# FIXED POINTS AND PERIODIC POINTS OF SEMIFLOWS OF HOLOMORPHIC MAPS

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Let $\phi$ be a semiflow of holomorphic maps of a bounded domain $D$ in a complex Banach space. The general question arises under which conditions the existence of a periodic orbit of $\phi$ implies that $\phi$ itself is periodic. An answer is provided, in the first part of this paper, in the case in which $D$ is the open unit ball of a $J^*$-algebra and $\phi$ acts isometrically. More precise results are provided when the $J^*$-algebra is a Cartan factor of type one or a spin factor. The second part of this paper deals essentially with the discrete semiflow $\phi$ generated by the iterates of a holomorphic map. It investigates how the existence of fixed points determines the asymptotic behaviour of the semiflow. Some of these results are extended to continuous semiflows.

## 1. Introduction

Let $D$ be a bounded domain in a complex Banach space $\mathscr{E}$ and let $\phi : \mathbb{R}_+ \times D \to D$ be a continuous semiflow of holomorphic maps acting on $D$.

Under which conditions does the existence of a periodic point of $\phi$ (with a positive period) imply that the semiflow $\phi$ itself is periodic?

An answer to this question was provided in [22] in the case in which $\mathscr{E}$ is a complex Hilbert space and $D$ is the open unit ball of $\mathscr{E}$, showing that, if the orbit of the periodic point spans a dense linear subspace of $\mathscr{E}$, then $\phi$ is the restriction to $\mathbb{R}_+$ of a continuous periodic flow of holomorphic automorphisms of $D$.

In the first part of this paper, a somewhat similar result will be established in the more general case in which $\mathscr{E}$ is a $J^*$-algebra and $D$ is the open unit ball $B$ of $\mathscr{E}$. The main result in this direction can be stated more easily in the case in which the periodic point is the center 0 of $B$. It will be shown that, if the points of the orbit of 0 which are collinear to extreme points of the closure $\overline{B}$ of $B$ span a dense linear subspace of $\mathscr{E}$, then the same conclusion of [22] holds,


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that is, $\phi$ is the restriction to $\mathbb{R}_+$ of a continuous periodic flow of holomorphic automorphisms of $B$.

If the $J^*$-algebra $\mathcal{E}$ is a Cartan factor of type one—that is, it is the Banach space $\mathcal{L}(\mathcal{H},\mathcal{K})$ of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$ with values in a complex Hilbert space $\mathcal{K}$—it was shown by Franzoni in [4] that any holomorphic automorphism of $B$ is essentially associated to a linear continuous operator preserving a Kreĭn space structure defined on the Hilbert space direct sum $\mathcal{K}\oplus\mathcal{H}$; a situation that has been further explored in [19, 20] in the case in which $\mathcal{K}\oplus\mathcal{H}$ carries the structure of a Pontryagin space.

Starting from a strongly continuous group $T:\mathbb{R}\to\mathcal{L}(\mathcal{K}\oplus\mathcal{H})$, inducing a continuous flow $\phi$ of holomorphic automorphisms of $B$, it will be shown that, if $\phi$ has a periodic point $x_0$, and if the orbit of $x_0$ is "sufficiently ample," a rescaled version of $T$ is periodic. A theorem of Bart [1] yields a complete description of the spectral structure of the infinitesimal generator $X$ of $T$.

The particular case in which $\mathcal{H}\simeq\mathbb{C}$ and $B$ is the open unit ball of $\mathcal{K}$, which was initially explored in [22], will be revisited, showing that the periodic flow $\phi$ fixes some point of $B$ and that, if $\phi$ is eventually differentiable, the dimension of $\mathcal{K}$ is finite.

As was shown in [17, 19], in the case in which $\mathcal{K}\oplus\mathcal{H}$ carries the structure of Pontryagin space, a Riccati equation defined on $B$ is canonically associated to $X$. The periodicity of $\phi$ implies then the periodicity of the integrals of this Riccati equation.

A similar investigation to the one carried out in Sections 3 and 4 for a Cartan factor of type one is developed in Section 5 in the case in which $\mathcal{E}$ is a spin factor. In this case, the norm in $\mathcal{E}$ is equivalent to a Hilbert space norm. Assuming again, for the sake of simplicity, that the periodic point is the center 0 of $D$, a hypothesis leading to the periodicity of $\phi$, consists in supposing that the points of the orbit of 0 which are collinear to scalar multiples of selfadjoint unitary operators acting on $\mathcal{E}$ span a dense linear submanifold of this latter space.

The case of fixed points of the semiflow $\phi$ acting on the bounded domain $D$ is considered in the second part of this paper, where, among other things, some results which were announced in [16] for discrete semiflows generated iterating a holomorphic map $f:D\to D$ are established in the general case. (One of the basic tools in this investigation was the Earle-Hamilton theorem (see [2] or, e.g., [5, 6, 9]). This theorem, coupled with the theory of complex geodesics for the Carathéodory distance, was also used by several authors (see, e.g., [10, 11, 15, 16, 23, 24, 25, 26, 27]) to investigate the geometry of the set of fixed points of $f$. Further references to fixed points of holomorphic maps can be found in [13].) Our main purpose is to obtain some information on the asymptotic behaviour of $\phi$ in terms of "local" properties.

In this direction, extending to the continuous case a result announced in [16] for the iteration of a holomorphic map, it is shown that, if there is a sequence $\{t_\nu\}$ in $\mathbb{R}_+$ diverging to infinity and such that $\{\phi_{t_\nu}\}$ converges, for the topology of local uniform convergence, to a function mapping $D$ into a set completely interior to $D$, then there exists a unique point $x_0 \in D$ which is fixed by the semiflow $\phi$; moreover, $\phi_s(x)$ tends to $x_0$ as $s \to +\infty$, for all $x \in D$.

If some point $x_0 \in D$ is fixed by the continuous semiflow $\phi$, the map $t \mapsto d\phi_t(x_0)$, where $d\phi_t(x_0) \in \mathscr{L}(\mathscr{E})$ is the Fréchet differential of $\phi_t(x)$ at $x = x_0$, defines a strongly continuous semigroup of bounded linear operators acting on $\mathscr{E}$.

Some situations are explored in which the behaviour of this semigroup determines the asymptotic behaviour of the semiflow $\phi$.

It is shown in Sections 7 and 8 that, if the spectral radius $\rho(d\phi_t(x_0))$ of $d\phi_t(x_0)$ is $\rho(d\phi_t(x_0)) < 1$ for some $t > 0$, then, as $s \to +\infty$, $\phi_s$ converges to the constant map $x \mapsto x_0$ for the topology of local uniform convergence.

The case in which $\rho(d\phi_t(x_0)) = 1$ at some $t > 0$ is considered in Sections 9 and 10, under the additional hypothesis that $d\phi_t(x_0)$ is an idempotent of $\mathscr{L}(\mathscr{E})$. As is well known, the spectrum $\sigma(d\phi_t(x_0))$ of $d\phi_t(x_0)$ consists of two eigenvalues in 0 and in 1 at most.

If

$$\sigma\big(d\phi_t(x_0)\big) = \{0\}, \tag{1.1}$$

then $d\phi_s(x_0) = \{0\}$ for all $s \geq t$. As a consequence of Sections 7 and 8, if $s \to +\infty$, $\phi_s$ converges to the constant map $x \mapsto x_0$ for the topology of local uniform convergence.

If

$$\sigma\big(d\phi_t(x_0)\big) = \{1\}, \tag{1.2}$$

then $\phi$ is the restriction to $\mathbb{R}_+$ of a periodic flow of holomorphic automorphisms of $D$.

Finally, if

$$1 \in \sigma\big(d\phi_t(x_0)\big), \tag{1.3}$$

and if there is some $t' > 0$, with $t'/t \notin \mathbb{Q}$, such that also $d\phi_{t'}(x_0)$ is an idempotent of $\mathscr{L}(\mathscr{E})$, then the semiflow $\phi$ is constant, that is, $\phi_t = \mathrm{id}$ (the identity map) for all $t \geq 0$.

## 2. The general case of a $J^*$-algebra

Let $\mathscr{E}$ be a complex Banach space, let $D$ be a domain in $\mathscr{E}$, and let

$$\phi : \mathbb{R}_+ \times D \longrightarrow D \tag{2.1}$$

be a semiflow of holomorphic maps of $D$ into $D$, that is, a map such that

$$\phi_0 = \mathrm{id}, \tag{2.2}$$

$$\phi_{t_1+t_2} = \phi_{t_1}\phi_{t_2}, \tag{2.3}$$

$$\phi_t \in \mathrm{Hol}(D), \tag{2.4}$$

for all $t, t_1, t_2 \in \mathbb{R}_+$, where $\mathrm{Hol}(D)$ is the semigroup of all holomorphic maps $D \to D$.

A point $x \in D$ is said to be a periodic point of $\phi$ with period $\tau > 0$ if $\phi_\tau(x) = x$ and $\phi_t(x) \neq x$ for all $t \in (0,\tau)$. The semiflow $\phi$ will be said to be periodic with period $\tau$ if $\phi_\tau = \mathrm{id}$ and, whenever $0 < t < \tau$, $\phi_t$ is not the identity map.

We begin by establishing the following elementary lemma, which is a consequence of Cartan's uniqueness theorem (see, e.g., [5]) and which might have some interest in itself.

Let $D$ be a hyperbolic domain in the Banach space $\mathscr{E}$ (or, more in general, a domain in $\mathscr{E}$ on which either the Carathéodory or the Kobayashi distances define equivalent topologies to the relative topology) and let $x_0 \in D$ be a fixed point of the semiflow $\phi$, that is, $\phi_t(x_0) = x_0$ for all $t \in \mathbb{R}_+$.

Lemma 2.1. *If there is a vector $\xi \in \mathscr{E}\setminus\{0\}$, for which the map $t \mapsto d\phi_t(x_0)\xi$ of $\mathbb{R}_+$ into $\mathscr{L}(\mathscr{E})$ is periodic with period $\tau > 0$, and there is a set $K \subset (0,\tau)$ such that $\{d\phi_t(x_0)\xi : t \in K\}$ spans a dense affine subspace $\tilde{K}$ of $\mathscr{E}$, then $\phi_\tau = \mathrm{id}$.*

*Proof.* Let $x_0 = 0$. Since

$$d\phi_\tau(0)\big(d\phi_t(0)\xi\big) = d\phi_{\tau+t}(0)\xi = d\phi_t(0)\xi \quad \forall t \geq 0, \tag{2.5}$$

then $d\phi_\tau(0) = \mathrm{id}$ on $\tilde{K}$ and therefore on $\mathscr{E}$. Cartan's identity theorem yields the conclusion. □

Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces and let $\mathscr{L}(\mathscr{H},\mathscr{K})$ be the complex Banach space of all continuous linear operators $\mathscr{H} \to \mathscr{K}$, endowed with the operator norm. For $A \in \mathscr{L}(\mathscr{H},\mathscr{K})$, $A^* \in \mathscr{L}(\mathscr{K},\mathscr{H})$ will denote the adjoint of $A$. A $J^*$-algebra [7] is a closed linear subspace $\mathscr{A}$ of $\mathscr{L}(\mathscr{H},\mathscr{K})$ such that

$$A \in \mathscr{A} \Longrightarrow AA^*A \in \mathscr{A}. \tag{2.6}$$

The roles of $\mathscr{E}$ and $D$ will now be played by a $J^*$-algebra $\mathscr{A}$ and by the open unit ball $B$ of $\mathscr{A}$.

Let $S$ be the set of all extreme points of the closure $\overline{B}$ of $B$. As was noted by Harris in [7], if $\mathscr{A}$ is weakly closed in $\mathscr{L}(\mathscr{H},\mathscr{K})$, then $S \neq \varnothing$.

Lemma 2.2. *Let $S \neq \varnothing$. If $0$ is a periodic point of the semiflow $\phi : \mathbb{R}_+ \times B \to B$, with period $\tau > 0$, and if there is a set $K \subset (0,\tau)$ such that, for every $t \in K$, $\phi_t(0)$ is collinear to some point of $S$, and the set $\{\phi_t(0) : t \in K\}$ spans a dense linear subspace of $\mathscr{A}$, then the semiflow $\phi$ is periodic with period $\tau$.*

*Proof.* Let $\Delta$ be the open unit disc of $\mathbb{C}$. For $t \in K$,

$$\Delta \ni \zeta \longmapsto \frac{\zeta}{||\phi_t(0)||}\phi_t(0) \tag{2.7}$$

is, up to parametrization, the unique complex geodesic whose support contains both 0 and $\phi_t(0)$. (For the Kobayashi or Carathéodory metrics on $B$, for the basic notions concerning complex geodesics, see, e.g., [14, 15].)

Since $\phi_\tau(0) = 0$ and

$$\phi_\tau(\phi_t(0)) = \phi_t(\phi_\tau(0)) = \phi_t(0), \tag{2.8}$$

then $\phi_\tau$ is the identity on the support of the complex geodesic (2.7). Hence

$$d\phi_\tau(0)(\phi_t(0)) = \phi_t(0) \quad \forall t \in K, \tag{2.9}$$

and therefore $d\phi_\tau(0) = I_{\mathcal{A}}$. Thus $d\phi_\tau(0)$ maps the set $S$ onto itself. By Harris' Schwarz lemma [7, Theorem 10], $\phi_\tau = d\phi_\tau(0) = \text{id}$. □

Let now $x_0 \in B$ be a periodic point of $\phi$ with period $\tau > 0$.

As was shown in [7], the Moebius transformation $M_{x_0}$ is a holomorphic automorphism of $B$ which maps any $x \in B$ to the point

$$\begin{aligned} M_{x_0}(x) &= (I - x_0 x_0{}^*)^{-1/2}(x + x_0)(I + x_0{}^* x)^{-1}(I - x_0{}^* x_0)^{1/2} \\ &= x_0 + (I - x_0 x_0{}^*)^{1/2} x (I + x_0{}^* x)^{-1}(I - x_0{}^* x_0)^{1/2}. \end{aligned} \tag{2.10}$$

Furthermore,

$$M_{x_0}(0) = x_0, \qquad M_{x_0}{}^{-1} = M_{-x_0}, \tag{2.11}$$

and $M_{x_0}$ is the restriction to $B$ of a holomorphic function on an open neighbourhood of $\overline{B}$ in $\mathcal{A}$, mapping $\partial B$ onto itself.

Applying Lemma 2.2 to the semiflow $t \mapsto \psi_t = M_{-x_0}\phi_t M_{x_0}$, we obtain the following theorem.

Theorem 2.3. *If $x_0 \in B$ is a periodic point of $\phi$ with period $\tau > 0$ and if there is a set $K \subset (0, \tau)$ such that*

(i) *for any $t \in K$, $M_{-x_0}(\phi_t(x_0))$ is collinear to some point in $S$;*
(ii) *the set $\{\phi_t(x_0) : t \in K\}$ spans a dense affine subspace of $\mathcal{A}$ (as was shown by Harris in* [7, Corollary 8], *$\overline{B}$ is the closed convex hull of $S$),*

*then the semiflow $\phi$ is periodic with period $\tau$.*

*Remark 2.4.* Under the hypotheses of Theorem 2.3, setting $\psi_t = \phi_t$ when $t \geq 0$, and $\psi_t = \phi_{-t}$ when $t \leq 0$, one defines a flow $\psi : \mathbb{R} \times B \to B$ of holomorphic automorphisms of $B$, whose restriction to $\mathbb{R}_+$ is $\phi$.

The flow $\psi$ is continuous if and only if the semiflow $\phi$ is continuous, that is, the map $\phi : \mathbb{R}_+ \times B \to B$ is continuous.

In the case in which $n = \dim_{\mathbb{C}} \mathcal{A} < \infty$, a similar statement to Theorem 2.3 holds for a discrete semiflow, that is to say, for the semiflow generated by the iterates $f^m = f \circ f \circ \cdots \circ f$ $(m = 1, 2, \ldots)$ of a holomorphic map $f : B \to B$.

Theorem 2.5. *If $f$ has a periodic point $x_0 \in B$, with period $p > n$ (i.e., $f^p(x_0) = x_0$, $f^q(x_0) \neq x_0$ if $q = 1, \ldots, p-1$), if $M_{-x_0}(f^q(x_0))$ is collinear to some point in the Shilov boundary of $\overline{B}$ for $q = 1, \ldots, p-1$, and if the orbit $\{f^q(x_0) : q = 1, \ldots, p-1\}$ of $x_0$ spans $\mathcal{A}$, then $f$ is periodic with period $p$.*

For example, let $f_1 : z \mapsto e^{2\pi i/3} z$ and let $f_2$ be another holomorphic function $\Delta \to \Delta$ such that $f_2(0) = 0$ but $f_2 \not\equiv 0$. Let $f : \Delta \times \Delta \to \Delta \times \Delta$ be the holomorphic map defined by

$$f(z_1, z_2) = \big(f_1(z_1), f_2(z_2)\big), \quad (z_1, z_2 \in \Delta). \tag{2.12}$$

If $f_2$ has a periodic point in $\Delta \setminus \{0\}$, and therefore is periodic, $f$ is periodic with period $\geq 3$. If $f_2$ is not periodic, $f$ is not periodic. However, every point $(z_1, 0)$ with $z_1 \in \Delta \setminus \{0\}$ is a periodic point of $f$ with period 3.

## 3. Cartan domains of type one

Let the $J^*$-algebra $\mathcal{A}$ be a Cartan factor of type one, $\mathcal{A} = \mathcal{L}(\mathcal{H}, \mathcal{K})$. Let

$$J = \begin{pmatrix} I_{\mathcal{K}} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix}, \tag{3.1}$$

and let $\Gamma(J)$ be the group of all linear continuous operators $A$ on $\mathcal{K} \oplus \mathcal{H}$ which are invertible in $\mathcal{L}(\mathcal{K} \oplus \mathcal{H})$ and such that

$$A^* J A = J. \tag{3.2}$$

It was shown by Franzoni in [4] that the group of all holomorphic automorphisms of the unit ball $B$ of $\mathcal{A}$, which is called a Cartan domain of type one, is isomorphic to a quotient of $\Gamma(J)$, up to conjugation when $\dim_{\mathbb{C}} \mathcal{H} = \dim_{\mathbb{C}} \mathcal{K}$.

To avoid conjugation, we will consider now the case in which $\infty \geq \dim_{\mathbb{C}} \mathcal{H} \neq \dim_{\mathbb{C}} \mathcal{K} \leq \infty$.

Let $T : \mathbb{R} \to \mathcal{L}(\mathcal{K} \oplus \mathcal{H})$ be a strongly continuous group such that

$$T(t)^* J T(t) = J, \tag{3.3}$$

or equivalently

$$T(t)JT(t)^* = J, \tag{3.4}$$

for all $t \in \mathbb{R}$.

If

$$T(t) = \begin{pmatrix} T_{11}(t) & T_{12}(t) \\ T_{21}(t) & T_{22}(t) \end{pmatrix} \tag{3.5}$$

is the representation of $T(t)$ in $\mathcal{K} \oplus \mathcal{H}$, with $T_{11}(t) \in \mathcal{L}(\mathcal{K})$, $T_{12}(t) \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $T_{21}(t) \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, and $T_{22}(t) \in \mathcal{L}(\mathcal{H})$, then (3.3) and (3.4) are equivalent to

$$\begin{aligned} T_{11}(t)^* T_{11}(t) - T_{21}(t)^* T_{21}(t) &= I_{\mathcal{K}}, \\ T_{22}(t)^* T_{22}(t) - T_{12}(t)^* T_{12}(t) &= I_{\mathcal{H}}, \\ T_{12}(t)^* T_{11}(t) - T_{22}(t)^* T_{21}(t) &= 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} T_{11}(t) T_{11}(t)^* - T_{12}(t) T_{12}(t)^* &= I_{\mathcal{K}}, \\ T_{22}(t) T_{22}(t)^* - T_{21}(t) T_{21}(t)^* &= I_{\mathcal{H}}, \\ T_{21}(t) T_{11}(t)^* - T_{22}(t) T_{21}(t)^* &= 0. \end{aligned} \tag{3.7}$$

Here $T_{11}(t)^* \in \mathcal{L}(\mathcal{K})$, $T_{12}(t)^* \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $T_{21}(t)^* \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, and $T_{22}(t)^* \in \mathcal{L}(\mathcal{H})$ are the adjoint operators of $T_{11}(t)$, $T_{12}(t)$, $T_{21}(t)$, and $T_{22}(t)$.

From now on, in this section, latin letters $x$ and $y$ indicate elements of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ and greek letters $\xi$ and $\eta$ indicate vectors in $\mathcal{H}$ and $\mathcal{K}$.

It was shown in [4], that, if $x \in B$, $T_{21}(t)x + T_{22}(t) \in \mathcal{L}(\mathcal{H})$ is invertible in $\mathcal{L}(\mathcal{H})$, and the function $\widetilde{T(t)}$, defined on $B$ by

$$\widetilde{T(t)} : x \longmapsto \big(T_{11}(t)x + T_{12}(t)\big)\big(T_{21}(t)x + T_{22}(t)\big)^{-1}, \tag{3.8}$$

is, for all $t \in \mathbb{R}$, a holomorphic automorphism of $B$.

Setting

$$\phi_t = \widetilde{T(t)} \tag{3.9}$$

for $t \in \mathbb{R}$, we define a continuous flow $\phi$ of holomorphic automorphisms of $B$. If $x_0 \in B$ is a periodic point of $\phi$ with period $\tau > 0$, and if the hypotheses of Theorem 2.3 are satisfied, $\phi$ is periodic with period $\tau$.

Since $\widetilde{T(\tau)} = \mathrm{id}$, then

$$T_{11}(\tau)x + T_{12}(\tau) = xT_{21}(\tau)x + xT_{22}(\tau) \quad \forall x \in \mathcal{L}(\mathcal{H}, \mathcal{K}), \tag{3.10}$$

whence

$$T_{12}(\tau) = 0, \qquad T_{21}(\tau) = 0, \tag{3.11}$$

and therefore, by (3.6),

$$\begin{aligned} T_{11}(\tau)^* T_{11}(\tau) &= T_{11}(\tau) T_{11}(\tau)^* = I_{\mathcal{K}}, \\ T_{22}(\tau)^* T_{22}(\tau) &= T_{22}(\tau) T_{22}(\tau)^* = I_{\mathcal{H}}, \end{aligned} \tag{3.12}$$

that is, $T_{11}(\tau)$ and $T_{22}(\tau)$ are unitary operators in the Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. Furthermore, (3.10) becomes

$$T_{11}(\tau)x = xT_{22}(\tau) \quad \forall x \in \mathcal{L}(\mathcal{H}, \mathcal{K}). \tag{3.13}$$

Since $T_{22}(\tau)$ is unitary, every point $e^{i\theta\tau}$ ($\theta \in \mathbb{R}$) in the spectrum $\sigma(T_{22}(\tau))$ of $T_{22}(\tau)$ is contained either in the point spectrum or in the continuous spectrum. In both cases, there exists a sequence $\{\xi_\nu\}$ in $\mathcal{H}$ (which may be assumed to be constant if $e^{i\theta\tau}$ is an eigenvalue), with $\|\xi_\nu\| = 1$, such that

$$\lim_{\nu \to +\infty} \left(T_{22}(\tau)\xi_\nu - e^{i\theta\tau}\xi_\nu\right) = 0. \tag{3.14}$$

Since, by the Schwarz inequality,

$$\begin{aligned} \left| \left(T_{22}(\tau)\xi_\nu | \xi_\nu\right) - e^{i\theta\tau} \right| &= \left| \left(T_{22}(\tau)\xi_\nu - e^{i\theta\tau}\xi_\nu | \xi_\nu\right) \right| \\ &\leq \left\| T_{22}(\tau)\xi_\nu - e^{i\theta\tau}\xi_\nu \right\|, \end{aligned} \tag{3.15}$$

then

$$\lim_{\nu \to +\infty} \left(T_{22}(\tau)\xi_\nu | \xi_\nu\right) = e^{i\theta\tau}. \tag{3.16}$$

Hence, letting, for any $\eta \in \mathcal{K}$, $x_\nu = \eta \otimes \xi_\nu \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, then $x_\nu(\xi_\nu) = \eta$ and

$$\lim_{\nu \to +\infty} x_\nu\left(T_{22}(\tau)\xi_\nu\right) = \lim_{\nu \to +\infty} \left(T_{22}(\tau)\xi_\nu | \xi_\nu\right)\eta = e^{i\theta\tau}\eta. \tag{3.17}$$

Thus, by (3.13),

$$T_{11}(\tau)\eta = \lim_{\nu \to +\infty} T_{11}(\tau)\left(x_\nu(\xi_\nu)\right) = \lim_{\nu \to +\infty} x_\nu\left(T_{22}(\tau)\xi_\nu\right) = e^{i\theta\tau}\eta \tag{3.18}$$

for all $\eta \in \mathcal{K}$. Therefore,

$$T_{11}(\tau) = e^{i\theta\tau} I_{\mathcal{K}}, \tag{3.19}$$

and (3.13) yields

$$T_{22}(\tau) = e^{i\theta\tau} I_{\mathcal{H}}. \tag{3.20}$$

In conclusion,

$$T(\tau) = e^{i\theta\tau} I_{\mathcal{K}\oplus\mathcal{H}}. \tag{3.21}$$

Thus, the rescaled group $L : \mathbb{R} \to \mathcal{L}(\mathcal{K} \oplus \mathcal{H})$, defined by

$$L(t) = e^{-i\theta t} T(t), \tag{3.22}$$

is periodic with period $\tau$.

Note that

$$L(t)^* J L(t) = J \quad \forall t \in \mathbb{R}. \tag{3.23}$$

If

$$L(t) = \begin{pmatrix} L_{11}(t) & L_{12}(t) \\ L_{21}(t) & L_{22}(t) \end{pmatrix} \tag{3.24}$$

is the representation of $L(t)$ in $\mathcal{K} \oplus \mathcal{H}$, with $L_{11}(t) \in \mathcal{L}(\mathcal{K})$, $L_{12}(t) \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $L_{21}(t) \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, and $L_{22}(t) \in \mathcal{L}(\mathcal{H})$, then

$$L_{\alpha,\beta}(t) = e^{-i\theta t} T_{\alpha,\beta}(t) \tag{3.25}$$

for $\alpha, \beta = 1, 2$. Therefore, setting, for $x \in B$,

$$\widetilde{L(t)}(x) : x \longmapsto \left(L_{11}(t)x + L_{12}(t)\right)\left(L_{21}(t)x + L_{22}(t)\right)^{-1}, \tag{3.26}$$

then

$$\widetilde{L(t)} = \phi_t \quad \forall t \in \mathbb{R}. \tag{3.27}$$

If $X : \mathcal{D}(X) \subset \mathcal{K} \oplus \mathcal{H} \to \mathcal{K} \oplus \mathcal{H}$ is the infinitesimal generator of the group $T$, the operator $X - i\theta I_{\mathcal{K}\oplus\mathcal{H}}$, with domain $\mathcal{D}(X)$, generates the group $L$.

The structure of the spectrum $\sigma(X - i\theta I_{\mathcal{K}\oplus\mathcal{H}})$ is described in [1] by a theorem of Bart, whereby

(i) $\sigma(X - i\theta I_{\mathcal{K}\oplus\mathcal{H}}) \subset i(2\pi/\tau)\mathbb{Z}$;

(ii) $\sigma(X - i\theta I_{\mathcal{K}\oplus\mathcal{H}})$ consists of simple poles of the resolvent function $\zeta \mapsto (\zeta I_{\mathcal{K}\oplus\mathcal{H}} - (X - i\theta I_{\mathcal{K}\oplus\mathcal{H}}))^{-1}$;

(iii) the eigenvectors of $X - i\theta I_{\mathcal{K}\oplus\mathcal{H}}$ span a dense linear subspace of $\mathcal{K} \oplus \mathcal{H}$.

According to [1], if $X$ is the infinitesimal generator of a strongly continuous group $T$, and if conditions (i), (ii), and (iii) hold, the group $L$ defined by (3.22) is periodic with period $\tau$.

Summing up, in view of Theorem 2.3, the following result has been established.

Theorem 3.1. *If there is a periodic point $x_0 \in B$ for $\phi$, with period $\tau > 0$, and if there is a set $K \subset (0,\tau)$ such that, for any $t \in K$, $M_{-x_0}(\phi_t(x_0))$ is collinear to some point of S, and the set $\{\phi_t(x_0) : t \in K\}$ spans a dense affine subspace of $\mathcal{L}(\mathcal{H},\mathcal{K})$, then there exist a strongly continuous group $T : \mathbb{R} \to \mathcal{L}(\mathcal{H},\mathcal{K})$ and a real number $\theta$ such that the rescaled group $\mathbb{R} \ni t \mapsto L(t)$ is a periodic group with period $\tau$.*

*If $X : \mathcal{D}(X) \subset \mathcal{K} \oplus \mathcal{H} \to \mathcal{K} \oplus \mathcal{H}$ is the infinitesimal generator of the group $T$, conditions (i), (ii), and (iii) characterize the periodicity of L with period $\tau$.*

Thus, if $X$ generates a strongly continuous group $T$, and if conditions (i), (ii), and (iii) hold, the group $L$ defined by (3.22) is periodic with period $\tau$. As was proved in [19, Proposition 4.1], the group $T$ satisfies (3.3) for all $t \in \mathbb{R}$ if and only if the operator $iJX$ is selfadjoint. If that is the case, setting

$$\mathcal{K}' \oplus 0 = (\mathcal{K} \oplus 0) \cap \mathcal{D}(X), \qquad 0 \oplus \mathcal{H}' = (0 \oplus \mathcal{H}) \cap \mathcal{D}(X), \tag{3.28}$$

[19, Lemma 5.3] implies that the linear spaces $\mathcal{K}'$ and $\mathcal{H}'$ are dense in $\mathcal{K}$ and $\mathcal{H}$.

We consider now the case in which the semigroup $T_{|\mathbb{R}_+}$ is eventually differentiable (i.e., there is $t^0 \geq 0$ such that the function $t \mapsto T(t)x$ is differentiable in $(t^0, +\infty)$ for all $x \in \mathcal{K} \oplus \mathcal{H}$). By (3.22), also $L_{|\mathbb{R}_+}$ is eventually differentiable.

According to a theorem by Pazy (see, e.g., [12]), there exist $a \in \mathbb{R}$ and $b > 0$ such that the set

$$\{\zeta \in \mathbb{C} : \Re\zeta \geq a - b\log|\Im\zeta|\} \tag{3.29}$$

is contained in the resolvent set of $X - i\theta I_{\mathcal{K}\oplus\mathcal{H}}$. Thus, the intersection of $\sigma(X - i\theta I_{\mathcal{K}\oplus\mathcal{H}})$ with the imaginary axis is bounded. Condition (i) implies then that $\sigma(X - i\theta I_{\mathcal{K}\oplus\mathcal{H}})$ is finite. But then, by [1, Proposition 3.2], $X - i\theta I_{\mathcal{K}\oplus\mathcal{H}} \in \mathcal{L}(\mathcal{K} \oplus \mathcal{H})$, and therefore $X \in \mathcal{L}(\mathcal{K} \oplus \mathcal{H})$, proving thereby the following proposition.

Proposition 3.2. *Under the hypotheses of Theorem 3.1, if moreover the semigroup $T_{|\mathbb{R}_+}$ is eventually differentiable, the group T is uniformly continuous.*

*Remark 3.3.* The above argument holds for any strongly continuous semigroup $T$ of linear operators, which is periodic, showing that, if $T$ is eventually differentiable, then $T$ is uniformly continuous.

If $T$ is eventually norm continuous, then (see, e.g., [3]) its infinitesimal generator $X$ is such that, for every $r \in \mathbb{R}$, the set

$$\{\zeta \in \sigma(X) : \Re\zeta \geq r\} \tag{3.30}$$

is bounded.

At this point, [1, Proposition 3.2] implies that, if $T$ is also periodic, then the operator $X$ is bounded, and therefore $T$ is uniformly continuous.

This conclusion holds, for example, if the periodic semigroup $T$ is eventually compact.

## 4. The unit ball of a Hilbert space

Theorem 3.1 has been established in [22] in the case in which $B$ is the open unit ball of the Hilbert space $\mathcal{K}$ (i.e., when $\mathcal{H} = \mathbb{C}$).

In this case, $T_{11}(t) \in \mathcal{L}(\mathcal{K})$ is invertible in $\mathcal{L}(\mathcal{K})$, $T_{12}(t) \in \mathcal{K}$, $T_{21}(t) = (\bullet | T_{12}(t))$, and $T_{22}(t) \in \mathbb{C}$ are characterized by the equations

$$\begin{gathered} |T_{22}(t)|^2 - \|T_{12}(t)\|^2 = 1, \\ T_{11}(t)^* T_{11}(t) = I + \frac{1}{|T_{22}(t)|^2} (\bullet | T_{11}(t)^* T_{12}(t)) T_{11}(t)^* T_{12}(t). \end{gathered} \tag{4.1}$$

As was shown in [22], there is a neighbourhood $U$ of $\overline{B}$ such that

$$(x | T_{11}(t)^* T_{12}(t)) + T_{22}(t) \neq 0 \quad \forall x \in U,\ t \in \mathbb{R}. \tag{4.2}$$

The orbit of $x_0 \in B$ is described by

$$\phi_t(x_0) = \widetilde{T(t)}(x_0) = \frac{1}{(x_0 | T_{11}(t)^* T_{12}(t)) + T_{22}(t)} (T_{11}(t)x_0 + T_{12}(t)). \tag{4.3}$$

The infinitesimal generator $X$ of $T$ is represented in $\mathcal{K} \oplus \mathbb{C}$ by the matrix

$$X = \begin{pmatrix} X_{11} & X_{12} \\ (\bullet | X_{12}) & iX_{22} \end{pmatrix}, \tag{4.4}$$

where $X_{12} \in \mathcal{K}$, $X_{22} \in \mathbb{R}$, $iX_{11}$ is a selfadjoint operator, and the domains $\mathcal{D}(X)$ and $\mathcal{D}(X_{11})$ of $X$ and of $X_{11}$ are related by

$$\mathcal{D}(X) = \mathcal{D}(X_{11}) \oplus \mathbb{C}. \tag{4.5}$$

Since $\phi_\tau$ is the identity, by [17, Proposition 7.3] and by (3.27), the set

$$\text{Fix}\,\phi = \{x \in B : \phi_t(x) = x \ \forall t \in \mathbb{R}\} \tag{4.6}$$

is nonempty.

The ball $B$ being homogeneous, there is no restriction in assuming $0 \in \text{Fix}\,\phi$. Thus, by (3.8), $T_{12}(t) = 0$ for all $t \in \mathbb{R}$, and therefore $X_{12} = 0$. Furthermore, as a consequence of (4.1),

$$T_{22}(t) = e^{iX_{22}t}, \tag{4.7}$$

and the skew-selfadjoint operator $X_{11}$ generates the strongly continuous group $T_{11} : t \mapsto T_{11}(t)$ of unitary operators in $\mathcal{K}$.

Equation (3.9), which now reads

$$\phi_t(x) = e^{-iX_{22}t}T_{11}(t), \tag{4.8}$$

yields the following lemma.

Lemma 4.1. *The set* Fix $\phi$ *is the intersection of* $B$ *with a closed affine subspace of* $\mathcal{H}$.

Because of (3.21),

$$X_{22} = \theta + \frac{2n\pi}{\tau} \tag{4.9}$$

for some $n \in \mathbb{Z}$, and therefore

$$\phi_t(x) = e^{-(2n\pi/\tau)it}L_{11}(t)x \tag{4.10}$$

for all $x \in B$ and some $n \in \mathbb{Z}$.

The strongly continuous periodic group $L_{11} : t \mapsto L_{11}(t)$, with period $\tau$, of unitary operators in $\mathcal{H}$ is generated by

$$Y_{11} := X_{11} - i\theta I_{\mathcal{H}} : \mathcal{D}(X_{11}) \subset \mathcal{H} \longrightarrow \mathcal{H}. \tag{4.11}$$

By [1], $\sigma(Y_{11}) \subset i(2\pi/\tau)\mathbb{Z}$ consists entirely of eigenvalues, and the corresponding eigenspaces, which are mutually orthogonal, span a dense linear subspace of $\mathcal{H}$.

For $m \in \mathbb{Z}$, let $P_m$ be the orthogonal spectral projector associated with $(2\pi/\tau)mi$. By [1, (3)], $L_{11}$ is expressed by

$$L_{11}(t)x = \sum_m e^{(2m\pi/\tau)it}P_m x \tag{4.12}$$

for all $x \in \mathcal{H}$ and all $t \in \mathbb{R}$. Thus $L_{11}(t)$ leaves invariant every space $P_m(\mathcal{H})$, and acts on it by the rotation

$$x \longmapsto e^{(2m\pi/\tau)it}x. \tag{4.13}$$

Hence, the following lemma follows.

Lemma 4.2. *If the orbit of* $x_0 \in B$ *spans a dense affine subspace of* $\mathcal{H}$, *then* $\dim_{\mathbb{C}} P_m(\mathcal{H}) \leq 1$ *for all* $m \in \mathbb{Z}$.

Since, by (3.25),

$$\sigma(Y_{11}) = \sigma(X_{11} - i\theta I_{\mathcal{H}}) \tag{4.14}$$

if $\sigma(X_{11})$ is finite, also $\sigma(Y_{11})$ is finite.

A similar argument to that leading to Proposition 3.2 yields now the following theorem.

Theorem 4.3. *If the continuous flow $\phi$ of holomorphic automorphisms of the open unit ball $B$ of $\mathcal{K}$ defined by a strongly continuous group $T : \mathbb{R} \to \mathcal{L}(\mathcal{K} \oplus \mathbb{C})$ has a periodic point whose orbit spans a dense affine subspace of $\mathcal{K}$, and if moreover $T$ is eventually differentiable, then* $\dim_{\mathbb{C}} \mathcal{K} < \infty$.

According to [17, Theorem VII], for any $\gamma > 0$ and every choice of $x_0 \in B \cap \mathcal{D}(X_{11})$, the function

$$\phi_\bullet(x_0)|_{[0,\gamma]} : [0,\gamma] \longrightarrow \mathcal{D}(X_{11}), \tag{4.15}$$

defined by (4.3) for $0 \le t \le \gamma$, is the unique continuously differentiable map $[0,\gamma] \to \mathcal{K}$ with $x([0,\gamma]) \subset \mathcal{D}(X_{11})$, which is continuous for the graph norm

$$x \longmapsto \|x\| + \|X_{11}x\| \tag{4.16}$$

on $\mathcal{D}(X_{11})$, and satisfies the Riccati equation

$$\frac{d}{dt}\phi_t(x_0) = X_{11}\phi_t(x_0) - ((\phi_t(x_0)|X_{12}) + iX_{22})\phi_t(x_0) + X_{12} \tag{4.17}$$

with the initial condition $\phi_0(x_0) = x_0 \in B \cap \mathcal{D}(X_{11})$.

Hence, Theorem 3.1 can be rephrased.

Proposition 4.4. *If the Riccati equation (4.17) has a periodic integral which spans a dense affine subspace of $\mathcal{K}$, (4.17) is periodic (i.e., all integrals of (4.17) satisfying the above regularity conditions are periodic).*

We consider now the case in which one of the two spaces $\mathcal{K}$ and $\mathcal{H}$ has a finite dimension, and therefore $J$ defines in $\mathcal{K} \oplus \mathcal{H}$ the structure of a Pontryagin space. Assuming

$$\infty > \dim_{\mathbb{C}} \mathcal{H} < \dim_{\mathbb{C}} \mathcal{K} \le \infty, \tag{4.18}$$

the extreme points of $\overline{B}$ are all the linear isometries $\mathcal{H} \to \mathcal{K}$; by [19, Theorem III], $X$ is represented by the matrix

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}{}^* & iX_{22} \end{pmatrix}, \tag{4.19}$$

where $X_{11} : \mathcal{D}(X_{11}) \subset \mathcal{K} \to \mathcal{K}$ and $X_{22} \in \mathcal{L}(\mathcal{K})$ are skew-selfadjoint, $X_{12} \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, and $\mathcal{D}(X) = \mathcal{D}(X_{11}) \oplus \mathcal{H}$.

The Riccati equation (4.17) is replaced in [19] by the operator-valued Riccati equation

$$\frac{d}{dt}x(t) = X_{11}x(t) - x(t)X_{22} - x(t)X_{22} - x(t)X_{12}{}^{*}x(t) + X_{12} \tag{4.20}$$

acting on $C^1$ maps of $[0,\gamma]$ into

$$\check{D} = \{x \in \mathcal{L}(\mathcal{H},\mathcal{K}) : x\xi \in \mathfrak{D}(X_{11})\ \forall \xi \in \mathcal{H}\} \tag{4.21}$$

which are continuous for the norm (4.16).

For any $\gamma > 0$, any choice of $u$ invertible in $\mathcal{L}(\mathcal{H})$ and of $v \in \check{D}$ such that $x_0 = vu^{-1} \in B$, the function $t \mapsto x(t)$ expressed by (3.8), with $x = x_0$, for $t \in [0,\gamma]$ is the unique solution of (4.20) satisfying the conditions stated above, with the initial condition $x(0) = x_0$.

Theorem 3.1 yields then the following proposition.

Proposition 4.5. *Let the integral $t \mapsto x(t)$ be periodic with period $\tau > 0$, and let there be a set $K \subset (0,\tau)$ such that $x(K)$ spans a dense affine subspace of $\mathcal{L}(\mathcal{H},\mathcal{K})$. If, for any $t \in K$, $M_{-x_0}(x(t))$ is collinear to some linear isometry of $\mathcal{H}$ into $\mathcal{K}$, the Riccati equation (4.20) is periodic.*

## 5. Spin factors

Similar results to some of those of Section 3 will now be established in the case in which the $J^*$-algebra $\mathcal{A}$ is a spin factor. In this section, $\mathcal{K}$ is, as before, a complex Hilbert space, and $C^*$ is the adjoint of $C \in \mathcal{L}(\mathcal{K})$. A Cartan factor of type four, also called a spin factor, is a closed linear subspace $\mathcal{A}$ of $\mathcal{L}(\mathcal{K})$ which is $^*$-invariant and such that $C \in \mathcal{A}$ implies that $C^2$ is a scalar multiple of $I_{\mathcal{K}}$.

Since, for $C_1, C_2 \in \mathcal{A}$, $C_1C_2{}^* + C_2{}^*C_1$ is a scalar multiple, $2(C_1|C_2)I_{\mathcal{K}}$, of the identity, then $C_1, C_2 \mapsto (C_1|C_2)$ is a positive-definite scalar product, with respect to which $\mathcal{A}$ is a complex Hilbert space. (For more details concerning spin factors, see, e.g., [7, 18, 21].) Denoting by $|||\cdot|||$ and by $\|\cdot\|$ the operator norm and the Hilbert space norm on $\mathcal{A}$, then

$$|||C|||^2 = \|C\|^2 + \sqrt{\|C\|^4 - |(C|C^*)|^2} \quad \forall C \in \mathcal{A}. \tag{5.1}$$

The open unit ball $B$ for the norm $|||\cdot|||$, also expressed by

$$B = \left\{C \in \mathcal{A} : \|C\|^2 < \frac{1 + |(C|C^*)|^2}{2} < 1\right\}, \tag{5.2}$$

is called a Cartan domain of type four. The set $S$ of all extreme points of $\overline{B}$ is the set of all multiples, by a constant factor of modulus one, of all selfadjoint unitary operators acting on the Hilbert space $\mathcal{K}$, which are contained in $\mathcal{A}$ [7, 21].

Changing again notations, we denote by $x, y$ elements of the spin factor $\mathscr{A}$, and $x \mapsto \overline{x}$ stands for the conjugation defined by the adjunction in the Hilbert space $\mathscr{A}$. For any $M \in \mathscr{L}(\mathscr{A})$, $M^t$ will indicate the transposed of $M$. The same notation will be used to indicate the canonical transposition in $\mathbb{C}^2$ and the transposition in $\mathscr{A} \oplus \mathbb{C}^2$.

According to [7, 21], any holomorphic automorphism $f$ of $B$ can be described as follows.

Let

$$J = \begin{pmatrix} I_{\mathscr{H}} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}, \tag{5.3}$$

and let $\Lambda$ be the semigroup consisting of all $A \in \mathscr{L}(\mathscr{A} \oplus \mathbb{C}^2)$ such that

$$A^t J A = J. \tag{5.4}$$

Every $A \in \Lambda$ is represented by a matrix

$$A = \begin{pmatrix} M & q_1 & q_2 \\ (\,\bullet\,|r_1) & e_{11} & e_{12} \\ (\,\bullet\,|r_2) & e_{21} & e_{22} \end{pmatrix}, \tag{5.5}$$

where $M \in \mathscr{L}(\mathscr{A})$ is a real operator, $q_1$, $q_2$, $r_1$, and $r_2$ are real vectors in $\mathscr{A}$, and

$$E := \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \tag{5.6}$$

is a real $2 \times 2$ matrix such that $\det E > 0$, and

$$M^t M - R^t R = I_{\mathscr{A}}, \tag{5.7}$$

$$M^t Q - R^t E = 0, \tag{5.8}$$

$$E^t E - Q^t Q = I_{\mathbb{C}^2}. \tag{5.9}$$

Here $R : \mathscr{A} \to \mathbb{C}^2$ and $Q : \mathbb{C}^2 \to \mathscr{A}$ are defined by

$$\begin{aligned} Rx &= \begin{pmatrix} (x|r_1) \\ (x|r_2) \end{pmatrix} \in \mathbb{C}^2 \quad \forall x \in \mathscr{A}, \\ Qz &= z_1 q_1 + z_2 q_2 \quad \forall z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2. \end{aligned} \tag{5.10}$$

It was shown in [18] that the set $\Lambda_0 = \{A \in \Lambda : \det E > 0\}$ is a subsemigroup of $\Lambda$.

For $x \in \mathscr{A}$, let

$$\delta(A,x) = 2(x|r_1 - r_2) + (e_{11} - e_{22} + i(e_{12} + e_{21}))(x|\overline{x}) + e_{11} + e_{22} + i(e_{21} - e_{12}). \tag{5.11}$$

One shows (see [18, 21]) that, if $A \in \Lambda_0$, $\delta(A,x) \neq 0$ for all $x$ in an open neighbourhood $U$ of $\overline{B}$. Hence, the map

$$\hat{A} : U \ni x \longmapsto \frac{1}{\delta(A,x)}\big(2Mx + \big(1+(x|\overline{x})\big)q_1 - i\big(1-(x|\overline{x})\big)q_2\big) \tag{5.12}$$

is holomorphic in $U$. Its restriction to $B$, which will be denoted by the same symbol $\hat{A}$, is the most general holomorphic isometry for the Carathéodory-Kobayashi metric of $B$ [21]. This isometry is a holomorphic automorphism of $B$ if, and only if, $A$ is invertible in $\mathscr{L}(\mathscr{A} \oplus \mathbb{C}^2)$.

If $\hat{A}(0) = 0$, then $q_1 - iq_2 = 0$, and therefore $q_1 = q_2 = 0$ because $q_1$ and $q_2$ are real vectors; (5.9) reads now $E \in \mathrm{SO}(2)$, and (5.8), which now becomes $R^t E = 0$, yields $r_1 = r_2 = 0$. Thus, by (5.7), $M$ is a real linear isometry of $\mathscr{A}$. Setting

$$E = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{5.13}$$

for some $\alpha \in \mathbb{R}$, then

$$\hat{A}(x) = e^{i\alpha} Mx \quad \forall x \in B. \tag{5.14}$$

As a consequence,

$$\hat{A}(x) = x \ \forall x \in B \Longleftrightarrow A = \begin{pmatrix} e^{-i\alpha} I_{\mathscr{A}} & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}. \tag{5.15}$$

Now, let $T : \mathbb{R}_+ \to \mathscr{L}(\mathscr{A} \oplus \mathbb{C}^2)$ be a strongly continuous semigroup such that $T(t) \in \Lambda_0$ for all $t \geq 0$. Setting

$$\phi_t = \widehat{T(t)} \tag{5.16}$$

for $t \geq 0$, one defines a continuous semiflow $\phi : \mathbb{R}_+ \times B \to B$ of holomorphic isometrics $B \to B$.

If $x_0 \in B$ is a periodic point of $\phi$ with period $\tau > 0$, and if the hypotheses of Theorem 2.3 are satisfied, then

(i) $\phi$ is the restriction to $\mathbb{R}_+$ of a continuous flow $\mathbb{R} \times B \to B$, which will be denoted by the same symbol $\phi$;

(ii) $T$ is the restriction to $\mathbb{R}_+$ of a strongly continuous group $\mathbb{R} \to \mathscr{L}(\mathscr{A} \oplus \mathbb{C}^2)$, which will be denoted by the same symbol $T$;

(iii) (5.16) holds for all $t \in \mathbb{R}$.

Since, $\widehat{T(\tau)}(x) = x$ for all $x \in B$, by (5.15), there is some $\alpha \in \mathbb{R}$ such that

$$T(\tau) = F(\tau), \tag{5.17}$$

where

$$F(\tau)=\begin{pmatrix} e^{-i\alpha\tau}I_{\mathcal{A}} & 0 & 0 \\ 0 & \cos(\alpha\tau) & -\sin(\alpha\tau) \\ 0 & \sin(\alpha\tau) & \cos(\alpha\tau) \end{pmatrix}. \tag{5.18}$$

Thus,

$$\sigma(T(\tau))=\sigma(F(\tau)). \tag{5.19}$$

Setting

$$L_{-}=\{(\zeta,i\zeta):\zeta\in\mathbb{C}\},\qquad L_{+}=\{(\zeta,-i\zeta):\zeta\in\mathbb{C}\}, \tag{5.20}$$

if $\alpha\tau\notin\pi\mathbb{Z}$, $\sigma(T(\tau))$ consists of the eigenvalue $e^{-i\alpha\tau}$, with the eigenspace $\mathcal{A}\oplus L_{-}\subset\mathcal{A}\oplus\mathbb{C}^2$, and of the eigenvalue $e^{i\alpha\tau}$, with the eigenspace $0\oplus L_{+}\subset\mathcal{A}\oplus\mathbb{C}^2$. If $\alpha\tau\in\pi\mathbb{Z}$, $T(\tau)=I_{\mathcal{A}\oplus\mathbb{C}^2}$ when $\alpha\tau/\pi$ is even , and $T(\tau)=-I_{\mathcal{A}\oplus\mathbb{C}^2}$ when $\alpha\tau/\pi$ is odd.

In conclusion, the following theorem has been established.

Theorem 5.1. *If there is a periodic point $x_0\in B$ for $\phi$, with period $\tau>0$, and if there is a set $K\subset(0,\tau)$ such that, for any $t\in K$, $M_{-x_0}(\phi_t(x_0))$ is collinear to a multiple, by a constant factor of modulus one, of a selfadjoint unitary operator which acts on the Hilbert space $\mathcal{K}$ and is contained in $\mathcal{A}$, and the set $\{\phi_t(x_0):t\in K\}$ spans a dense affine subspace of $\mathcal{A}$, then there exist a strongly continuous group $T:\mathbb{R}\to\mathcal{L}(\mathcal{A}\oplus\mathbb{C}^2)$ and a real number $\alpha$ for which (5.17) and (5.18) hold.*

*The infinitesimal generator*

$$X:\mathfrak{D}(X)\subset\mathcal{A}\oplus\mathbb{C}^2\longrightarrow\mathcal{A}\oplus\mathbb{C}^2 \tag{5.21}$$

*of the group $T$ has a pure point spectrum, consisting of at least one and at most two distinct eigenvalues.*

If $\alpha\tau\notin\pi\mathbb{Z}$, $\sigma(T(\tau))$ consists of the eigenvalue $e^{-i\alpha\tau}$, with the eigenspace $\mathcal{A}\oplus L_{-}$, and of the eigenvalue $e^{i\alpha\tau}$ with the one-dimensional eigenspace $0\oplus L_{+}$.

If $\alpha\tau\in\pi\mathbb{Z}$, the group $T$ is periodic with period $\tau$ when $\alpha\tau/\pi$ is even, and period $2\tau$ when $\alpha\tau/\pi$ is odd.

According to [18, Theorem 4.1], $\mathfrak{D}(X)=\mathfrak{D}\oplus\mathbb{C}^2$, where $\mathfrak{D}$ is a dense linear subspace of $\mathcal{A}$, and $X$ is expressed by the matrix

$$X=\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ (\bullet\,|X_{12}) & 0 & X_{23} \\ (\bullet\,|X_{13}) & -X_{23} & 0 \end{pmatrix}, \tag{5.22}$$

where $X_{23}\in\mathbb{R}$, $X_{12}$ and $X_{13}$ are real vectors in $\mathcal{A}$, and $X_{11}$ is a real, skew-selfadjoint operator on $\mathcal{A}$ with domain $\mathfrak{D}$.

Similar results to those established in Propositions 4.4 and 4.5 for (4.17) and (4.20) hold for the Riccati equation

$$\frac{d}{dt}\phi_t(x_0) = (X_{11} + iX_{23}I)\phi_t(x_0) + \frac{1}{2}(X_{12} + iX_{13})(\phi_t(x_0)\,|\,\overline{\phi_t(x_0)}) \\ - (\phi_t(x_0)\,|\,X_{12} - iX_{13})\phi_t(x_0) + \frac{1}{2}(X_{12} - iX_{13}) \tag{5.23}$$

with initial conditions $\phi_0(x_0) = x_0 \in B \cap \mathfrak{D}(X_{11})$.

## 6. Fixed points of semiflows

The next sections will be devoted to investigating the fixed points of a continuous semiflow $\phi : \mathbb{R}_+ \times D \to D$ of holomorphic maps of a bounded domain $D$ in a complex Banach space $\mathscr{E}$, that is to say, the points $x \in D$ such that $\phi_t(x) = x$ for all $x \in \mathbb{R}_+$.

Actually, some of the results we are going to establish hold under slightly weaker conditions. Namely, $\phi$ will be a map of $\mathbb{R}_+^* \times D$ into $D$ satisfying (2.3) and (2.4) for all $t, t_1, t_2 \in \mathbb{R}_+^*$ and such that the map $t \mapsto \phi_t(y)$ is continuous on $\mathbb{R}_+^*$ for all $y \in \mathscr{E}$.

A set $S \subset D$ is said to be *completely interior to D*, in symbols $S \Subset D$ if $\inf\{\|x - y\| : x \in D,\ y \in \mathscr{E} \backslash D\} > 0$.

Since

$$\phi_{t+s} = \phi_t(\phi_s(D)) \subset \phi_t(D) \quad \forall t, s > 0, \tag{6.1}$$

if

$$\phi_t(D) \Subset D, \tag{6.2}$$

then

$$\phi_r(D) \Subset D \quad \forall r \geq t. \tag{6.3}$$

Let $\phi_{t_0}(D) \Subset D$ for some $t_0 > 0$, and let $t \geq t_0$. By the Earle-Hamilton theorem (see [2] or, e.g., [5, Theorem V.5.2]), there is a unique point $x_t \in D$ such that $\phi_t(x_t) = x_t$. Hence $x_t$ is the unique point in $D$ such that

$$\phi_{nt}(x_t) = x_t \quad \forall n = 1, 2 \ldots. \tag{6.4}$$

Moreover, by the Earle-Hamilton theorem,

$$\lim_{n \to +\infty} \phi_{nt}(x) = x_t \quad \forall x \in D. \tag{6.5}$$

Let $p$, $q$ be positive integers, with $p \geq q$. There is a unique point $x_{(p/q)t} \in D$ such that

$$\phi_{(p/q)t}(x_{(p/q)t}) = x_{(p/q)t}. \tag{6.6}$$

Since

$$\phi_{n(p/q)t}(x_{(p/q)t}) = x_{(p/q)t} \tag{6.7}$$

for $n = 1, 2, \ldots$, choosing $n = mq$, $m = 1, 2, \ldots$ yields

$$\phi_{mpt}(x_{(p/q)t}) = x_{(p/q)t}. \tag{6.8}$$

Since, by (6.5),

$$\lim_{m \to +\infty} \phi_{mpt}(x_{(p/q)t}) = x_t, \tag{6.9}$$

then

$$x_{(p/q)t} = x_t \tag{6.10}$$

for all positive integers $p \geq q = 1, 2, \ldots$.

The continuity of $t \mapsto \phi_t(y)$ implies that

$$\phi_{rt}(x_t) = x_t \tag{6.11}$$

for all real numbers $r \geq 1$. Hence there is a point $x_0 \in D$ which is the unique fixed point of $\phi_t$ for every $t \geq t_0$.

Let $t_0 > 0$ and choose $s \in (0, t_0)$ and $t \geq t_0$. Then

$$\phi_s(x_0) = \phi_s(\phi_t(x_0)) = \phi_{t+s}(x_0) = x_0 \tag{6.12}$$

because $t + s > t_0$.

In conclusion, the first part of the following theorem has been established.

THEOREM 6.1. *Let $\phi : \mathbb{R}_+^* \times D \to D$ satisfy (2.3) and (2.4), and be such that $t \mapsto \phi_t(x)$ is continuous on $\mathbb{R}_+^*$ for all $x \in D$. If $D$ is bounded, and if $\phi_t(D) \Subset D$ for some $t > 0$, there exists $x_0 \in D$ which is the unique fixed point of $\phi_s$ for every $s > 0$, and*

$$\lim_{s \to +\infty} \phi_s(x) = x_0 \quad \forall x \in D. \tag{6.13}$$

*Proof.* Let $k_D$ be the Kobayashi distance in $D$. To complete the proof of the theorem note that, given $x \in D$ and $s > 0$, for every $\epsilon > 0$ there exists a positive integer $n_0$ such that, whenever $n \geq n_0$,

$$k_D(x_0, \phi_{ns}(x)) < \epsilon. \tag{6.14}$$

If $n \geq n_0$ and $t > ns$,

$$\begin{aligned} k_D(x_0, \phi_t(x)) &= k_D(x_0, \phi_{ns+t-ns}(x)) \\ &= k_D(\phi_{t-ns}(x_0), \phi_{t-ns}(\phi_{ns}(x))) \\ &\leq k_D(x_0, \phi_{ns}(x)) < \epsilon. \end{aligned} \tag{6.15}$$

□

Corollary 6.2. *Under the hypotheses of Theorem 6.1, $x_0$ is the only $\omega$-stable point of $\phi$. (That means that, for every $\epsilon > 0$ and every $\tau > 0$, there is some $t \geq \tau$ for which $k_D(x_0, \phi_t(x_0)) < \epsilon$.)*

Theorem 6.3. *Let $D$ be bounded and let $\phi : \mathbb{R}_+^* \times D \to D$ satisfy the hypotheses of Theorem 6.1. If there exist a sequence $\{t_\nu\} \subset \mathbb{R}_+^*$ diverging to $+\infty$ and a map $g : D \to D$ such that $\lim_{\nu \to +\infty} \phi_{t_\nu} = g$ for the topology of local uniform convergence and if $g(D) \Subset D$, then there exists a unique point $x_0 \in D$ such that $\phi_t(x_0) = x_0$ for all $t > 0$ and $\lim_{t \to +\infty} \phi_t(x) = x_0$ for all $x \in D$.*

*Proof.* Since $g$ is holomorphic and $g(D) \Subset D$, the Earle-Hamilton theorem implies that there is a unique point $x_0 \in D$ which is fixed by $g$.

If $\phi_t(y) = y$ for some $y \in D$ and some $t > 0$, then, if $s > t$,

$$\phi_s(y) = \phi_{s-t+t}(y) = \phi_{s-t}(\phi_t(y)) = \phi_{s-t}(y), \tag{6.16}$$

and therefore

$$\phi_t(\phi_s(y)) = \phi_t(\phi_{s-t}(y)) = \phi_s(y). \tag{6.17}$$

But then

$$g(y) = \lim_{\nu \to +\infty} \phi_{t_\nu}(y) = y, \tag{6.18}$$

and therefore $y = x_0$. Hence, either $\operatorname{Fix} \phi_t = \varnothing$ for all $t > 0$, or $\operatorname{Fix} \phi_t = \{x_0\}$ when $t \gg 0$.

Let $R > 0$ be such that

$$B(x_0, R) \Subset D. \tag{6.19}$$

Since the Kobayashi distance $k_D$ and $\|\cdot\|$ are equivalent on $B(x_0, R)$, there exist real constants $c > b > 0$ such that

$$b\|x - y\| \leq k_D(x, y) \leq c\|x - y\| \quad \forall x, y \in B(x_0, R). \tag{6.20}$$

Let $r > 0$ be such that

$$B_{k_D}(x_0, r) \subset B(x_0, R). \tag{6.21}$$

For every $\epsilon > 0$, there is $\nu_0$ such that

$$\nu \geq \nu_0 \Longrightarrow ||\phi_{t_\nu}(x) - g(x)|| < \epsilon \quad \forall x \in B(x_0, R) \tag{6.22}$$

(because the sequence $\{\phi_{t_\nu}\}$ converges to $g$ for the topology of local uniform convergence).

Since $g(D) \Subset D$, there exists $a \in (0, 1)$ such that

$$\begin{aligned} k_D(\phi_{t_\nu}(x), x_0) &\leq k_D(\phi_{t_\nu}(x), g(x)) + k_D(g(x), x_0) \\ &\leq c||\phi_{t_\nu}(x) - g(x)|| + ak_D(x, x_0) \\ &< c\epsilon + ar. \end{aligned} \tag{6.23}$$

Let $\ell \in (a, 1)$ and $\epsilon$ be such that

$$0 < \epsilon < \frac{\ell - a}{c} r. \tag{6.24}$$

Then

$$c\epsilon + ar < (\ell - a)r + ar = \ell r, \tag{6.25}$$

and therefore

$$\phi_{t_\nu}(B_{k_D}(x_0, r)) \subset B_{k_D}(x_0, \ell r) \quad \forall \nu \geq \nu_0. \tag{6.26}$$

It turns out that

$$B_{k_D}(x_0, \ell r) \Subset B_{k_D}(x_0, r). \tag{6.27}$$

Indeed, if $x \in B_{k_D}(x_0, \ell r)$ and $y \in B(x_0, R) \backslash B_{k_D}(x_0, r)$,

$$\|x - y\| \geq \frac{1}{c} k_D(x, y) \geq \frac{1}{c}(k_D(y, x_0) - k_D(x_0, x)) > \frac{1 - \ell}{c} r. \tag{6.28}$$

As a consequence of (6.27),

$$\phi_{t_\nu}(B_{k_D}(x_0, r)) \Subset B_{k_D}(x_0, r) \quad \forall \nu \geq \nu_0. \tag{6.29}$$

If $t > t_{\nu_0}$,

$$\begin{aligned}\phi_t(B_{k_D}(x_0,r)) &= \phi_{t-t_{\nu_0}+t_{\nu_0}}(B_{k_D}(x_0,r)) = \phi_{t_{\nu_0}}(\phi_{t-t_{\nu_0}}(B_{k_D}(x_0,r))) \\ &\subset \phi_{t_{\nu_0}}(B_{k_D}(x_0,r)) \Subset B_{k_D}(x_0,r).\end{aligned} \tag{6.30}$$

Hence,

$$\operatorname{Fix}\phi_t = \{x_0\} \quad \forall t \geq t_{\nu_0}. \tag{6.31}$$

Thus,

$$\lim_{t\to+\infty} \phi_t(x) = x_0 \tag{6.32}$$

for all $x \in B_{k_D}(x_0,r)$. In particular,

$$\lim_{\nu\to+\infty} \phi_{t_\nu}(x) = x_0 \tag{6.33}$$

for all $x \in B_{k_D}(x_0,r)$. Hence, $g(x) = x_0$ on $B_{k_D}(x_0,r)$ and therefore also on $D$ (because the open set $D$ is connected and $g$ is holomorphic on $D$), and (6.32) holds for all $x \in D$. □

## 7. Convergence of iterates and its consequences

The following theorem was announced in [16] without proof.

Theorem 7.1. *Let $D$ be a bounded domain in the complex Banach space $\mathscr{E}$, and let $f : D \to D$ be a holomorphic map fixing a point $x_0 \in D$. If the sequence $\{f^n\}$ of the iterates of $f$ converges for the topology of local uniform convergence on $D$, then either*

$$\sigma(df(x_0)) \subset \Delta \tag{7.1}$$

*or*

$$\sigma(df(x_0)) = \{1\} \cup (\Delta \cap \sigma(df(x_0))), \tag{7.2}$$

*and 1 is an isolated point of $\sigma(df(x_0))$ at which the resolvent function $(\bullet I - df(x_0))^{-1}$ has a pole of order one.*

Since $df^n(x_0) = (df(x_0))^n$ for $n = 0, 1, \ldots,$ and $\{df^n(x_0)\}$ converges in the operator topology, Theorem 7.1 is a consequence of the following proposition, also announced in [16] without proof.

Proposition 7.2. *Let $A$ and $P$ be elements of $\mathscr{L}(\mathscr{E})$. If*

$$\lim_{n\to+\infty} ||A^n - P|| = 0, \tag{7.3}$$

*there exists $k \in \mathbb{R}_+^*$, for which,*

$$||A^n|| \leq k \quad \forall n = 1, 2, \ldots, \tag{7.4}$$

*and therefore the spectral radius of $A$ is*

$$\rho(A) \leq 1. \tag{7.5}$$

*If $\rho(A) < 1$, then $P = 0$. If $\rho(A) = 1$, then*

$$\sigma(A) \cap \partial\Delta = \{1\}, \tag{7.6}$$

*and 1 is an isolated point of $\sigma(A)$ which is a pole of order one of the resolvent function $(\bullet I - A)^{-1}$. Furthermore, $P$ is the projector associated to the spectral set $\{1\}$ in the spectral resolution of $A$.*

*Proof.* For any integer $m \geq 0$,

$$A^m P = P A^m = P, \tag{7.7}$$

and therefore

$$P^2 = \lim_{m \to +\infty} A^m P = P, \tag{7.8}$$

that is, $P$ is an idempotent of $\mathcal{L}(\mathcal{E})$.

For $m = 1$, $(A - I)P = 0$, and this fact, together with (7.3), yields

$$\ker(A - I) = \operatorname{Ran} P. \tag{7.9}$$

Thus, $P \neq 0$ if, and only if, 1 is an eigenvalue of $A$.

Since

$$\big| ||A^n|| - \|P\| \big| \leq ||A^n - P||, \tag{7.10}$$

(7.3) implies (7.4), for a finite constant $k > 0$, and therefore implies (7.5) as well.

Recall that $\sigma(P) \subset \{0, 1\}$ and that $\sigma(P) = \{0\}$ if, and only if, $P = 0$, $\sigma(P) = \{1\}$ if, and only if, $P = I$. By the upper semicontinuity of the spectrum, for any open neighbourhood $U$ of $\sigma(P)$, there is an integer $n_0 \geq 0$ such that, whenever $n \geq n_0$, $\sigma(A^n) \subset U$, and therefore the image of $\sigma(A)$ by the map $\zeta \mapsto \zeta^n$ is contained in $U$. Hence,

$$P = 0 \Longrightarrow \rho(A) < 1, \tag{7.11}$$

and if $1 \in \sigma(P)$, then (7.3) and the upper semicontinuity imply (7.6).

Choosing a neighbourhood $U$ of the pair $\{0,1\}$ consisting of two mutually disjoint open discs $\Delta(0,r_1)$ and $\Delta(1,r_2)$ centered at the points 0 and 1, with radii $r_1>0$ and $r_2>0$, and using again the upper semicontinuity of the spectrum, we see that 1 is an isolated point of $\sigma(A)$ and

$$\sigma(A)=\{1\}\cup\big(\sigma(A)\cap\Delta\big). \tag{7.12}$$

What is left to prove is the final part of the proposition.

(a) It will be shown first that, for any open, relatively compact neighbourhood $U$ in $\mathbb{C}$ of $\{0,1\}$ and for any compact set $K\subset\mathbb{C}$ such that $K\cap\overline{U}=\varnothing$, there exist a constant $k_1>0$ and an integer $n_1\geq 1$ such that

$$\sup\{\|(\zeta I-A^n)^{-1}\|:\zeta\in K,\ n\geq n_1\}\leq k_1. \tag{7.13}$$

Let now $r_1$ and $r_2$ be such that $0<r_1<r_1+r_2<1$, so that

$$\overline{\Delta(0,r_1)}\cup\overline{\Delta(1,r_2)}\subset U. \tag{7.14}$$

There is $n_2\geq n_1$ such that

$$\sigma(A^n)\cap\Delta\subset\Delta(0,r_1)\quad\forall n\geq n_2. \tag{7.15}$$

Given $n\geq n_2$, choose $r_3\in(0,r_2)$ so small that the image by the map $\zeta\mapsto\zeta^n$ of $\overline{\Delta(1,r_3)}$ be contained in $\Delta(1,r_2)$. Then, for any $\zeta\in K$,

$$\begin{aligned}(\zeta I-A^n)^{-1}=\frac{1}{2\pi i}\Bigg\{&\int_{|\tau|=r_1}\frac{1}{\zeta-\tau^n}(\tau I-A)^{-1}\,d\tau\\&+\int_{|\tau-1|=r_3}\frac{1}{\zeta-\tau^n}(\tau I-A)^{-1}\,d\tau\Bigg\}.\end{aligned} \tag{7.16}$$

Let $d$ be the Euclidean distance in $\mathbb{C}$. If $\zeta\in K$, then $|\zeta|>r_1$ and, for any $|\tau|=r_1$,

$$\begin{aligned}|\zeta-\tau^n|&\geq\big||\zeta|-|\tau|^n\big|=|\zeta|-|\tau|^n\\&\geq|\zeta|-|\tau|\geq d(\zeta,\overline{\Delta(0,r_1)})\geq d(K,U).\end{aligned} \tag{7.17}$$

If $\tau\in\overline{\Delta(1,r_3)}$, then

$$|\zeta-\tau^n|\geq d(\zeta,\overline{\Delta(1,r_2)})\geq d(K,U). \tag{7.18}$$

Thus, (7.16) yields

$$\|(\zeta I - A^n)^{-1}\| \leq \frac{2}{d(K,U)} \sup\{\|(\tau I - A)^{-1}\| : \tau \in U\} \tag{7.19}$$

for all $\zeta \in K$ and all $n \geq n_1$, proving thereby (7.13).

(b) Let

$$k_2 = \sup\{\|\zeta I - P\| : \zeta \in K\}. \tag{7.20}$$

For $\zeta \in K$,

$$\begin{aligned} \|(\zeta I - A^n)^{-1} - (\zeta I - P)^{-1}\| &= \|(\zeta I - A^n)^{-1}(\zeta I - P - (\zeta I - A^n))(\zeta I - P)^{-1}\| \\ &= \|(\zeta I - A^n)^{-1}(A^n - P)(\zeta I - P)^{-1}\| \\ &\leq \|(\zeta I - A^n)^{-1}\|\|A^n - P\|\|(\zeta I - P)^{-1}\| \\ &\leq k_1 k_2 \|A^n - P\|. \end{aligned} \tag{7.21}$$

In the following, $K = \partial\Delta(1,r)$, and $r \in (0,1)$ will be chosen in such a way that

$$\overline{\Delta(1,r)} \cap \sigma(A) = \emptyset. \tag{7.22}$$

Let

$$(\zeta I - A^n)^{-1} = \sum_{\nu=-\infty}^{+\infty} (\zeta - 1)^\nu A^n{}_\nu, \tag{7.23}$$

with $A^n{}_\nu \in \mathcal{L}(\mathcal{E})$, be the Laurent expansion of $(\zeta I - A^n)^{-1}$ at 1.

Let $P_\nu \in \mathcal{L}(\mathcal{E})$ be the coefficient of $(\zeta - 1)^\nu$ in the Laurent expansion of $(\zeta I - P)^{-1}$ at 1.

Then, by (7.21), for $\nu \geq 1$,

$$\begin{aligned} \|A^n{}_{-\nu} - P_{-\nu}\| &\leq \frac{1}{2\pi}\left\| \int_{|\zeta-1|=r} (\zeta - 1)^{\nu-1}\left((\zeta I - A^n)^{-1} - (\zeta I - P)^{-1}\right) d\zeta \right\| \\ &\leq \frac{1}{2\pi}\int_{|\zeta-1|=r} |\zeta - 1|^{\nu-1}\|(\zeta I - A^n)^{-1} - (\zeta I - P)^{-1}\| d\zeta \\ &\leq r^{\nu-1} k_1 k_2 \|A^n - P\|, \end{aligned} \tag{7.24}$$

and therefore

$$\lim_{n\to+\infty} \|A^n{}_{-\nu} - P_{-\nu}\| = 0 \tag{7.25}$$

for $\nu = 1, 2, \ldots$. But, since

$$(\zeta I - P)^{-1} = \frac{1}{\zeta - 1} P + \frac{1}{\zeta}(I - P), \tag{7.26}$$

$P_{-1} = P$ and $P_{-\nu} = 0$ for $\nu \geq 2$. Hence,

$$\lim_{n \to +\infty} ||A^n{}_{-1} - P|| = 0, \tag{7.27}$$

$$\lim_{n \to +\infty} ||A^n{}_{-\nu}|| = 0 \tag{7.28}$$

for $\nu = 2, 3, \ldots$.

(c) Choose $r_1$ and $r_2$ in such a way that $0 < r_1 < r_1 + r_2 < 1$, and $\sigma(A) \cap \Delta \subset \Delta(0, r_1)$. For any $n \geq 1$, choose $r_3$ such that $0 < r_3 < r_2$ and that the image of $\overline{\Delta(1, r_3)}$ by the map $\zeta \mapsto \zeta^n$ be contained in $\Delta(1, r_2)$.

For any $\nu \geq 1$, Dunford's integral and Fubini's theorem yield

$$\begin{aligned} A^n{}_{-\nu} &= \frac{1}{(2\pi i)^2} \int_{|\zeta - 1| = r_2} (\zeta - 1)^{\nu - 1} \\ &\qquad \times \left\{ \int_{|\tau| = r_1} \frac{1}{\zeta - \tau^n} (\tau I - A)^{-1} d\tau \right. \\ &\qquad \left. + \int_{|\tau - 1| = r_3} \frac{1}{\zeta - \tau^n} (\tau I - A)^{-1} d\tau \right\} d\zeta \\ &= \frac{1}{(2\pi i)^2} \left\{ \int_{|\tau| = r_1} \left( \int_{|\zeta - 1| = r_2} \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n} d\zeta \right) (\tau I - A)^{-1} d\tau \right. \\ &\qquad \left. + \int_{|\tau - 1| = r_3} \left( \int_{|\zeta - 1| = r_2} \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n} d\zeta \right) (\tau I - A)^{-1} d\tau \right\}. \end{aligned} \tag{7.29}$$

For $|\tau| = r_1$, the function

$$\zeta \longmapsto \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n} \tag{7.30}$$

is holomorphic in a neighbourhood of $\overline{\Delta(1, r_2)}$. Hence, by the Cauchy integral theorem,

$$\int_{|\zeta - 1| = r_2} \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n} d\zeta = 0. \tag{7.31}$$

On the other hand, the Cauchy integral formula yields

$$\frac{1}{2\pi i} \int_{|\zeta - 1| = r_3} \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n} d\zeta = (\tau^n - 1)^{\nu - 1}. \tag{7.32}$$

Hence, for $\nu \geq 1$,

$$A^n{}_{-\nu} = \frac{1}{2\pi i}\int_{|\tau-1|=r_3} (\tau^n - 1)^{\nu-1}(\tau I - A)^{-1}\,d\tau = (A^n - I)^{\nu-1}A_{-1}, \tag{7.33}$$

and (7.27) yields

$$A^n{}_{-1} = P \quad \text{for } n = 1, 2, \dots. \tag{7.34}$$

Since

$$A_{-\nu} = (A - I)^{\nu-1}P \quad \forall \nu = 1, 2, \dots, \tag{7.35}$$

(7.9) yields $A_{-\nu} = 0$ for $\nu = 2, 3, \dots$. □

A part of Proposition 7.2 follows also from the following lemma.

Lemma 7.3. *If (7.4) holds, if $\partial\Delta \cap \sigma(A) \ni e^{i\theta}$ for some $\theta \in \mathbb{R}$, and if $e^{i\theta}$ is an isolated point of $\sigma(A)$ which is a pole of the resolvent function $(\bullet I - A)^{-1}$, then $e^{i\theta}$ is a pole of order one.*

*Proof.* There is no restriction in assuming $e^{i\theta} = 1$. If $n > 0$ is the order of the pole, the resolvent function is represented in a neighbourhood of 1 by the Laurent series

$$(\zeta I - A)^{-1} = \sum_{\nu=-n}^{+\infty} (\zeta - 1)^\nu A_\nu, \tag{7.36}$$

and the range $\mathrm{Ran}(A_{-1})$ of $A_{-1}$ is related to $\ker(I - A)^m$ by

$$\mathrm{Ran}\,(A_{-1}) = \ker(I - A)^m \quad \text{for } m = n, n+1, \dots. \tag{7.37}$$

Being

$$\ker(I - A) \subset \ker(I - A)^2 \subset \cdots, \tag{7.38}$$

(7.37) holds for $m = 1$ if, and only if,

$$Ax = x \quad \forall x \in \mathrm{Ran}\,(A_{-1}). \tag{7.39}$$

To see that this latter condition actually holds, assume that there is some $y \in \mathrm{Ran}(A_{-1})$ such that $(A - I)y \neq 0$, and let $\lambda$ be a continuous linear form on $\mathscr{E}$ such that

$$\langle (A - I)y, \lambda\rangle \neq 0. \tag{7.40}$$

By (7.35), $(A-I)^n y = 0$, and therefore

$$A^N y = (A-I+I)^N y = \sum_{p=0}^{N} \binom{N}{p} (A-I)^p y = \sum_{p=0}^{n-1} \binom{N}{p} (A-I)^p y \tag{7.41}$$

for all $N \geq n$. Thus

$$\langle A^N y, \lambda \rangle = \sum_{p=0}^{n-1} \binom{N}{p} \langle (A-I)^p y, \lambda \rangle, \tag{7.42}$$

and therefore

$$\lim_{N\to+\infty} |\langle A^N y, \lambda\rangle| = \infty, \tag{7.43}$$

contradicting the fact that, in view of (7.4),

$$|\langle A^N y, \lambda\rangle| \leq \|\lambda\| \|A^N\| \|y\| \leq k\|\lambda\| \|y\| \tag{7.44}$$

for all $N > 0$.

Thus (7.39) holds, and (7.35) yields $A_{-\nu} = 0$ for $\nu = 2, 3, \ldots$. □

If the hypotheses of Lemma 7.3 are satisfied with $e^{i\theta} = 1$, $\sigma(A)$ splits as the union of the two disjoint spectral sets $\{1\}$ and $\sigma(A) \cap \Delta$. The corresponding spectral projectors are $P = A_{-1}$ and $I - P$; moreover, $(A-I)P = 0$.

Setting

$$C = A(I-P) = A - P, \tag{7.45}$$

then $\sigma(C) = (\sigma(A) \cap \Delta) \cup \{0\}$.

Since $CP = PC$, then

$$A^n = P + C^n \quad \text{for } n = 1, 2, \ldots. \tag{7.46}$$

Being $\rho(C) < 1$, there exist $\epsilon \in (0,1)$ and $n_0 \geq 1$ such that

$$\|C^n\|^{1/n} \leq 1 - \epsilon, \tag{7.47}$$

that is,

$$\|C^n\| \leq (1-\epsilon)^n \quad \forall n \geq n_0, \tag{7.48}$$

and therefore, by (7.46), (7.3) holds.

In conclusion, the following proposition has been established.

PROPOSITION 7.4. *If (7.4) and (7.6) hold and if* 1 *is an isolated point of* $\sigma(A)$ *which is also a pole of the resolvent function* $(\bullet I - A)^{-1}$, *then (7.3) holds, where* $P$ *is the spectral projector associated to the spectral set* $\{1\}$ *in the spectral resolution of* $A$.

It will be shown in Section 8 that, if (7.1) holds, Theorem 7.1 can be inverted.

## 8. Sufficient conditions for the convergence of iterates

Let $D$ be a bounded domain in the complex Banach space $\mathscr{E}$, and let $f : D \to D$ be a holomorphic map fixing a point $x_0 \in D$. As was noticed already, since $D$ is bounded, $\sigma(df(x_0)) \subset \overline{\Delta}$ (see [5]).

THEOREM 8.1. *If* $\sigma(df(x_0)) \subset \Delta$, *the sequence* $\{f^n\}$ *of the iterates of* $f$ *converges to the constant map* $x \mapsto x_0$ *for the topology of local uniform convergence on* $D$.

Obviously, there is no restriction in assuming $D$ to be a bounded, connected, open neighbourhood of $x_0 = 0$.

Let $R > 0$ be such that

$$D \subset B(0,R). \tag{8.1}$$

Let

$$f(x) = Ax + A_2(x,x) + \cdots + A_N(x,\dots,x) + \cdots \tag{8.2}$$

be the power series expansion of $f$ in 0, where $A \in \mathscr{L}(\mathscr{E})$ and $A_N$ is a continuous, homogeneous, polynomial of degree $N = 2,3,\dots$ on $\mathscr{E}$, with values in $\mathscr{E}$, that is, the restriction to the diagonal of $\mathscr{E} \times \cdots \times \mathscr{E}$ ($n$ times) of a continuous $N$-linear symmetric map, which will be denoted by the same symbol $A_N$, of $\mathscr{E} \times \cdots \times \mathscr{E}$ into $\mathscr{E}$. If

$$r = \inf\{\|y\| : y \notin D\}, \tag{8.3}$$

the power series (8.2) converges uniformly on $\overline{B(0,s)}$ whenever $0 < s < r$.

The $n$th iterate $f^n$ ($n = 2,3,\dots$) of $f$ has a power series expansion in 0 which converges uniformly on $B(0,s)$ and is expressed by

$$f^n(x) = A^n x + C_2^{(n)}(x,x) + \cdots + C_N^{(n)}(x,\dots,x) + \cdots, \tag{8.4}$$

where $C_N^{(n)}$ is a continuous homogeneous polynomial of degree $N = 2,3,\dots$ on $\mathscr{E}$ with values in $\mathscr{E}$.

An induction argument on $n$ will show now that, for all $x \in \mathscr{E}$, $N = 2, 3, \ldots$ and $n = 2, 3, \ldots$,

$$\begin{aligned} C_N^{(n)}(x,\ldots,x) &= \sum_{q=0}^{n-1} A^q\big(A_N\big(A^{n-q-1}x,\ldots,A^{n-q-1}x\big)\big) \\ &\quad + \sum_{m=1}^{n-1}\sum_{q=2}^{N-1}\sum_{(q,N)} C_q^{(m)}\big(A_{p_1}\big(A^{n-m-1}x,\ldots,A^{n-m-1}x\big),\ldots, \\ &\qquad\qquad A_{p_q}\big(A^{n-m-1}x,\ldots,A^{n-m-1}x\big)\big), \end{aligned} \tag{8.5}$$

where $x \in \mathscr{E}$, $C_q^{(1)} = A_q$, and the sum $\sum_{(q,N)}$ is extended to all positive integers $p_1, \ldots, p_q$ such that $p_1 + \cdots + p_q = N$.

First of all, a simple induction on $n$ yields

$$C_2^{(n)}(x,x) = \sum_{q=0}^{n-1} A^q\big(A_2\big(A^{n-q-1}x, A^{n-q-1}x\big)\big), \tag{8.6}$$

which coincides with (8.5) when $N = 2$.

Assuming (8.5) to hold, then

$$\begin{aligned} C_N^{(n+1)}(x,\ldots,x) &= A^n\big(A_N(x,\ldots,x)\big) + \sum_{q=2}^{N}\sum_{(q,N)} C_q^{(n)}\big(A_{p_1}(x,\ldots,x),\ldots,A_{p_q}(x,\ldots,x)\big) \\ &= A^n\big(A_N(x,\ldots,x)\big) + C_N^{(n)}(Ax,\ldots,Ax) \\ &\quad + \sum_{q=2}^{N-1}\sum_{(q,N)} C_q^{(n)}\big(A_{p_1}(x,\ldots,x),\ldots,A_{p_q}(x,\ldots,x)\big) \\ &= A^n\big(A_N(x,\ldots,x)\big) + \sum_{q=0}^{n-1} A^q\big(A_N\big(A^{n-q-1}Ax,\ldots,A^{n-q-1}Ax\big)\big) \\ &\quad + \sum_{m=1}^{n-1}\sum_{q=2}^{N-1}\sum_{(q,N)} C_q^{(m)}\big(A_{p_1}\big(A^{n+1-m-1}x,\ldots,A^{n+1-m-1}x\big),\ldots, \\ &\qquad\qquad A_{p_q}\big(A^{n+1-m-1}x,\ldots,A^{n+1-m-1}x\big)\big) \\ &\quad + \sum_{q=2}^{N-1}\sum_{(q,N)} C_q^{(n)}\big(A_{p_1}(x,\ldots,x),\ldots,A_{p_q}(x,\ldots,x)\big) \\ &= \sum_{q=0}^{n+1-1} A^q\big(A_N\big(A^{n+1-q-1}Ax,\ldots,A^{n+1-q-1}Ax\big)\big) \\ &\quad + \sum_{m=1}^{n+1-1}\sum_{q=2}^{N-1}\sum_{(q,N)} C_q^{(m)}\big(A_{p_1}\big(A^{n+1-m-1}x,\ldots,A^{n+1-m-1}x\big),\ldots, \\ &\qquad\qquad A_{p_q}\big(A^{n+1-m-1}x,\ldots,A^{n+1-m-1}x\big)\big). \end{aligned} \tag{8.7}$$

This inductive argument shows that (8.5) holds for $N = 2, 3, \ldots$ and $n = 2, 3, \ldots$.

Lemma 8.2. *If* $\|A\| < 1$, *for* $N = 2, 3, \ldots$, *there is a positive constant* $c_N$ *such that*

$$||C_N^{(n)}|| \leq c_N \|A\|^{n-N+1} \quad \forall n \geq N-1. \tag{8.8}$$

Here, $\|C_N^{(n)}\|$ is the norm of the continuous polynomial $x \mapsto C_N^{(n)}(x, \ldots, x)$

$$||C_N^{(n)}|| = \sup\{||C_N^{(n)}(x, \ldots, x)|| : \|x\| \leq 1\}, \tag{8.9}$$

and is related to the norm

$$|||C_N^{(n)}||| = \sup\{||C_N^{(n)}(x, \ldots, y)|| : \|x\| \leq 1, \ldots, \|y\| \leq 1\} \tag{8.10}$$

of the continuous, symmetric $N$-linear map $(x, \ldots, y) \mapsto C_N^{(n)}(x, \ldots, y)$ by the inequalities (see, e.g., [5])

$$||C_N^{(n)}|| \leq |||C_N^{(n)}||| \leq \frac{N^N}{N!}||C_N^{(n)}||. \tag{8.11}$$

*Proof of Lemma 8.2.* By (8.5),

$$C_2^{(n)}(x, x) = \sum_{q=0}^{n-1} A^q\left(A_2\left(A^{n-q-1}x, A^{n-q-1}x\right)\right), \tag{8.12}$$

and therefore

$$\begin{aligned} ||C_2^{(n)}(x, x)|| &\leq ||A_2|| \sum_{q=0}^{n-1} \|A\|^{2n-2q-2+q} \|x\|^2 \\ &= ||A_2|| \|A\|^{n-1} \sum_{q=0}^{n-1} \|A\|^{n-q+1} \|x\|^2 \\ &= ||A_2|| \|A\|^{n-1} \frac{1 - \|A\|^n}{1 - \|A\|} \|x\|^2 \\ &\leq ||A_2|| \frac{\|A\|^{n-1}}{1 - \|A\|} \|x\|^2. \end{aligned} \tag{8.13}$$

Assuming the lemma to hold for $q = 2, 3, \dots, N-1$, and choosing $n \ge N-1$, then

$$\begin{aligned}
&||C_N^{(n)}(x,\dots,x)|| \\
&\quad \le \Bigg\{ ||A_N|| \sum_{q=0}^{n-1} \|A\|^q \|A\|^{N(n-q-1)} \\
&\qquad + \sum_{m=1}^{n-1} \sum_{q=2}^{N-1} \sum_{(q,N)} \frac{q^q}{q!} ||C_q^{(m)}|| ||A_{p_1}|| \cdots ||A_{p_q}|| \|A\|^{N(n-m-1)} \Bigg\} \|x\|^N \\
&\quad \le \Bigg\{ ||A_N|| \|A\|^{n-1} \frac{1-\|A\|^{n(N-1)}}{1-\|A\|^{N-1}} \\
&\qquad + \sum_{m=1}^{n-1} \sum_{q=2}^{N-1} \sum_{(q,N)} \frac{q^q}{q!} c_q \|A\|^{m-q+1} ||A_{p_1}|| \cdots ||A_{p_q}|| \|A\|^{N(n-m-1)} \Bigg\} \|x\|^N \\
&\quad = \Bigg\{ ||A_N|| \|A\|^{n-1} \frac{1-\|A\|^{n(N-1)}}{1-\|A\|^{N-1}} + \frac{1-\|A\|^{(n-1)(N-1)}}{1-\|A\|^{N-1}} \\
&\qquad + \sum_{q=2}^{N-1} c_q \frac{q^q}{q!} \|A\|^{n-q} \sum_{(q,N)} ||A_{p_1}|| \cdots ||A_{p_q}|| \Bigg\} \|x\|^N \\
&\quad \le \Bigg\{ ||A_N|| \|A\|^{n-1} + \sum_{q=2}^{N-1} \left( \frac{q^q}{q!} c_q \sum_{(q,N)} ||A_{p_1}|| \cdots ||A_{p_q}|| \right) \|A\|^{n-q} \Bigg\} \frac{\|x\|^N}{1-\|A\|^{N-1}}.
\end{aligned} \tag{8.14}$$

Since $\|A\| < 1$, then

$$\|A\|^{n-q} \le \|A\|^{n-N+1} \quad \text{for } q = 1, 2, \dots, N-1. \tag{8.15}$$

Hence,

$$||C_N^{(n)}(x,\dots,x)|| \le c_N \|A\|^{n-N+1} \|x\|^N, \tag{8.16}$$

with

$$c_N = ||A_N|| + \sum_{q=2}^{N-1} \left( \frac{q^q}{q!} c_q \sum_{(q,N)} ||A_{p_1}|| \cdots ||A_{p_q}|| \right) \frac{1}{1-\|A\|^{N-1}}. \tag{8.17}$$

□

In view of (8.1), the Cauchy inequalities yield

$$||C_N^{(n)}|| \le \frac{R}{r^n} \quad \forall N \ge 1,\ n \ge 1. \tag{8.18}$$

Hence, if $s \in (0,1)$ is sufficiently small, in such a way that $B(0,s) \subset D$, and if $x \in B(0,s/2)$, $n \geq 1$, and $N_0 \geq 2$,

$$\begin{aligned}||f^n(x)|| &\leq ||A^n x|| + ||C_2^{(n)}(x,x)|| + \cdots + ||C_{N_0}^{(n)}(x,\ldots,x)|| + R \sum_{N=N_0+1}^{+\infty} \left(\frac{\|x\|}{s}\right)^N \\ &\leq ||A^n x|| + ||C_2^{(n)}(x,x)|| + \cdots + ||C_{N_0}^{(n)}(x,\ldots,x)|| \\ &\quad + R\left(\frac{\|x\|}{s}\right)^{N_0+1} \frac{1}{1-\|x\|/s} \\ &\leq \|A\|^n \|x\| + c_2 \|A\|^{n-1} \|x\|^2 + \cdots + c_{N_0} \|A\|^{n-N_0+1} \|x\|^{N_0} + \frac{R}{2^{N_0}}.\end{aligned} \tag{8.19}$$

Let $c = \max\{1, c_2, \ldots, c_{N_0}\}$. Then

$$\begin{aligned}||f^n(x)|| &\leq \|A\|^{n-N_0+1}(\|A\|^{N_0-1} + \|A\|^{N_0-2} + \cdots + 1)s + \frac{R}{2^{N_0}} \\ &\leq c\frac{\|A\|^{n-N_0+1}}{1-\|A\|}s + \frac{R}{2^{N_0}}.\end{aligned} \tag{8.20}$$

For $\epsilon > 0$, choosing $N_0 \gg 0$ and $n_0 \gg 0$ in such a way that

$$\frac{Rr^{N_0+1}}{1-r} < \frac{\epsilon}{2}, \qquad c\frac{\|A\|^{n-N_0+1}}{1-\|A\|}r < \frac{\epsilon}{2} \quad \forall n \geq n_0, \tag{8.21}$$

then

$$||f^n(x)|| < \epsilon \quad \forall x \in B\left(0, \frac{s}{2}\right), \ \forall n \geq n_0. \tag{8.22}$$

That proves the following lemma.

Lemma 8.3. *If* $\|A\| < 1$, *for any* $\epsilon > 0$ *and any* $s \in (0,1)$ *such that* $B(0,s) \subset D$, *there is* $n_0 \geq 1$ *such that (8.22) holds.*

Proposition 8.4. *If* $\sigma(A) \subset \Delta$, *for any* $\epsilon > 0$ *and any* $s \in (0,1)$ *such that* $B(0,s) \subset D$, *there is* $n_0 \geq 1$ *such that (8.22) holds.*

*Proof.* There is $n_1 \geq 1$ such that $\|A^{n_1}\| < 1$. By Lemma 8.3, there is $n_2 \geq 1$ such that

$$||f^{n_1 n}(x)|| < \epsilon \quad \forall x \in B\left(0, \frac{s}{2}\right), \ n \geq n_2. \tag{8.23}$$

Let $\omega$ be the Poincaré distance in $\Delta$. Since holomorphic maps contract the Kobayashi distance, for $m \geq 1$, $n \geq n_2$, and $x \in B(0, s/2)$, then

$$\begin{aligned}\omega\left(0, \frac{\|f^{n_1 n+m}(x)\|}{R}\right) &= k_{B(0,R)}(0, f^{n_1 n+m}(x)) \leq k_D(0, f^{n_1 n+m}(x)) \\ &\leq k_D(0, f^{n_1 n}(x)) \leq k_{B(0,s)}(0, f^{n_1 n}(x)) \\ &= \omega\left(0, \frac{\|f^{n_1 n}(x)\|}{s}\right) < \omega\left(0, \frac{\epsilon}{s}\right).\end{aligned} \tag{8.24}$$

Thus, the sequence $\{f^n\}$ converges to 0 uniformly on $B(0, s/2)$, and therefore converges to zero everywhere on $D$ by Vitali's theorem [8, Theorem 3.18.1]. The convergence being uniform on $B(0, s/2)$, the sequence $\{f^n\}$ tends to zero for the topology of local uniform convergence on $D$ [5, page 104].

The proof of Theorem 8.1 is complete. □

As in Section 6, let $\phi$ be a map of $\mathbb{R}_+^* \times D$ into $D$ satisfying (2.3) and (2.4) for all $t, t_1, t_2 \in \mathbb{R}_+^*$, and such that the map $t \mapsto \phi_t(x)$ is continuous on $\mathbb{R}_+^*$ for all $x \in \mathscr{E}$.

Let $\phi_t(x_0) = x_0$ for all $t > 0$ and for some point $x_0$ in the bounded domain $D \subset \mathscr{E}$.

If $\sigma(d\phi_{t_0}(x_0)) \subset \Delta$ for some $t_0 > 0$, Theorem 8.1 applied to the function $f = \phi_{t_0}$, implies that, as $n \to +\infty$, the sequence $\{\phi_{nt_0} : n = 1, 2, \ldots\}$ converges to the constant map $x \mapsto x_0$ for the topology of local uniform convergence.

Let $r > 0$ be such that

$$B_{k_D}(x_0, r) \Subset D. \tag{8.25}$$

Since the distances $\| \ \|$ and $k_D$ are equivalent on $B_{k_D}(x_0, r)$, for any $\epsilon > 0$, there is $n_0 \geq 1$ such that

$$\phi_{n_0 t_0}(B_{k_D}(x_0, r)) \subset B_{k_D}(x_0, \epsilon), \tag{8.26}$$

whenever $n \geq n_0$. For all $t > n_0 t_0$,

$$\begin{aligned}\phi_t(B_{k_D}(x_0, r)) &= \phi_{t - n_0 t_0 + n_0 t_0}(B_{k_D}(x_0, r)) = \phi_{t - n_0 t_0}(\phi_{n_0 t_0}(B_{k_D}(x_0, r))) \\ &\subset \phi_{t - n_0 t_0}(B_{k_D}(x_0, \epsilon)) \subset B_{k_D}(x_0, \epsilon)\end{aligned} \tag{8.27}$$

because holomorphic maps contract the Kobayashi distance.

Thus the following theorem holds.

Theorem 8.5. *If $\phi : \mathbb{R}_+^* \times D \to D$ fixes a point $x_0 \in D$ of the bounded domain $D$, and if $\sigma((d\phi_{t_0})(x_0)) \subset \Delta$ for some $t_0 > 0$, then, as $t \to +\infty$, $\phi_t$ converges to the constant map $x \mapsto x_0$ for the topology of local uniform convergence.*

## 9. Fixed points and idempotents

As at the beginning of Section 8, let $D$ be a bounded domain in $\mathscr{E}$ and let $f : D \to D$ be a holomorphic map fixing a point $x_0 \in D$.

If $f$ is an idempotent of the semigroup $\mathrm{Hol}(D)$, a direct inspection of the power series expansion of $f$ at $x_0$ shows that $df(x_0)$ is an idempotent of $\mathscr{L}(\mathscr{E})$.

In this section, we show that, if the geometry of $D$ satisfies suitable conditions, the fact that $df(x_0)$ is an idempotent of $\mathscr{L}(\mathscr{E})$ implies that the iterates of $f$ converge for the topology of local uniform convergence to an idempotent of $\mathrm{Hol}(D)$.

As before, let $D$ be a bounded, open, connected neighbourhood of 0, and let $f(0) = 0$. Let $f$ be expressed in $B(0, r)$ by the power series (8.2) (and $r$ is given by (8.3)).

Let $A = df(0)$ be an idempotent of $\mathscr{L}(\mathscr{E})$.

Since $A^2 = A$, (8.12) reads, for $n \geq 2$,

$$C_2^{(n)}(x,x) = AA_2(x,x) + A_2(Ax,Ax) + (n-2)AA_2(Ax,Ax) \tag{9.1}$$

for all $x \in \mathscr{E}$. If $AA_2(Ax,Ax) \not\equiv 0$, there are $y \in \mathscr{E}$ and $\lambda \in \mathscr{E}'$ (the topological dual of $\mathscr{E}$) such that

$$\langle AA_2(Ay,Ay), \lambda \rangle \neq 0. \tag{9.2}$$

The Cauchy inequalities (8.18) yield, for $N = 2$ and $n = 1, 2, \ldots,$

$$\left| \langle AA_2(y,y) + A_2(Ay,Ay) + (n-2)AA_2(Ay,Ay), \lambda \rangle \right| \leq \frac{R}{r^2} \|y\|^2 |\lambda| \tag{9.3}$$

for all $n = 2, 3, \ldots,$ contradicting (9.2). Hence, $AA_2(Ax,Ax) = 0$ for all $x \in \mathscr{E}$, and therefore

$$C_2^{(n)}(x,x) = AA_2(x,x) + A_2(Ax,Ax) \tag{9.4}$$

for all $n = 2, 3, \ldots,$ and all $x \in \mathscr{E}$.

Thus, $C_2^n(x,x)$ does not depend on $n \geq 2$. Proceeding by induction on $N$, we show that $C_N^{(n)}(x, \ldots, x)$ is independent of $n \geq N$ for all $N$.

Assuming this fact to hold for $C_2, \ldots, C_N$, then

$$f^N(x) = Ax + C_2(x,x) + \cdots + C_N(x, \ldots, x) + F_{N+1}(x, \ldots, x) + \cdots \tag{9.5}$$

for all $x \in B(0,r)$, where $F_{N+1}$ is a homogeneous, continuous polynomial of degree $N+1$ from $\mathscr{E}$ to $\mathscr{E}$.

Then, setting $A_1 = A$,

$$
\begin{aligned}
f^{N+1}(x) = Ax &+ \sum_{q=2}^{N} C_q(x,\dots,x) + AA_{N+1}(x,\dots,x) \\
&+ \sum_{q=2}^{N} \sum_{(q,N)} C_q\big(A_{p_1}(x,\dots,x),\dots,A_{p_q}(x,\dots,x)\big) \\
&+ F_{N+1}(Ax,\dots,Ax) + \cdots, \\
f^{N+2}(x) = Ax &+ \sum_{q=2}^{N} C_q(x,\dots,x) + AA_{N+1}(x,\dots,x) \\
&+ \sum_{q=2}^{N} \sum_{(q,N)} C_q\big(A_{p_1}(x,\dots,x),\dots,A_{p_q}(x,\dots,x)\big) \\
&+ F_{N+1}(Ax,\dots,Ax) + AA_{N+1}(Ax,\dots,Ax) \\
&+ \sum_{q=2}^{N} \sum_{(q,N)} C_q\big(A_{p_1}(Ax,\dots,Ax),\dots,A_{p_q}(Ax,\dots,Ax)\big) + \cdots, \\
&\vdots \\
f^{N+\ell}(x) = Ax &+ \sum_{q=2}^{N} C_q(x,\dots,x) + AA_{N+1}(x,\dots,x) \\
&+ \sum_{q=2}^{N} \sum_{(q,N)} C_q\big(A_{p_1}(x,\dots,x),\dots,A_{p_q}(x,\dots,x)\big) + F_{N+1}(Ax,\dots,Ax) \\
&+ (\ell-1)\Bigg[AA_{N+1}(Ax,\dots,Ax) \\
&\qquad + \sum_{q=2}^{N} \sum_{(q,N)} C_q\big(A_{p_1}(Ax,\dots,Ax),\dots,A_{p_q}(Ax,\dots,Ax)\big)\Bigg] \\
&+ \cdots
\end{aligned}
\tag{9.6}
$$

for all $x \in B(0,r)$ and all $\ell = 2, 3, \dots$.

A similar argument to that devised for $C_2$ implies that

$$
AA_{N+1}(Ax,\dots,Ax) + \sum_{q=2}^{N} \sum_{(q,N)} C_q\big(A_{p_1}(Ax,\dots,Ax),\dots,A_{p_q}(Ax,\dots,Ax)\big) = 0
\tag{9.7}
$$

for all $x \in \mathscr{E}$.

The inductive argument is now complete, showing that

$$f^n(x) = Ax + C_2(x,x) + \cdots + C_N(x,\ldots,x) + O(\|x\|^{N+1}) \tag{9.8}$$

for all $x \in B(0,r)$ and all $n \geq N = 1, 2, \ldots$, with

$$C_{N+1}(x,\ldots,x) = AA_{N+1}(x,\ldots,x) + \sum_{q=2}^{N} \sum_{(q,N)} C_q(A_{p_1}(x,\ldots,x),\ldots,A_{p_q}(x,\ldots,x)) + F_{N+1}(Ax,\ldots,Ax). \tag{9.9}$$

Since, by the Cauchy inequalities,

$$||(d^N f^n)(0)|| \leq \frac{R}{r^N} N! \tag{9.10}$$

for all $N \geq 0$, $n > 0$, and therefore

$$\limsup_N \left\|\frac{1}{n!}(d^N f^n)(0)\right\|^{1/N} \leq \frac{1}{r}, \tag{9.11}$$

the Cauchy-Hadamard formula implies that the power series

$$Ax + \sum_{N=2}^{+\infty} B_N(x,\ldots,x) \tag{9.12}$$

converges uniformly on $\overline{B(0,s)}$ whenever $0 < s < r$. Let $g$ be the holomorphic function on $B(0,r)$ represented by this power series.

By the Cauchy inequalities, if $\|x\| \leq s < r$,

$$\begin{aligned} ||g(x) - f^n(x)|| &\leq \sum_{N=n+1}^{+\infty} ||C_N(x,\ldots,x) - C_N^{(n)}(x,\ldots,x)|| \\ &\leq \sum_{N=n+1}^{+\infty} (||C_N(x,\ldots,x)|| + ||C_N^{(n)}(x,\ldots,x)||) \\ &\leq \sum_{N=n+1}^{+\infty} (||C_N|| + ||C_N^{(n)}||)\,\|x\|^N \\ &\leq 2R \sum_{N=n+1}^{+\infty} \left(\frac{\|x\|}{r}\right)^N \\ &\leq 2R \sum_{N=n+1}^{+\infty} \left(\frac{s}{r}\right)^N \\ &= 2R\left(\frac{s}{r}\right)^{N+1} \frac{1}{1 - s/r}. \end{aligned} \tag{9.13}$$

Hence, the sequence $\{f^n\}$ converges to $g$ uniformly on $\overline{B(0,s)}$. By Vitali's theorem [8, Theorem 3.18.1], the sequence $\{f^n(x)\}$ converges for all $x \in D$, and the limit is a holomorphic map $h : D \to \mathscr{E}$. Clearly, $h|_{B(0,r)} = g$.

The convergence being uniform on $\overline{B(0,s)}$, the sequence $\{f^n\}$ tends to $h$ for the topology of local uniform convergence.

In conclusion, the following theorem has been established.

Theorem 9.1. *Let $f$ be a holomorphic map of a bounded domain $D$ into itself. If $f$ fixes a point $x_0 \in D$, and if $df(x_0)$ is an idempotent of $\mathscr{L}(\mathscr{E})$, the sequence $\{f^n\}$ converges for the topology of local uniform convergence to a holomorphic map $h : D \to \mathscr{E}$.*

Obviously, $h(D) \subset \overline{D}$, $h(x_0) = x_0$,

$$dh(x_0) = df(x_0), \tag{9.14}$$

and $h \circ f = h$. Furthermore,

$$f \circ h = h, \tag{9.15}$$

and therefore $\operatorname{Fix} f = h(D)$, provided that $h(D) \subset D$. This latter condition is fulfilled if $D$ satisfies the following principle.

*Maximum principle.* Whenever a holomorphic function $h : D \to \mathscr{E}$ is such that $h(D) \subset \overline{D}$ and $h(D) \cap \partial D \neq \varnothing$, then $h(D) \subset \partial D$.

*Example 9.2.* If the bounded domain $D$ is convex, its support function is plurisubharmonic [14]. Thus, $D$ satisfies the maximum principle.

Summing up, the following proposition holds.

Proposition 9.3. *Under the hypotheses of Theorem 9.1, and if moreover $D$ satisfies the maximum principle, $h$ is an idempotent of the semigroup of all holomorphic maps of $D$ into $D$ which commute with $f$ and is such that $h(D) = \operatorname{Fix} f$.*

If $df(x_0)$ is an idempotent of $\mathscr{L}(\mathscr{E})$, then

$$\sigma(df(x_0)) = p\sigma(df(x_0)) \subset \{0,1\}, \tag{9.16}$$

$$\sigma(df(x_0)) = \{0\} \Longrightarrow df(x_0) = 0, \tag{9.17}$$

$$\sigma(df(x_0)) = \{1\} \Longrightarrow df(x_0) = I. \tag{9.18}$$

Since $D$ is bounded, by Cartan's identity theorem, (9.18) holds if, and only if, $f = \mathrm{id}$.

Theorem 8.1 and (9.17) yield the following proposition.

Proposition 9.4. *If $D$ is bounded, if $f(x_0) = x_0$, and if $df(x_0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$ with $\sigma(df(x_0)) = \{0\}$, then the sequence $\{f^n\}$ converges to the constant map $x \mapsto x_0$ for the topology of local uniform convergence on $D$.*

Theorem 9.5 [16]. *Let $D$ be a bounded, open, convex neighbourhood of 0, and let $f \in \mathrm{Hol}(D)$ be such that $f(0) = 0$ and $df(0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$. If $\partial D \cap \mathrm{Ran}\, df(0)$ consists of complex extreme points of $\overline{D}$, then $h(D) = D \cap \mathrm{Ran}\, df(0)$.*

*Proof.* Let $A = df(0)$ and $\mathcal{F} = \ker(I-A) = \mathrm{Ran}\, A$. As a consequence of the strong maximum principle [15, Corollary 5.4], if $x \in \mathcal{F} \cap D$, $f(x) = Ax = x$, and with the same notations of (8.2),

$$A_2(x,x) = A_3(x,x,x) = \cdots = 0 \quad \forall x \in \mathcal{F}. \tag{9.19}$$

Therefore,

$$A_2(Ax,Ax) = A_3(Ax,Ax,Ax) = \cdots = 0 \quad \forall x \in \mathcal{E}. \tag{9.20}$$

Thus, by (9.4),

$$C_2(x,x) = AA_2(x,x) \quad \forall x \in \mathcal{E}. \tag{9.21}$$

Similarly, for any $N = 2,3,\ldots$, if $x \in \mathcal{F} \cap D$, then $f^N(x) = Ax = x$, and

$$C_2(Ax,Ax) = \cdots = C_N(Ax,\ldots,Ax) = F_{N+1}(Ax,\ldots,Ax) = 0 \quad \forall x \in \mathcal{E}. \tag{9.22}$$

Assuming that there are continuous polynomials $x \mapsto \tilde{C}_2(x,x),\ldots,x \mapsto \tilde{C}_N(x,\ldots,x)$ such that $C_2 = A\tilde{C}_2,\ldots,C_N = A\tilde{C}_N$, (9.9) yields

$$C_{N+1} = A\tilde{C}_{N+1} \tag{9.23}$$

with

$$\tilde{C}_{N+1}(x,\ldots,x) = A_{N+1}(x,\ldots,x) + \sum_{q=2}^{N} \sum_{(q,N)} \tilde{C}_q\big(A_{p_1}(x,\ldots,x),\ldots,A_{p_q}(x,\ldots x)\big). \tag{9.24}$$

This inductive argument shows that $h(B(0,r)) \subset \mathcal{F}$, and therefore $h(D) \subset \mathcal{F} \cap D$. Since, on the other hand, $\mathcal{F} \cap D \subset \mathrm{Fix}\, f = h(D)$, the conclusion follows. □

## 10. Extensions to semiflows

In this section, we apply the results of Section 8 to the case in which $f$ is an element of a semiflow. Thus, let $x_0 \in D$ be a fixed point of a semiflow $\phi : \mathbb{R}_+ \times D \to D$ acting by holomorphic maps $\phi_t$ on a domain $D$ of $\mathcal{E}$. Denoting by $d\phi_t(x) \in \mathcal{L}(\mathcal{E})$ the Fréchet differential of $\phi_t$ at $x$, then

$$d\phi_{t_1+t_2}(x_0) = d\phi_{t_1}(x_0)\,d\phi_{t_2}(x_0) \quad \forall t_1,t_2 \in \mathbb{R}_+, \qquad d\phi_0(x_0) = I. \tag{10.1}$$

Lemma 10.1. *If the semiflow $\phi$ is continuous, the semigroup $d\phi_\bullet(x_0):\mathbb{R}_+\to\mathcal{L}(\mathcal{E})$ is strongly continuous.*

*If the domain $D$ is bounded, the semigroup is uniformly bounded.*

*Proof.* Choose $r>0$ so small that $B(x_0,r)\subset D$.

If $\xi\in\mathcal{E}$, choose $s>0$ in such a way that $\phi_t(x_0+\zeta\xi)\in B(x_0,r)$ whenever $|\zeta|\leq s$ and for any $t$ in a neighbourhood of 0 in $\mathbb{R}_+$.

If $\lambda\in\mathcal{E}'$, the Cauchy integral formula yields

$$\langle d\phi_t(x_0)\xi,\lambda\rangle=\frac{1}{2\pi i}\int_{\partial\Delta(0,s)}\frac{\langle\phi_t(x_0+\zeta\xi),\lambda\rangle}{\zeta^2}\,d\zeta. \tag{10.2}$$

Since, for $\zeta\in\partial\Delta(0,s)$,

$$\left|\frac{\langle\phi_t(x_0+\zeta\xi),\lambda\rangle}{\zeta^2}\right|\leq\frac{r\|\lambda\|}{s^2}, \tag{10.3}$$

the dominated convergence theorem implies that

$$\lim_{t\downarrow 0}\langle d\phi_t(x_0)\xi-\xi,\lambda\rangle=0, \tag{10.4}$$

that is, the semigroup $d\phi_\bullet(x_0)$ is weakly, hence strongly, continuous.

The uniform boundedness of the semigroup follows from the Cauchy inequalities. □

Let $Z:\mathcal{D}(Z)\subset\mathcal{E}\to\mathcal{E}$ be the infinitesimal generator of the strongly continuous semigroup $d\phi_\bullet(x_0):\mathbb{R}_+\to\mathcal{L}(\mathcal{E})$.

Let $D$ be a bounded domain in $\mathcal{E}$, and let $\phi:\mathbb{R}_+\times D\to D$ be a continuous semiflow of holomorphic maps of $D$ into $D$ fixing a point $x_0\in D$.

If $\phi_{2t_0}=\phi_{t_0}$, for some $t_0>0$, then $d\phi_{t_0}$ is an idempotent of $\mathcal{L}(\mathcal{E})$.

If $\sigma(d\phi_{t_0}(x_0))=\{0\}$, (9.17) applied to $f=\phi_{t_0}$ shows that the semigroup $d\phi_\bullet(x_0)$ is nilpotent. Theorem 8.5 implies that, as $t\to+\infty$, $\phi_t$ converges to the constant map $x\mapsto x_0$ for the topology of local uniform convergence.

If $\sigma(d\phi_{t_0}(x_0))=\{1\}$, (9.18) applied to $f=\phi_{t_0}$, coupled with Cartan's identity theorem, implies that $\phi_{t_0}=\mathrm{id}$, and therefore $\phi$ is the restriction to $\mathbb{R}_+$ of a continuous periodic flow with period $t_0/p$ for some positive integer $p$.

How many values of the semigroup $d\phi_\bullet(x_0)$ can be idempotent in $\mathcal{L}(\mathcal{E})$?

Clearly, if $d\phi_{t_0}(x_0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$, then $d\phi_{nt_0}(x_0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$ for $n=1,2,\ldots$.

If $d\phi_{t_0}(x_0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$ for some $t_0>0$, and if $1\in\sigma(d\phi_{t_0}(x_0))$, then $2n\pi i/t_0\in p\sigma(Z)$ for some $n\in\mathbb{Z}$. Letting

$$V:=\left\{n\in\mathbb{Z}:\frac{2n\pi i}{t_0}\in p\sigma(Z)\right\}, \tag{10.5}$$

then $V \neq \emptyset$,

$$\sigma(Z)\backslash\{0\} = p\sigma(Z)\backslash\{0\} = \frac{2\pi i}{t_0}V,$$
$$\ker\left(I - d\phi_{t_0}(x_0)\right) = \bigvee_{n\in\mathbb{Z}} \ker\left(\frac{2n\pi i}{t_0}I - Z\right). \tag{10.6}$$

For any $t > 0$ and $n \in V$

$$e^{2n\pi it/t_0} \in p\sigma\left(d\phi_t(x_0)\right). \tag{10.7}$$

Hence, if $d\phi_{t_1}(x_0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$ for some $t_1 > 0$, for any $n \in V$,

$$e^{2n\pi it_1/t_0} = 1, \tag{10.8}$$

that is, there is $m \in \mathbb{Z}$ such that

$$\frac{2n\pi it_1}{t_0} = 2\pi im, \tag{10.9}$$

that is,

$$nt_1 = mt_0. \tag{10.10}$$

As a consequence, if $t_1/t_0 \notin \mathbb{Q}$, then $n = m = 0$. Hence, $V = \{0\}$, therefore

$$p\sigma\left(d\phi_t(x_0)\right) = \{1\}, \tag{10.11}$$
$$\operatorname{Ran} d\phi_t(x_0) = \ker\left(I - d\phi_t(x_0)\right) = \ker Z \quad \forall t \in \mathbb{R}_+. \tag{10.12}$$

Thus, since $d\phi_{t_0}(x_0)$ is an idempotent,

$$\mathcal{E} = \ker\left(d\phi_{t_0}(x_0)\right) \oplus \ker Z. \tag{10.13}$$

Let $\Pi$ and $\Lambda = I - \Pi$ be the projectors, with ranges $\ker d\phi_t(x_0)$ and $\ker Z$, associated to this direct sum decomposition of $\mathcal{E}$.

Since, for any $x \in \mathcal{E}$ and any $t \geq t_0$,

$$d\phi_t(x_0)\Pi x = d\phi_{t-t_0}(x_0)\left(d\phi_{t_0}(x_0)\Pi x\right) = 0, \tag{10.14}$$

then, by (10.12),

$$d\phi_t(x_0)x = d\phi_t(x_0)\Lambda x = \Lambda x, \tag{10.15}$$

and therefore

$$d\phi_{2t}(x_0)x = d\phi_t(x_0)\Lambda x = \Lambda x = d\phi_t(x_0)x. \tag{10.16}$$

Hence, if $d\phi_{t_0}(x_0)$ and $d\phi_{t_1}(x_0)$ are idempotents of $\mathcal{L}(\mathcal{E})$, and if $t_1/t_0 \notin \mathbb{Q}$, then

$$d\phi_t(x_0) = d\phi_{t_0}(x_0) \quad \forall t \geq \min\{t_0, t_1\}. \tag{10.17}$$

Let $0 < t < t_0$. If $x \in \ker d\phi_{t_0}(x_0)$ and $d\phi_t(x_0)x \neq 0$, then

$$\Lambda d\phi_t(x_0)x \in \ker Z \setminus \{0\}, \tag{10.18}$$

and therefore

$$0 = d\phi_{t_0+t}(x_0)x = d\phi_t(x_0)(\Lambda d\phi_t(x_0)x) = \Lambda d\phi_t(x_0)x \neq 0. \tag{10.19}$$

This contradiction proves that if $x \in \ker d\phi_{t_0}(x_0)$, then $x \in d\phi_t(x_0)$ for all $t \in (0, t_0]$.

Summing up, if $1 \in \sigma(d\phi_{t_0}(x_0))$ and if $t_1/t_0 \notin \mathbb{Q}$, then $d\phi_t(x_0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$ which is independent of $t > 0$. The strong continuity of the semigroup $d\phi_\bullet(x_0)$ implies then that $d\phi_t(x_0) = I$ for all $t \geq 0$.

Since $D$ is a bounded domain, Cartan's identity theorem yields the following theorem.

Theorem 10.2. *If $d\phi_{t_0}(x_0)$ and $d\phi_{t_1}(x_0)$, with $t_1/t_0 \notin \mathbb{Q}$, are idempotents of $\mathcal{L}(\mathcal{E})$, and if $1 \in \sigma(d\phi_{t_0}(x_0))$, then $\phi_t = \mathrm{id}$ for all $t \in \mathbb{R}_+$.*

As in Section 4, and with the same notations, let $D$ be the open unit ball $B$ of the complex Hilbert space $\mathcal{K}$, and let $\phi$ be the periodic continuous semiflow, with period $\tau$, of holomorphic automorphisms of $B$, defined by the group $T$.

If $0 \in \mathrm{Fix}\phi$, (4.8) shows that $\phi$ is (the restriction to $B$ of) a strongly continuous group of linear operators on $\mathcal{K}$,

$$\phi_t = d\phi_t(0)_{|B}, \tag{10.20}$$

and $Z = X_{11} - iX_{22}I_{\mathcal{K}}$.

If $0 \in p\sigma(Z)$ and $x \in \ker Z \setminus \{0\}$, then

$$\phi_t(x) = d\phi_t(0)x = x \quad \forall t \in \mathbb{R}. \tag{10.21}$$

Vice versa, if $\phi_t(x) = x$ for some $x \in B \setminus \{0\}$ and all $t \in \mathbb{R}$, Bart's theorem in [1] implies that $0 \in p\sigma(Z)$. That proves the following lemma.

Lemma 10.3. *Let $0 \in \mathrm{Fix}\phi$. Then $\{0\} = \mathrm{Fix}\phi$ if, and only if, $0 \notin p\sigma(Z)$.*

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