued mapping called monotone operator. A set-valued mapping T from X into Y is denoted by $T : X \rightrightarrows Y$.

Let E be a real Banach space. A set-valued mapping $A : E \rightrightarrows E^*$ is called a *monotone operator* if, for any $x, y \in E$ and $x^*, y^* \in E^*$ with $x^* \in Ax$ and $y^* \in Ay$,

$$\langle x^* - y^*, x - y\rangle \geq 0. \tag{5.1}$$

If a monotone operator A has no monotone extension, then A is said to be *maximal monotone*.

For a maximal monotone operator A and a real number λ with $0 < \lambda < \infty$, we define a set-valued mapping $J_\lambda : E^* \rightrightarrows E$ by

$$J_\lambda : E^* \ni x^* \longmapsto (J + \lambda A)^{-1} x^* \subset E. \tag{5.2}$$

It is known that J_λ is a single-valued mapping if E is reflexive, smooth, and strictly convex.

First we show the following lemma.

Lemma 5.1. *Let E be a reflexive Banach space and C a nonempty closed convex subset of E. Let $\{x_n\}$ be a sequence of E converging weakly to $x_0 \in C$. For a sequence $\{C_n\}$ of nonempty closed convex subsets of E such that* $\text{M-lim}_{n\to\infty} C_n = C$, *it follows that*

$$C = \operatorname*{M-lim}_{n\to\infty} \overline{\text{co}}(\{x_n\} \cup C_n). \tag{5.3}$$

Proof. We write $D_n = \overline{\text{co}}(\{x_n\} \cup C_n)$ for all $n \in \mathbb{N}$. Fix $y \in \text{w-Ls}_n D_n$. Then there exist $\{y_i \in D_{n_i}\}$, $\{z_i \in C_{n_i}\}$, and $\{\alpha_i\} \subset [0,1]$ such that

$$\begin{aligned} y_i = \alpha_i x_{n_i} + (1-\alpha_i) z_i; \qquad & \operatorname*{w-lim}_{i\to\infty} y_i = y; \\ \operatorname*{w-lim}_{i\to\infty} z_i = z_0 \in C; \qquad & \lim_{i\to\infty} \alpha_i = \alpha_0 \in [0,1]. \end{aligned} \tag{5.4}$$

Hence, we have $y = \alpha_0 x_0 + (1-\alpha_0)z_0 \in C$ and therefore $\text{w-Ls}_n D_n \subset C$. On the other hand, it is obvious that

$$C \subset \operatorname*{s-Li}_n C_n \subset \operatorname*{s-Li}_n D_n. \tag{5.5}$$

Thus we have $C = \text{M-lim}_{n\to\infty} D_n = \text{M-lim}_{n\to\infty} \overline{\text{co}}(\{x_n\} \cup C_n)$. □

Theorem 5.2. *Let E be a reflexive, smooth, and strictly convex Banach space and let $\{A_0, A_1, A_2, \dots\}$ be a sequence of maximal monotone operators from E into E^*. Suppose that $\text{M-lim}_{n\to\infty} A_n^{-1}0 = A_0^{-1}0 \neq \emptyset$ and that*

$$\operatorname*{w-Ls}_n A_n^{-1} y_n^* \subset A_0^{-1}0 \tag{5.6}$$

for any $\{y_n^*\} \subset E^*$, converging strongly to 0. For $x^* \in E^*$ and $\{\lambda_n\} \in]0,\infty[$ with $\lambda_n \to \infty$, define a single-valued mapping $J_{\lambda_n}(x^*) = (J+\lambda_n A_n)^{-1}x^*$. Then $J_{\lambda_n}x^*$ converges weakly to $\pi^*_{A_0^{-1}0}x^*$.

Proof. For the sake of simplicity, we write $x_n = J_{\lambda_n}x^*$ for each $n \in \mathbb{N}$. Since $J(x_n) + \lambda_n A_n x_n \ni x^*$, there exists $w_n^* \in A_n x_n$ such that

$$J(x_n) + \lambda_n w_n^* = x^* \quad \forall n \in \mathbb{N}. \tag{5.7}$$

From the assumption, there exists a bounded sequence $\{u_n\}$ such that $u_n \in A_n^{-1}0$ for each $n \in \mathbb{N}$. Since A_n is monotone, we have

$$\begin{aligned} \langle J(x_n) - J(u_n), x_n - u_n\rangle &= \langle x^* - \lambda_n w_n^* - J(u_n), x_n - u_n\rangle \\ &= \langle x^* - J(u_n), x_n - u_n\rangle - \lambda_n\langle w_n^*, x_n - u_n\rangle \\ &\leq \langle x^* - J(u_n), x_n - u_n\rangle. \end{aligned} \tag{5.8}$$

Thus we get

$$||x_n||^2 - 2||x_n||||u_n|| + ||u_n||^2 \leq ||x^* - J(u_n)||(||x_n|| + ||u_n||). \tag{5.9}$$

Suppose that $\{x_n\}$ is not bounded. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i}\| \to \infty$. It follows that

$$||x_{n_i}|| - 2||u_{n_i}|| + \frac{||u_{n_i}||^2}{||x_{n_i}||} \leq ||x^* - J(u)||\left(1 + \frac{||u_{n_i}||}{||x_{n_i}||}\right) \tag{5.10}$$

for a sufficiently large number $i \in \mathbb{N}$. As $i \to \infty$, we obtain $+\infty \leq \|x^* - J(u)\| < +\infty$. This is a contradiction. Hence we have that $\{x_n\}$ is bounded.

Fix an arbitrary subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to x_0. Since $J(x_{n_i}) + \lambda_{n_i} A_{n_i} x_{n_i} \ni x^*$, we have

$$x_{n_i} \in A_{n_i}^{-1}\left(\frac{x^* - J(x_{n_i})}{\lambda_{n_i}}\right). \tag{5.11}$$

Using (5.6), we get

$$x_0 = \operatorname*{w-lim}_{i\to\infty} x_{n_i} \in \operatorname*{M-lim}_{n\to\infty} A_n^{-1}0. \tag{5.12}$$

Let $C_i = \overline{\mathrm{co}}(\{x_{n_i}\} \cup A_{n_i}^{-1}0)$ for each $i \in \mathbb{N}$. Then Lemma 5.1 implies that $A_0^{-1}0 = \text{M-lim}_{i\to\infty} A_{n_i}^{-1}0 = \text{M-lim}_{i\to\infty} C_i$. Now we fix $i \in \mathbb{N}$. For any $v \in C_i$, there exist $\alpha \in [0,1]$ and $u \in A_{n_i}^{-1}0$ such that $v = \alpha x_{n_i} + (1-\alpha)u$. Since A_{n_i} is monotone, we obtain

$$\left\langle \frac{x^* - J(x_{n_i})}{\lambda_{n_i}} - 0, x_{n_i} - u \right\rangle \geq 0. \tag{5.13}$$

This implies that $\langle x^* - J(x_{n_i}), x_{n_i} - v\rangle \geq 0$. Hence, we have $x_{n_i} = \pi_{C_i}^* x^*$. Using Theorem 3.1, we obtain $\text{w-lim}_{i\to\infty} x_{n_i} = \pi^*_{A_0^{-1}0} x^*$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of a bounded sequence $\{x_n\}$, it follows that

$$\operatorname*{w-lim}_{n\to\infty} x_n = \pi^*_{A_0^{-1}0} x^*. \tag{5.14}$$

This completes the proof. □

Assuming that E has the Kadec-Klee property, we obtain a strong convergence theorem. The proof is almost the same as the previous one.

Theorem 5.3. *Let E be a smooth Banach space and suppose that E^* has a Fréchet differentiable norm. Let $\{A_0, A_1, A_2, \dots\}$, x^*, $\{\lambda_n\}$, $\{J_{\lambda_n}\}$ be the same as Theorem 5.2 and suppose that (5.6) holds. Then $J_{\lambda_n} x^*$ converges strongly to $\pi^*_{A_0^{-1}0} x^*$.*

We can apply Theorems 5.2 and 5.3 to a single maximal monotone operator A with $A^{-1}0 \neq \varnothing$. Namely, for an arbitrary sequence $\{y_n^*\}$ of E^* converging to 0, it holds that

$$\operatorname*{w-Ls}_{n} A^{-1} y_n^* \subset A^{-1}0. \tag{5.15}$$

Indeed, for $x \in \text{w-Ls}_n A^{-1} y_n^*$, there exists a sequence $\{x_i\}$ such that $x_i \in A^{-1} y_{n_i}^*$ for each $i \in \mathbb{N}$ and that x_i converges weakly to x. For any $v \in E$ and $v^* \in E^*$ satisfying $v^* \in Av$, we have

$$\langle y_{n_i}^* - v^*, x_{n_i} - v\rangle \geq 0. \tag{5.16}$$

As $i \to \infty$, it follows that

$$\langle 0 - v^*, x - v\rangle \geq 0 \tag{5.17}$$

and hence $x \in A^{-1}0$.

References

[1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes in Pure and Applied Mathematics, vol. 178, Dekker, New York, 1996, pp. 15–50.

[2] G. Beer, *Topologies on Closed and Closed Convex Sets*, Mathematics and Its Applications, vol. 268, Kluwer Academic Publishers Group, Dordrecht, 1993.

[3] Y. Kimura and W. Takahashi, *Strong convergence of sunny nonexpansive retractions in Banach spaces*, Panamer. Math. J. **9** (1999), no. 4, 1–6.

[4] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Advances in Math. **3** (1969), 510–585.

[5] S. Reich, *Constructive techniques for accretive and monotone operators*, Applied Nonlinear Analysis (Proc. Third Internat. Conf., Univ. Texas, Arlington, Tex, 1978), Academic Press, New York, 1979, pp. 335–345.

[6] W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, 2000.

[7] M. Tsukada, *Convergence of best approximations in a smooth Banach space*, J. Approx. Theory **40** (1984), no. 4, 301–309.

Takanori Ibaraki: Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan

Current address: Institute of Economic Research, Hitotsubashi University, Tokyo 186-8603, Japan

E-mail address: ibaraki@is.titech.ac.jp, ibaraki@ier.hit-u.ac.jp

Yasunori Kimura: Institute of Economic Research, Hitotsubashi University, Tokyo 186-8603, Japan

Current address: Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan

E-mail address: yasunori@ier.hit-u.ac.jp, yasunori@is.titech.ac.jp

Wataru Takahashi: Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan

E-mail address: wataru@is.titech.ac.jp