MINIMAX THEOREMS ON C^1 MANIFOLDS VIA EKELAND VARIATIONAL PRINCIPLE

MABEL CUESTA

Received 10 January 2003

We prove two minimax principles to find almost critical points of C^1 functionals restricted to globally defined C^1 manifolds of codimension 1. The proof of the theorems relies on Ekeland variational principle.

1. Introduction

Let X be a Banach space and $\Phi : X \to \mathbb{R}$ of class C^1. We are interested in finding critical points for the restriction of Φ to the manifold $M = \{u \in X : G(u) = 1\}$, where $G : X \to \mathbb{R}$ is a C^1 function having 1 as a regular value. A point $u \in M$ is a critical point of the restriction of Φ to M if and only if $d\Phi(u)_{|_{T_u M}} = 0$ (see the definition in Section 2).

Our purpose is to prove two general minimax principles to find *almost critical points* of Φ restricted to M. A compactness condition of (PS) type will then imply the existence of a critical point.

The applications of minimax principles in the theory of elliptic PDEs are well known and the reader is referred, for instance, to [15] for a thorough introduction to the subject. For applications of minimax principles on C^1 manifolds, we refer, for instance, to [2, 8, 11, 13, 16].

In this paper, we present two general minimax principles, Theorems 2.1 and 2.6, for functionals Φ restricted to M. The first one, Theorem 2.1, is a theorem of "mountain-pass type" and the second one, Theorem 2.6, is a theorem of "Ljusternik-Schnirelman type."

A standard approach to prove such results is to first derive a deformation lemma on the manifold M. In the case of Theorem 2.6, one would ask furthermore the deformation to be symmetric, that is, equivariant under the action of the group $\mathbb{Z}_2$. Classically the deformation homotopy is constructed with the help of integral lines of a pseudogradient vector field of Φ on M. Since the construction of the integral lines requires the vector field to be locally Lipschitz

Abstract and Applied Analysis 2003:13 (2003) 757–768
2000 Mathematics Subject Classification: 58E05, 58E30, 35J20
URL: http://dx.doi.org/10.1155/S1085337503303100

continuous, it seems necessary to assume that M is at least $C^{1,1}$. Deformation lemmas and their equivariant versions for $C^{1,1}$ manifolds are well-established results and we refer to [15].

However, in some applications the manifold M is merely of class C^1 and then one has to construct the deformation more carefully. As a matter of fact, several deformation lemmas on C^1 manifolds have already been proved by [3, 4, 7, 13] and, precisely, the deformation lemma of [7] could be used to prove Theorem 2.1. According to our knowledge, the only symmetric version of the deformation lemma on C^1 manifolds has been proved by [3, 4]. The equivariant deformation lemmas of [13, 15] are stated for manifolds of class $C^{1,1}$ and the symmetric deformation lemma of [7] is stated on Banach spaces. These deformation theorems do not seem to apply directly in the proof of Theorem 2.6 or Proposition 2.7.

The main novelty of this paper is that we present a proof that relies mainly on the variational principle of Ekeland *without any use of a deformation lemma*. The only cost of this approach is that we need to assume the space X to be *uniformly convex*. This is not however a restriction for the applications that we have in mind where $X = W_0^{1,p}(\Omega)$ or $L^p(\Omega)$ for $1 < p < \infty$ and Ω is an open set of $\mathbb{R}^n$.

This approach via Ekeland principle to prove a minimax principle similar to Theorem 2.1 has already been used by [5, 9, 14] in the case of no constraint, that is, when $M = X$. Our proof follows the general lines of [9]. Our approach also seems to be new in proving the analogue of Theorem 2.6 in the case of no constraint or for the proofs of Theorems 2.1 and 2.6 in the case of more regular manifolds.

2. Statement of the theorems

Let X be a Banach space with norm $\|\cdot\|_X$, X^* a dual space, and $\langle\cdot,\cdot\rangle$ a duality pairing between X^* and X. We will assume throughout this paper that X is uniformly convex (see [10]).

Let $G: X \to \mathbb{R}$ be given and assume that $G \in C^1(X,\mathbb{R})$ and 1 is a regular value of G. We consider the C^1 manifold $M \stackrel{\text{def}}{=} \{u \in X : G(u) = 1\}$. For $\Phi \in C^1(X,\mathbb{R})$, the norm of the derivative at $u \in M$ of the restriction $\tilde{\Phi}$ of Φ to M is defined as $\|\tilde{\Phi}'(u)\|_* \stackrel{\text{def}}{=} \|d\Phi(u)\|_{(T_uM)^*}$, where $T_uM = \{v \in X : \langle dG(u), v\rangle = 0\}$ denotes the tangent space to M at u and $\|\cdot\|_{(T_uM)^*}$ denotes the norm on the dual space $(T_uM)^*$.

In what follows K is a given compact metric space and $K_0 \subset K$ is a closed subset.

Theorem 2.1. *Let $\Phi \in C^1(X,\mathbb{R})$ and let $h_0 \in C(K_0,M)$ be fixed. Consider the family $\Gamma = \{h \in C(K,M) : h_{|K_0} = h_0\}$ and assume that $\Gamma \neq \emptyset$. Assume further that the following condition holds:*

$$\max_{z\in K_0} \Phi(h_0(z)) < \max_{z\in K} \Phi(h(z)) \tag{2.1}$$

for all $h \in \Gamma$. *Define* $c \overset{\text{def}}{=} \inf_{h\in\Gamma} \max_{z\in K} \Phi(h(z))$. *Let* $\epsilon > 0$ *and* $h \in \Gamma$ *be such that* $\max_{z\in K} \Phi(h(z)) < c+\epsilon^2$. *Then there exists* $u \in M$ *such that*

$$c \leq \Phi(u) \leq c+\epsilon^2, \qquad \operatorname{dist}(u, h(K)) \leq \epsilon, \qquad ||\tilde{\Phi}'(u)||_* \leq \epsilon. \tag{2.2}$$

The following propositions follow directly from Theorem 2.1. We recall that Φ is said to satisfy the (PS) *condition on* M at level c ($(\text{PS})_{c,M}$ for short) if any sequence $u_n \in M$, such that $\lim_{n\to\infty} \Phi(u_n) = c$ and $\lim_{n\to\infty} \|\tilde{\Phi}'(u_n)\|_* = 0$, possesses a convergent subsequence.

Proposition 2.2. *Let* Φ, Γ, *and* c *be as in Theorem 2.1 and assume that (2.1) holds. If* Φ *satisfies* $(\text{PS})_{c,M}$, *then there exists* $u \in M$ *such that* $\Phi(u) = c$ *and* $\tilde{\Phi}'(u) = 0$.

Proposition 2.3. *Let* Φ, Γ, *and* c *be as in Theorem 2.1 and assume that (2.1) holds. Assume further that there exists a path* $h \in \Gamma$ *such that* $\max_{z\in K} \Phi(h(z)) = c$. *Then there exists* $u \in h(K)$ *such that* $\Phi(u) = c$ *and* $\tilde{\Phi}'(u) = 0$.

Remark 2.4. Theorem 2.1 with the stronger condition

$$\max_{z\in K_0} \Phi(h_0(z)) < \inf_{h\in\Gamma} \max_{z\in K} \Phi(h(z)) \tag{2.3}$$

instead of condition (2.1) has been proved by [13, Lemma 3.7 and Theorem 3.2] using a deformation lemma.

Remark 2.5. The result of Proposition 2.3 was already observed by [5] in the case of no constraint and it can also be proved using a deformation argument. Notice that the $(\text{PS})_{c,M}$ condition is not required in Proposition 2.3 to get a critical point.

Next we state a second minimax principle that will give almost critical points of Φ restricted to M when we minimize along continuous odd maps defined on spheres of finite dimension. To that effect, we assume that the map G is *even*, so in particular, $-M = M$.

For any $k \in \mathbb{N}$, we denote by S^k the unit sphere of $\mathbb{R}^{k+1}$. We also denote

$$C_o(S^k, M) := \{h \in C(S^k, M) : h \text{ is odd}\}. \tag{2.4}$$

Theorem 2.6. *Let* $\Phi \in C^1(X, \mathbb{R})$ *be an even function. We define*

$$d \overset{\text{def}}{=} \inf_{h\in C_o(S^k,M)} \max_{z\in S^k} \Phi(h(z)) \tag{2.5}$$

and assume that $d \in \mathbb{R}$. *Let* $\epsilon > 0$ *and* $h \in C_o(S^k, M)$ *be such that*

$$\max_{z \in S^k} \Phi(h(z)) < d + \epsilon^2. \tag{2.6}$$

Then there exists $u \in M$ *such that*

$$d \le \Phi(u) \le d + \epsilon^2, \qquad \operatorname{dist}(u, h(S^k)) \le \epsilon, \qquad \|\tilde{\Phi}'(u)\|_* \le \epsilon. \tag{2.7}$$

As a consequence of the theorem, we have the following results.

Proposition 2.7. *Let* Φ *and* d *be as in Theorem 2.6. If* Φ *satisfies* $(\mathrm{PS})_{d,M}$, *then there exists* $u \in M$ *such that* $\Phi(u) = d$ *and* $\tilde{\Phi}'(u) = 0$.

Proposition 2.8. *Let* Φ *and* d *be as in Theorem 2.6 and assume that there exists a path* $h \in C_o(S^k, M)$ *such that* $\max_{z \in S^k} \Phi(h(z)) = d$. *Then there exists* $u \in h(S^k)$ *such that* $\Phi(u) = d$ *and* $\tilde{\Phi}'(u) = 0$.

3. Proof of Theorem 2.1

Before starting the proof of Theorem 2.1, we will give a result concerning the existence of C^1 paths in $C(K, M)$ with a prescribed derivative.

In the sequel we will consider the complete metric spaces $C(K, X)$ and $C(K, \mathbb{R})$ endowed with the supremum norms $\|\cdot\|_{X,\infty}$ and $\|\cdot\|_\infty$, respectively. The space $C(K, M)$ will be inherited with the norm of $C(K, X)$.

Lemma 3.1. *Let* $f \in C(K, M)$ *and let* $q \in C(K, X)$ *be such that* $q(z) \in T_{f(z)}M$ *for all* $z \in K$. *Then there exist* $r_0 > 0$ *and* $\gamma \in C^1((-r_0, r_0), C(K, M))$ *such that*

$$\begin{gathered} \gamma(0) = f, \\ \forall r \in [-r_0, r_0],\ \forall z \in K, \quad \gamma(r)(z) = f(z) \text{ iff } q(z) = 0 \text{ or } r = 0, \\ \gamma'(0) = q. \end{gathered} \tag{3.1}$$

Proof. Since X is uniformly convex, the duality map $J : X^* \to X$ defined by $\langle x, J(x)\rangle = \|x\|_{X^*}^2$ and $\|J(x)\| = \|x\|_{X^*}$ is well defined and uniformly continuous on bounded sets (see [10]). For each $u \in M$, we define

$$\mathcal{N}(u) \stackrel{\text{def}}{=} \frac{J(dG(u))}{\|dG(u)\|_{X^*}^2}. \tag{3.2}$$

Thus $\langle dG(u), \mathcal{N}(u)\rangle = 1$. We denote

$$n(z) = \mathcal{N}(f(z)) \tag{3.3}$$

for each $z \in K$. We decompose $f(z)$ as follows: $f(z) = v_0(z) + t_0(z)n(z)$, where

$$t_0(z) = \langle dG(f(z)), f(z)\rangle, \qquad v_0(z) = f(z) - \langle dG(f(z)), f(z)\rangle n(z). \tag{3.4}$$

Thus $v_0(\cdot) \in T_{f(\cdot)}M$ and it is clear from the definitions that $v_0 \in C(K,X)$, $n \in C(K,X)$, and $t_0 \in C(K,\mathbb{R})$.

We consider the map $F : C(K,X) \times C(K,\mathbb{R}) \to C(K,\mathbb{R})$ defined by $F(v,t) = G(v+tn)$. Using that $G \in C^1$ and the uniform continuity on compact sets of G and dG, one can prove that F is of class C^1. Furthermore,

$$\frac{\partial F}{\partial t}(v,t) = \langle dG(v+tn), n\rangle Id, \qquad \frac{\partial F}{\partial v}(v,t) = dG(v+tn), \tag{3.5}$$

and consequently $(\partial F/\partial t)(v_0,t_0) = Id$ is an invertible map. By the implicit function theorem (see, e.g., [1]), there exist two open sets $\mathcal{V}$, $\mathcal{U}$ such that $v_0 \in \mathcal{V} \subset C(K,X)$, $t_0 \in \mathcal{U} \subset C(K,\mathbb{R})$ and there exists a C^1 map $\phi : \mathcal{V} \to \mathcal{U}$ such that

$$\begin{gathered} \phi(v_0) = t_0, \qquad F(v,\phi(v)) = \mathbf{1}, \\ F(v,t) = \mathbf{1}, \qquad (v,t) \in \mathcal{V} \times \mathcal{U} \Longrightarrow t = \phi(v), \end{gathered} \tag{3.6}$$

where $\mathbf{1}$ denotes the constant function 1. We now take q satisfying the conditions of the lemma and let $r_0 > 0$ be such that $v_0 + rq \in \mathcal{V}$ for all $r \in (-r_0, r_0)$. We define the C^1 path $\gamma : (-r_0,r_0) \to C(K,X)$ as follows:

$$\gamma(r) = v_0 + rq + \phi(v_0 + rq)n. \tag{3.7}$$

Using that $\mathbf{1} = F(v_0 + rq, \phi(v_0+rq)) = G(v_0 + rq + \phi(v_0+rq)n)$, it follows that $\gamma(r)(z) \in M$ for all $z \in K$, that is, $\gamma \in C^1((-r_0,r_0), C(K,M))$. It also follows from the definition of γ that

$$\gamma(0) = v_0 + \phi(v_0)n = v_0 + t_0 n = f. \tag{3.8}$$

Moreover $\gamma(r)(z) = f(z)$ for some $z \in K$ and some $r \neq 0$ if and only if

$$rq(z) + \phi(v_0 + rq)(z)n(z) = t_0(z)n(z). \tag{3.9}$$

Applying $dG(f(z))$ to the above identity and using that $\langle dG(f), n\rangle = \mathbf{1}$, we find $\phi(v_0+rq)(z) = t_0(z)$ and then $rq(z) = 0$.

Finally we differentiate with respect to v the second identity of (3.6) at $v = v_0$. We find $d\phi(v_0) = -dG(f)$ and hence

$$\gamma'(0) = q + \langle d\phi(v_0), q\rangle n = q \tag{3.10}$$

and the proof is complete. □

Proof of Theorem 2.1. We introduce a functional $\Theta : C(K,\mathbb{R}) \to \mathbb{R}$ defined by $\Theta(x) \overset{\text{def}}{=} \max_{z\in K} x(z)$ and a functional $\Psi : \Gamma \to \mathbb{R}$ defined by $\Psi(f) \overset{\text{def}}{=} \Theta(\Phi \circ f)$. The family Γ is a complete metric space with the norm inherited from $C(K,X)$ and Ψ is continuous. (To show that Ψ is continuous, one uses the uniform continuity of Φ on $f(K)$.) Obviously Ψ is bounded from below and $\inf_{f\in\Gamma} \Psi(f) = c$. By the

Ekeland variational principle [12], there exists $f \in \Gamma$ such that

(a) $c \leq \Psi(f) \leq \Psi(h)$,

(b) $\|f - h\|_{X,\infty} \leq \epsilon$,

(c) $\Psi(f) < \Psi(g) + \epsilon\|f - g\|_{X,\infty}$ for all $g \neq f, g \in \Gamma$.

Our theorem will be proved if we show the existence of some $z \in K$ such that

$$\Phi(f(z)) = \Psi(f), \qquad ||\tilde{\Phi}'(f(z))||_* \leq \epsilon. \tag{3.11}$$

This will follow from five different claims.

For a given $u \in M$, we denote by P_u the map from X to T_uM defined as $P_u(v) \stackrel{\text{def}}{=} v - \langle dG(u), v\rangle n(u)$.

Claim 3.2. Let $\Lambda_0 \stackrel{\text{def}}{=} \{l \in C(K,X) : l_{|_{K_0}} \equiv 0\}$. Then

$$\sup_{l \in \Lambda_0} \min_{\mu \in \partial\Theta(\Phi \circ f)} \langle \mu, \langle d\Phi(f), P_f(l)\rangle\rangle \leq \epsilon||P_f(l)||_{X,\infty}, \tag{3.12}$$

where $\partial\Theta(x)$ stands for the subdifferential of Θ at x (see, e.g., [9]).

Proof of Claim 3.2. For simplicity we write $x = \Phi \circ f$. We fix $l \in \Lambda_0$. We can assume that $P_f(l) \neq 0$, otherwise there is nothing to prove. Consider the C^1 path $\gamma : (-r_0, r_0) \to C(K,M)$ given by Lemma 3.1 such that $\gamma(0) = f$ and $\gamma'(0) = P_f(l)$. Since $P_f(l) \equiv 0$ in K_0, then $\gamma(r)(z) = f(z)$ for all $z \in K_0$, and consequently $\gamma(r) \in \Gamma$ for all $r \in (-r_0, r_0)$. Moreover, since $P_f(l) \neq 0$, then $\gamma(r) \neq f$ for all $r \neq 0$. It follows from (c) with $g = \gamma(r)$, $0 < r < r_0$, that

$$\frac{\Theta(x) - \Theta(\Phi(\gamma(r)))}{r} \leq \epsilon \frac{1}{r}||f - \gamma(r)||_{X,\infty}. \tag{3.13}$$

We compute the limit as $r \to 0$ of both sides of inequality (3.13). The term on the left-hand side can be written as

$$\frac{\Theta(x) - \Theta(x + ry)}{r} + \frac{\Theta(x + ry) - \Theta(\Phi(\gamma(r)))}{r}, \tag{3.14}$$

where for the sake of simplicity we have denoted $y = \langle d\Phi(f), P_f(l)\rangle$. Using [9, Proposition 5.4], the first term of (3.14) goes to $-\max_{\mu \in \partial\Theta(x)}\langle \mu, y\rangle$ as $r \searrow 0$. The second term of (3.14) goes to 0 because Θ is Lipschitz continuous and the limit

$$\lim_{r \searrow 0} \frac{\Phi(\gamma(r)) - \Phi(f)}{r} = \langle d\Phi(f), \gamma'(0)\rangle = y \tag{3.15}$$

holds uniformly in K.

The limit as $r \searrow 0$ of the right-hand side of (3.13) gives $\|\gamma'(0)\|_{X,\infty} = \|P_f(l)\|_{X,\infty}$. Putting all together and passing to the limit in (3.13), we have

$$-\max_{\mu \in \partial\Theta(x)} \langle \mu, y\rangle \leq \epsilon||P_f(l)||_{X,\infty}. \tag{3.16}$$

The claim follows by replacing l by $-l$ and taking the supremum over all $l \in \Lambda_0$. □

Claim 3.3. Let $B = \{l \in C(K,X) : \|P_f(l)\|_{X,\infty} \leq 1\}$. Then

$$\min_{\mu \in \partial\Theta(x)} \sup_{l \in B \cap \Lambda_0} \langle \mu, \langle d\Phi(f), P_f(l) \rangle \rangle \leq \epsilon. \tag{3.17}$$

Proof of Claim 3.3. It is clear from Claim 3.2 that

$$\sup_{l \in B \cap \Lambda_0} \min_{\mu \in \partial\Theta(x)} \langle \mu, \langle d\Phi(f), P_f(l) \rangle \rangle \leq \epsilon. \tag{3.18}$$

We can interchange the sup and the min above because of Ky Fan-von Neumann minimax theorem (see [6]). Indeed, the map $\mathcal{F} : \mathcal{M}(K,\mathbb{R}) \times C(K,X) \to \mathbb{R}$ defined by $\mathcal{F}(\mu, l) = \langle \mu, \langle d\Phi(f), P_f(l) \rangle \rangle$ is bilinear and continuous. Here $\mathcal{M}(K,\mathbb{R})$ is the set of Borel measures endowed with the ω^*-topology. Moreover the set $\partial\Theta(x)$ is a compact convex and $B \cap \Lambda_0$ is convex. Claim 3.3 is proved. □

Claim 3.4. It holds that

$$\min_{\mu \in \partial\Theta(x)} \sup_{l \in B} \langle \mu, \langle d\Phi(f), P_f(l) \rangle \rangle \leq \epsilon. \tag{3.19}$$

Proof of Claim 3.4. We recall (see [9, Proposition 5.6]) that

$$\partial\Theta(x) = \{\mu \in \mathcal{M}(K,\mathbb{R}) : \mu \geq 0,\ \langle \mu, \mathbf{1} \rangle = 1,\ \operatorname{supp}\mu \subset K_1\}, \tag{3.20}$$

where $K_1 \stackrel{\text{def}}{=} \{z \in K : \Phi(f(z)) = \Theta(x)\}$. By (2.1) K_1 and K_0 are disjoint. Then we can find a continuous map $\varphi : K \to [0,1]$ such that $\varphi \equiv 1$ on K_1 and $\varphi \equiv 0$ on K_0. Given any $l \in B$ consider $l_1 = \varphi l$. Then $l_1 \in \Lambda_0$ and by linearity $\|P_f(l_1)\|_{X,\infty} = \|\varphi P_f(l)\|_{X,\infty} \leq 1$. Thus $l_1 \in B \cap \Lambda_0$. Moreover, since $\operatorname{supp}\mu \subset K_1$, we have

$$\langle \mu, \langle d\Phi(f), P_f(l) \rangle \rangle = \langle \mu, \langle d\Phi(f), P_f(l_1) \rangle \rangle \tag{3.21}$$

and the claim follows. □

Claim 3.5. It holds that

$$\min_{\mu \in \partial\Theta(x)} \left\langle \mu, \sup_{l \in B} \langle d\Phi(f), P_f(l) \rangle \right\rangle \leq \epsilon. \tag{3.22}$$

Proof of Claim 3.5. Let $\delta > 0$ and $z_0 \in K$. Then there exist $l_{z_0} \in B$ and an open neighborhood $\mathcal{U}_{z_0}$ of z_0 such that for all $z \in \mathcal{U}_{z_0}$,

$$\sup_{l \in B} \langle d\Phi(f(z_0)), P_{f(z)}(l(z)) \rangle - \delta \leq \langle d\Phi(f(z)), P_{f(z)}(l_{z_0}(z)) \rangle. \tag{3.23}$$

By compactness we can cover K with a finite subcovering $\mathcal{U}_{z_1} \cdots \mathcal{U}_{z_n}$. Let φ_i, $i = 1, \ldots, n$, be a continuous partition of unity subordinate to $\mathcal{U}_{z_i}$, that is, φ_i is continuous, $0 \leq \varphi_i \leq 1$, with support on $\mathcal{U}_{z_i}$ and $\sum_{i=1}^n \varphi_i = 1$ on K. We consider the function $l_1 = \sum_{i=1}^n \varphi_i l_{z_i}$. Writing (3.23) for $z_0 = z_i$ and adding from $i = 1$ to n, we have

$$\sup_{l\in B} \langle d\Phi(f), P_f(l)\rangle - \delta \mathbf{1} \leq \langle d\Phi(f), P_f(l_1)\rangle. \tag{3.24}$$

Composing with μ and using that $\langle \mu, \mathbf{1}\rangle = 1$ and $\mu \geq 0$, we get

$$\left\langle \mu, \sup_{l\in B} \langle d\Phi(f), P_f(l)\rangle \right\rangle - \delta \leq \langle \mu, \langle \Phi(f), P_f(l_1)\rangle\rangle. \tag{3.25}$$

Now observe that $l_1 \in B$ and consequently the right-hand side of the above inequality is less than or equal to $\sup_{l\in B}\langle \mu, \langle d\Phi(f), P_f(l)\rangle\rangle$. Letting $\delta \searrow 0$, we obtain

$$\left\langle \mu, \sup_{l\in B} \langle d\Phi(f), P_f(l)\rangle \right\rangle \leq \sup_{l\in B} \langle \mu, \langle d\Phi(f), P_f(l)\rangle\rangle. \tag{3.26}$$

The result now follows by taking the minimum over all $\mu \in \partial\Theta(X)$ and using Claim 3.4. □

Claim 3.6. We have

$$\min_{\mu\in\partial\Theta(x)} \langle \mu, ||\tilde{\Phi}'(f(\cdot))||_*\rangle \leq \epsilon. \tag{3.27}$$

Proof of Claim 3.6. The proof of this claim follows easily from Claim 3.5 and the identity $\sup_{l\in B}\langle d\Phi(f), P_f(l)\rangle = \|\tilde{\Phi}'(f)\|_*$. □

Now let $\overline{\mu} \in \partial\Theta(x)$ realize the minimum of Claim 3.6. Since $\overline{\mu}$ has mass equal to 1 and is supported in K_1, there exists $z \in K_1$ such that

$$||\tilde{\Phi}'(f(z))||_* \leq \epsilon. \tag{3.28}$$

Then (3.11) hold for $u = f(z)$ and the proof of the theorem is complete. □

4. Proof of Theorem 2.6

The proof of Theorem 2.6 goes along the same lines as the proof of Theorem 2.1. We indicate the necessary modifications.

First we give the symmetric version of Lemma 3.1. We will denote by $C_o(S^k, \mathbb{R})$ the subspace of odd functions of $C(S^k, \mathbb{R})$ and by $C_e(S^k, X)$ and $C_e(S^k, \mathbb{R})$ the subspaces of even functions of $C(S^k, X)$ and $C(S^k, \mathbb{R})$, respectively.

Lemma 4.1. *Assume that G is even. Let $f \in C_o(S^k, M)$ and let $q \in C_o(S^k, X)$ be such that $q(z) \in T_{f(z)}M$ for all $z \in S^k$. Then there exist $r_0 > 0$ and $\gamma \in C^1((-r_0, r_0), C_o(S^k, M))$ such that (3.1) is satisfied.*

Proof. It is easy to see that the duality map $J : X^* \to X$ and the map $\mathcal{N} : M \to \mathbb{R}$ are odd. Consequently, $t_0 \in C_e(S^k, \mathbb{R})$, $v_0 \in C_o(S^k, X)$, and $n \in C_o(S^k, \mathbb{R})$ as defined in (3.3) and (3.4).

We consider the map $F : C_o(S^k, X) \times C_e(S^k, \mathbb{R}) \to C_e(S^k, \mathbb{R})$ defined by $F(v, t) = G(v + tn)$. Observe that F is well defined because

$$\begin{aligned} G(v(z) + t(z)n(z)) &= G(-v(-z) - t(-z)n(-z)) \\ &= G(v(-z) + t(-z)n(-z)) \end{aligned} \tag{4.1}$$

for all $z \in K$, $v \in C_o(S^k, X)$, and $t \in C_e(S^k, \mathbb{R})$. By the implicit function theorem applied to F, there exist two open sets $\mathcal{V}$, $\mathcal{U}$ such that $v_0 \in \mathcal{V} \subset C_o(S^k, X)$, $t_0 \in \mathcal{U} \subset C_e(S^k, \mathbb{R})$ and there exists a C^1 map $\phi : \mathcal{V} \to \mathcal{U}$ such that (3.6) is satisfied. It is then clear that the map γ defined by (3.7) satisfied (3.1) and also that $\gamma \in C^1((-r_0, r_0), C_o(S^k, M))$. □

Proof of Theorem 2.6. We consider the functional $\Psi : C_o(S^k, M) \to \mathbb{R}$ defined by $\Psi(f) \stackrel{\text{def}}{=} \Theta(\Phi \circ f)$. The functional Ψ is continuous and bounded from below with $\inf_{f \in C_o(S^k, M)} \Psi(f) = c$. By the Ekeland variational principle, there exists $f \in C_o(S^k, M)$ satisfying (a), (b), and (c). We proceed now to prove four claims to show that there exists some $z \in S^k$ satisfying (3.11).

Observe that if $f \in C_o(S^k, M)$ and $l \in C_o(S^k, X)$, then $P_f(l) \in C_o(S^k, X)$.

Claim 4.2. We have

$$\sup_{l \in C_o(S^k, X)} \min_{\mu \in \partial\Theta(\Phi \circ f)} \langle \mu, \langle d\Phi(f), P_f(l) \rangle \rangle \le \epsilon \| P_f(l) \|_{X, \infty}. \tag{4.2}$$

Proof of Claim 4.2. The same proof as in Theorem 2.1 applies since we can choose the path γ of Lemma 4.1 with $\gamma(0) = f$ and $\gamma'(0) = P_f(l)$. Then $\gamma \in C^1((-r_0, r_0), C_o(S^k, M))$. □

Claim 4.3. Let $B = \{ l \in C(S^k, X) : \| P_f(l) \|_{X, \infty} \le 1 \}$. Then

$$\min_{\mu \in \partial\Theta(x)} \sup_{l \in B \cap C_o(S^k, X)} \langle \mu, \langle d\Phi(f), P_f(l) \rangle \rangle \le \epsilon. \tag{4.3}$$

Proof of Claim 4.3. The proof is the same as its analogue in Theorem 2.1. Notice that in this case, $B \cap C_o(S^k, X)$ is convex as well. □

Claim 4.4. It holds that

$$\min_{\mu \in \partial\Theta(x)} \left\langle \mu, \sup_{l \in B \cap C_o(S^k, X)} \langle d\Phi(f), P_f(l) \rangle \right\rangle \le \epsilon. \tag{4.4}$$

Proof of Claim 4.4. We proceed as in Claim 3.5. The only problem is to find a function l_1 in $B \cap C_o(S^k, X)$ satisfying (3.21). For any $\delta > 0$ and any $z_0 \in S^k$, let $l_{z_0} \in B \cap C_o(S^k, X)$ and let $\mathcal{V}_{z_0}$ be an open neighborhood in S^k such that (3.23) is satisfied for all $z \in \mathcal{V}_{z_0}$. We can assume that $\mathcal{V}_{z_0} = -\mathcal{V}_{-z_0}$ and $\mathcal{V}_{z_0} \cap \mathcal{V}_{-z_0} = \varnothing$ by taking, for instance, $\mathcal{V}_{z_0} = \{z \in S^k : \|z - z_0\| < \rho\}$ with ρ small. Let $\mathcal{U}_{z_0} = \mathcal{V}_{z_0} \cup \mathcal{V}_{-z_0}$. Hence $\mathcal{U}_{z_0} = -\mathcal{U}_{z_0}$. For any $z \in \mathcal{U}_{z_0}$, we have that either $z \in \mathcal{V}_{z_0}$ from which we deduce that

$$\sup_{l \in B \cap C_o(S^k, X)} \langle d\Phi(f(z_0)), P_{f(z)}(l(z))\rangle - \delta \leq \langle d\Phi(f(z)), P_{f(z)}(l_{z_0}(z))\rangle \tag{4.5}$$

or $z \in \mathcal{V}_{-z_0}$ in which case we have

$$\sup_{l \in B \cap C_o(S^k, X)} \langle d\Phi(f(-z_0)), P_{f(z)}(l(z))\rangle - \delta \leq \langle d\Phi(f(z)), P_{f(z)}(l_{-z_0}(z))\rangle. \tag{4.6}$$

Writing $\langle d\Phi(f(-z_0)), P_{f(z)}(l(z))\rangle = \langle d\Phi(f(z_0)), P_{f(z)}(-l(z))\rangle$, we see that (4.6) is equivalent to

$$\sup_{l \in B \cap C_o(S^k, X)} \langle d\Phi(f(z_0)), P_{f(z)}(l(z))\rangle - \delta \leq \langle d\Phi(f(z)), P_{f(z)}(l_{-z_0}(z))\rangle. \tag{4.7}$$

From (4.5) and (4.7), we find that for all $z \in \mathcal{U}_{z_0}$,

$$\sup_{l \in B \cap C_o(S^k, X)} \langle d\Phi(f(z_0)), P_{f(z)}(l(z))\rangle - \delta \leq \langle d\Phi(f(z)), P_{f(z)}(\bar{l}_{z_0})\rangle, \tag{4.8}$$

where $\bar{l}_{z_0}(z) := (l_{z_0}(z) + l_{-z_0}(z))/2$ belongs to $B \cap C_o(S^k, X)$.

We now cover S^k with a finite subcovering $\mathcal{U}_{z_1} \cdots \mathcal{U}_{z_n}$. Let φ_i, $i = 1, \ldots, n$, be a continuous partition of unity subordinate to $\mathcal{U}_{z_i}$ and consider the even part of φ_i, $\varphi_i^e(z) = 1/2(\varphi_i(z) + \varphi_i(-z))$. Clearly, $0 \leq \varphi_i^e \leq 1$, $\operatorname{supp}\varphi_i^e \subset \mathcal{U}_{z_i}$, and $\sum_{i=1}^n \varphi_i^e = 1$. We finally consider the odd function $l_1 = \sum_{i=1}^n \varphi_i^e \bar{l}_{z_i}$ and observe that $l_1 \in B$. The remaining part of the proof is similar to that of Claim 3.5. □

Claim 4.5. It follows that

$$\min_{\mu \in \partial\Theta(x)} \langle \mu, \|\tilde{\Phi}'(f(\cdot))\|_*\rangle \leq \epsilon. \tag{4.9}$$

Proof of Claim 4.5. We show that

$$\sup_{l \in B \cap C_o(S^k, X)} \langle d\Phi(f), P_f(l)\rangle = \|\tilde{\Phi}'(f)\|_*. \tag{4.10}$$

The inequality $\leq$ is clear because $P_{f(l(\cdot))} \in T_{f(\cdot)}M$ and $\|P_f(l)\|_{X,\infty} \leq 1$. The inequality $\geq$ comes from the following. Fix $z \in S^k$ and let $\delta > 0$. Then there exists $v \in T_{f(z)}M$, $\|v\|_X \leq 1$, such that

$$\langle d\Phi(f(z)), v\rangle \geq \|\tilde{\Phi}'(f(z))\|_* - \delta. \tag{4.11}$$

Take $l \in C_o(S^k, X)$ such that $l(z) = v$ and $\|l\|_\infty \leq 1$ (for instance, if $z = (1, 0, \ldots, 0)$, one can take $l(x_1, x_2, \ldots, x_{k+1}) = x_1 v$). Hence

$$||\tilde{\Phi}'(f(z))||_* - \delta \leq \sup_{l \in B \cap C_o(S^k, X)} \langle d\Phi(f), P_f(l) \rangle. \tag{4.12}$$

Letting δ go to 0, we obtain the desired inequality. Claim 4.5 is proved. □

The remaining part of the proof of the theorem is identical to its corresponding part in Theorem 2.1. □

Acknowledgment

We wish to thank J. Campos, J.-P. Gossez, and E. Lami-Dozo for their useful suggestions.

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Mabel Cuesta: Université du Littoral Côte d'Opale (ULCO), 50 rue F. Buisson, BP 699, 62228 Calais Cedex, France
E-mail address: cuesta@lmpa.univ-littoral.fr