SOLUTIONS TO H-SYSTEMS BY TOPOLOGICAL AND ITERATIVE METHODS

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We study H-systems with a Dirichlet boundary data g. Under some conditions, we show that if the problem admits a solution for some (H_0, g_0), then it can be solved for any (H, g) close enough to (H_0, g_0). Moreover, we construct a solution of the problem applying a Newton iteration.

1. Introduction

We consider the Dirichlet problem in a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^2$ for a vector function $X : \overline{\Omega} \to \mathbb{R}^3$ which satisfies the equation of prescribed mean curvature

$$\begin{aligned} \Delta X &= 2H(u, v, X) X_u \wedge X_v \quad \text{in } \Omega, \\ X &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\wedge$ denotes the exterior product in $\mathbb{R}^3$, $H : \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function, and the boundary data g is smooth. Problem (1.1) above arises in the Plateau and Dirichlet problems for the prescribed mean curvature equation that has been studied, for example, in [1, 2, 3, 4, 5].

In Section 2, we prove the following theorem.

Theorem 1.1. *Let $X_0 \in W^{2,p}(\Omega, \mathbb{R}^3)$ be a solution of (1.1) for some (H_0, g_0) with $g_0 \in W^{2,p}(\Omega, \mathbb{R}^3)$ $(2 < p < \infty)$ and H_0 continuously differentiable with respect to X over the graph of X_0. Set*

$$k = -2 \inf_{(u,v,Y) \in \Omega \times \mathbb{R}^3, |Y|=1} \left(\frac{\partial H_0}{\partial X}(u, v, X_0) Y \right) \left((X_{0_u} \wedge X_{0_v}) Y \right) \tag{1.2}$$

and assume that

$$k + 2\sqrt{\lambda_1} \| H_0(\cdot, X_0) \|_\infty \| \nabla X_0 \|_\infty < \lambda_1, \tag{1.3}$$

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where λ_1 is the first eigenvalue of $-\Delta$. Then there exists a neighborhood $\mathcal{B}$ of (H_0,g_0) in the space $C(\overline{\Omega}\times\mathbb{R}^3,\mathbb{R})\times W^{2,p}(\Omega,\mathbb{R}^3)$ such that (1.1) is solvable for any $(H,g)\in\mathcal{B}$.

Remark 1.2. It is clear that

$$\begin{aligned} 0 &\le -2\inf_{(u,v)\in\Omega}\frac{\partial H_0}{\partial X}(u,v,X_0)\left(X_{0_u}\wedge X_{0_v}\right) \\ &\le k \le 2\left\|\frac{\partial H_0}{\partial X}(\cdot,X_0)\right\|_\infty \|X_{0_u}\wedge X_{0_v}\|_\infty. \end{aligned} \tag{1.4}$$

Moreover, a simple computation shows that $k=0$ if and only if $(\partial H_0/\partial X)(\cdot,X_0)$ and $X_{0_u}\wedge X_{0_v}$ are linearly dependent, with $(\partial H_0/\partial X)(u,v,X_0)(X_{0_u}\wedge X_{0_v})\ge 0$ for every $(u,v)\in\Omega$.

In Section 3, we show that the solution provided by Theorem 1.1 can be obtained by a Newton iteration. For simplicity, we consider the case where H does not depend on X and prove the following theorem.

Theorem 1.3. *Let $X_0\in W^{2,p}(\Omega,\mathbb{R}^3)$ be a solution of (1.1) for some (H_0,g_0) with $g_0\in W^{2,p}(\Omega,\mathbb{R}^3)$ $(2<p<\infty)$ and H_0 continuous, and assume that*

$$2\|H_0\|_\infty\|\nabla X_0\|_\infty<\sqrt{\lambda_1}. \tag{1.5}$$

Then, if H and g are close enough to H_0 and g_0, respectively, the sequence given by

$$\begin{gathered} \Delta X_{n+1}=2H\left[\left(X_{n_u}\wedge\left(X_{n+1}-X_n\right)_v+\left(X_{n+1}-X_n\right)_u\wedge X_{n_v}\right)-X_{n_u}\wedge X_{n_v}\right], \\ X_{n+1}|_{\partial\Omega}=g \end{gathered} \tag{1.6}$$

is well defined and converges in $W^{2,p}(\Omega,\mathbb{R}^3)$ to a solution of (1.1).

2. Proof of Theorem 1.1

First we will prove a slight extension of a well-known result for linear elliptic second-order operators.

Lemma 2.1. *Let $L: W^{2,p}(\Omega,\mathbb{R}^3)\to L^p(\Omega,\mathbb{R}^3)$ be the linear elliptic operator given by $LX=\Delta X+AX_u+BX_v+CX$ with $A,B,C\in L^\infty(\Omega,\mathbb{R}^{3\times3})$ $(2<p<\infty)$, and assume that $r:=((\||A|^2+|B|^2\|_\infty)/\lambda_1)^{1/2}<1$ and that $CY\cdot Y\le\kappa|Y|^2$ for every $Y\in\mathbb{R}^3$ with $\kappa<\lambda_1(1-r)$. Then $L|_{W_0^{1,p}(\Omega,\mathbb{R}^3)}: W^{2,p}\cap W_0^{1,p}(\Omega,\mathbb{R}^3)\to L^p(\Omega,\mathbb{R}^3)$ is an isomorphism.*

Proof. Let $Z_n \in W^{2,p} \cap W_0^{1,p}(\Omega, \mathbb{R}^3)$ be a sequence such that $\|LZ_n\|_p \to 0$. Then $\|LZ_n\|_2 \to 0$, and from the inequalities

$$\begin{aligned} -\int LZ_nZ_n &\geq \|\nabla Z_n\|_2^2 - \left\|\left(|A|^2+|B|^2\right)^{1/2}\right\|_\infty \|\nabla Z_n\|_2\|Z_n\|_2 - \int CZ_nZ_n \\ &\geq \left(1-r-\frac{\kappa}{\lambda_1}\right)\|\nabla Z_n\|_2^2, \end{aligned} \tag{2.1}$$

we deduce that $\|\nabla Z_n\|_2 \to 0$. Thus, $\|Z_n\|_2 \to 0$ and hence $\|\Delta Z_n\|_2 \to 0$. From the invertibility of Δ, there exists a subsequence (still denoted Z_n) such that $\|Z_n\|_{2,2} \to 0$. By Sobolev imbedding, $\|Z_n\|_{1,p} \to 0$ and we conclude that $\|\Delta Z_n\|_p \to 0$. In order to prove that L is onto, it suffices to consider for any $\varphi \in L^p(\Omega)$, the homotopy

$$\Delta X = \sigma(\varphi - AX_u - BX_v - CX) \tag{2.2}$$

and apply a Leray-Schauder argument. □

Now we are able to prove Theorem 1.1. Consider a pair (H, g) with $\|g - g_0\|_{2,p} < \delta$ and $\|(H-H_0)|_K\|_\infty < \varepsilon$ for some compact K containing a neighborhood of the graph of X_0. Setting $Y = X - X_0$, equation (1.1) is equivalent to the problem

$$\begin{aligned} LY &= F(u,v,Y,Y_u,Y_v) \quad \text{in } \Omega, \\ Y &= g - g_0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

where L is the linear operator given by

$$LY = \Delta Y - 2H_0(u,v,X_0)[X_{0_u} \wedge Y_v + Y_u \wedge X_{0_v}] - 2\left(\frac{\partial H_0}{\partial X}(u,v,X_0)Y\right)X_{0_u} \wedge X_{0_v} \tag{2.4}$$

and

$$\begin{aligned} F&(u,v,Y,Y_u,Y_v) \\ &:= 2\bigg(H(u,v,X_0+Y)Y_u \wedge Y_v \\ &\qquad + [H(u,v,X_0+Y) - H_0(u,v,X_0)](X_{0_u} \wedge Y_v + Y_u \wedge X_{0_v}) \\ &\qquad + \left[H(u,v,X_0+Y) - H_0(u,v,X_0) - \frac{\partial H_0}{\partial X}(u,v,X_0)Y\right]X_{0_u} \wedge X_{0_v}\bigg). \end{aligned} \tag{2.5}$$

We define an operator $T : C^1(\overline{\Omega}, \mathbb{R}^3) \to C^1(\overline{\Omega}, \mathbb{R}^3)$ given by $T(\overline{Y}) = Y$ where Y is the unique solution of the linear problem

$$\begin{aligned} LY &= F(u,v,\overline{Y},\overline{Y}_u,\overline{Y}_v) \quad \text{in } \Omega, \\ Y &= g - g_0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.6}$$

As L satisfies the hypothesis of Lemma 2.1, it is immediate to prove that T is well defined and continuous. Furthermore, the range of a bounded set is bounded with $\| \ \|_{2,p}$, and by Sobolev imbedding, we conclude that T is compact. More precisely, for $\|\overline{Y}\|_{1,\infty} \leq R$, we obtain

$$\begin{aligned} \|T(\overline{Y})\|_{1,\infty} &\leq \|g-g_0\|_{1,\infty} + c\|T(\overline{Y}) - (g-g_0)\|_{2,p} \\ &\leq \|g-g_0\|_{1,\infty} + c_1\left(\|L(T(\overline{Y}))\|_p + \|L(g-g_0)\|_p\right) \\ &\leq k_0\delta + c_1\|F(\cdot,\overline{Y},\overline{Y}_u,\overline{Y}_v)\|_p \end{aligned} \tag{2.7}$$

for some constants k_0 and c_1.

On the other hand, a simple computation shows that

$$\|F(\cdot,\overline{Y},\overline{Y}_u,\overline{Y}_v)\|_p \leq k_1R^2 + k_2\varepsilon R + k_3\varepsilon \tag{2.8}$$

for some constants k_1, k_2, and k_3. Hence, if δ and ε are small, it is possible to choose R such that $T(B_R) \subset B_R$ and the result follows by Schauder's Theorem.

3. A Newton iteration for problem (1.1)

In this section, we apply a Newton iteration to (1.1). For simplicity, we will assume that H does not depend on X.

Let X_0 be a solution of (1.1) for some H_0 and g_0 with

$$2\|H_0\|_\infty \|\nabla X_0\|_\infty < \sqrt{\lambda_1}. \tag{3.1}$$

In order to define a sequence that converges to a solution of (1.1) for (H,g) close to (H_0,g_0), we consider the function $F: g + (W^{2,p} \cap W_0^{1,p}(\Omega,\mathbb{R}^3)) \to L^p(\Omega,\mathbb{R}^3)$ given by

$$F(X) = \Delta X - 2HX_u \wedge X_v. \tag{3.2}$$

Thus, the problem is equivalent to find a zero of F. The well-known Newton method consists in defining a recursive sequence

$$X_{n+1} = X_n - (DF(X_n))^{-1}(F(X_n)) \tag{3.3}$$

or equivalently

$$DF(X_n)(X_{n+1} - X_n) = -F(X_n). \tag{3.4}$$

A simple computation shows that in this case,

$$DF(X)(Y) = \Delta Y - 2H(X_u \wedge Y_v + Y_u \wedge X_v). \tag{3.5}$$

According to this, we start at X_0 and define the sequence $\{X_n\}$ from the following problem:

$$\Delta X_{n+1} - 2H(X_{n_u} \wedge (X_{n+1} - X_n)_v + (X_{n+1} - X_n)_u \wedge X_{n_v}) = 2HX_{n_u} \wedge X_{n_v} \tag{3.6}$$

with Dirichlet condition

$$X_{n+1}|_{\partial\Omega} = g. \tag{3.7}$$

We will prove that if H and g are close enough to H_0 and g_0, respectively, this sequence is well defined (i.e., $DF(X_n)$ is invertible for every n) and converges.

Fix a positive R such that

$$R < \frac{\sqrt{\lambda_1}}{2\|H_0(\cdot, X_0)\|_\infty} - \|\nabla X_0\|_\infty \tag{3.8}$$

and set

$$\mathscr{C} = \left\{X \in W^{2,p}(\Omega, \mathbb{R}^3) : X|_{\partial\Omega} = g,\ \|X - X_0\|_{2,p} \leq R\right\}. \tag{3.9}$$

We will assume that

$$\|H - H_0\|_\infty < \varepsilon, \qquad \|g - g_0\|_{2,p} < \delta \leq R \tag{3.10}$$

with

$$\varepsilon < \frac{\sqrt{\lambda_1}}{2(\|\nabla X_0\|_\infty + R)} - \|H(\cdot, X_0)\|_\infty. \tag{3.11}$$

For each $X \in \mathscr{C}$, we define the linear operator L_X given by

$$L_X Y = \Delta Y - 2H(X_u \wedge Y_v + Y_u \wedge X_v). \tag{3.12}$$

By Lemma 2.1, $L_X|_{W_0^{1,p}(\Omega)}$ is invertible for any $X \in \mathscr{C}$. Furthermore, we claim that $\|L_X^{-1}\|$ is bounded over $\mathscr{C}$. Indeed, for $Z \in W^{2,p} \cap W_0^{1,p}(\Omega, \mathbb{R}^3)$ and $X, Y \in \mathscr{C}$, we have

$$\begin{aligned} \|L_Y Z\|_p &\geq \|L_X Z\|_p - \|(L_X - L_Y)Z\|_p \\ &\geq \left(\frac{1}{\|L_X^{-1}\|} - 2\|H\|_\infty \|\nabla(X - Y)\|_\infty\right) \|Z\|_{2,p}. \end{aligned} \tag{3.13}$$

Taking, for example, Y such that $\|\nabla(Y-X)\|_\infty \le 1/(4\|H\|_\infty\|L_X^{-1}\|) := R_X$, we obtain

$$||L_Y^{-1}|| \le 2||L_X^{-1}||. \tag{3.14}$$

By compactness, there exist $X^1,\dots,X^n \in \mathscr{C}$ such that

$$\mathscr{C} \subset \bigcup_{i=1}^{n} \{Y : ||\nabla(Y-X^i)||_\infty \le R_{X^i}\} \tag{3.15}$$

and hence,

$$||L_X^{-1}|| \le 2 \max_{1\le i\le n} ||L_{X^i}^{-1}||. \tag{3.16}$$

Let $Z_n = X_{n+1} - X_n$. For $n=0$, we have

$$\begin{aligned} ||Z_0||_{2,p} &\le ||g-g_0||_{2,p} + ||Z_0-(g-g_0)||_{2,p} \\ &\le ||g-g_0||_{2,p} + c\Big(||L_{X_0}Z_0||_p + ||L_{X_0}(g-g_0)||_p\Big) \\ &\le 2\delta(1+\|H\|_\infty||\nabla X_0||_\infty) + c||L_{X_0}Z_0||_p. \end{aligned} \tag{3.17}$$

As

$$||L_{X_0}Z_0||_p = ||2(H-H_0)X_{0_u}\wedge X_{0_v}||_{2p}^2 \le \varepsilon||\nabla X_0||_p, \tag{3.18}$$

we conclude that

$$||Z_0||_{2,p} \le 2\delta(1+(||H_0||_\infty+\varepsilon)||\nabla X_0||_\infty) + c\varepsilon||\nabla X_0||_{2p}^2 := c(\delta,\varepsilon). \tag{3.19}$$

Then we may establish a more precise version of Theorem 1.3.

Theorem 3.1. *With the previous notations, assume that*

$$c(\delta,\varepsilon) \le \frac{R}{1+Rc_0c(||H_0||_\infty+\varepsilon)}, \tag{3.20}$$

where c_0 is the constant of the imbedding $W^{2,p}(\Omega,\mathbb{R}^3)\hookrightarrow C^1(\overline{\Omega},\mathbb{R}^3)$. Then the sequence given by (1.6) is well defined and converges in $W^{2,p}(\Omega,\mathbb{R}^3)$ to a solution of (1.1).

Proof. By (3.20), we have that $\|Z_0\|_{2,p} \le c(\delta,\varepsilon) \le R$, proving that $X_1 \in \mathscr{C}$. For $n>0$, we assume as inductive hypothesis that $X_k \in \mathscr{C}$ for $k\le n$, and then

$$\begin{aligned} ||Z_n||_{2,p} &\le c||L_{X_n}Z_n||_p = 2c||HZ_{n-1_u}\wedge Z_{n-1_v}||_p \\ &\le c\|H\|_\infty||\nabla Z_{n-1}||_\infty||\nabla Z_{n-1}||_p \\ &\le c_0c\|H\|_\infty||Z_{n-1}||_{2,p}^2. \end{aligned} \tag{3.21}$$

Inductively,

$$||Z_n||_{2,p} \leq (c_0 c\|H\|_\infty)^{2^n-1}||Z_0||_{2,p}^{2^n} = A^{2^n-1}||Z_0||_{2,p}, \tag{3.22}$$

where $A = c_0 c\|H\|_\infty \|Z_0\|_{2,p}$. By hypothesis, it is immediate that $A < 1$, and hence

$$||X_{n+1} - X_0||_{2,p} \leq \sum_{j=0}^{n} ||Z_j||_{2,p} \leq ||Z_0||_{2,p} \frac{1}{1-A} \leq R. \tag{3.23}$$

Thus, $X_n \in \mathscr{C}$ for every n, and

$$||X_{n+k} - X_n||_{2,p} \leq \frac{A^{2^n-1}}{1-A} \tag{3.24}$$

for every $k \geq 0$. Then X_n is a Cauchy sequence, and the result follows. □

Remark 3.2. It is clear from definition that $c(\delta,\varepsilon) \to 0$ for $(\delta,\varepsilon) \to (0,0)$.

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References

[1] P. Amster, M. C. Mariani, and D. F. Rial, *Existence and uniqueness of H-system's solutions with Dirichlet conditions*, Nonlinear Anal., Ser. A: Theory Methods **42** (2000), no. 4, 673–677.

[2] H. Brezis and J.-M. Coron, *Multiple solutions of H-systems and Rellich's conjecture*, Comm. Pure Appl. Math. **37** (1984), no. 2, 149–187.

[3] S. Hildebrandt, *On the Plateau problem for surfaces of constant mean curvature*, Comm. Pure Appl. Math. **23** (1970), 97–114.

[4] M. Struwe, *Plateau's Problem and the Calculus of Variations*, Mathematical Notes, vol. 35, Princeton University Press, New Jersey, 1988.

[5] G. F. Wang, *The Dirichlet problem for the equation of prescribed mean curvature*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (1992), no. 6, 643–655.

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