SOLVABILITY OF NONLINEAR DIRICHLET PROBLEM FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

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We prove an existence result for solution to a class of nonlinear degenerate elliptic equation associated with a class of partial differential operators of the form $Lu(x) = \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x))$, with $D_j = \partial/\partial x_j$, where $a_{ij} : \Omega \to \mathbb{R}$ are functions satisfying suitable hypotheses.

1. Introduction

In this paper, we prove the existence of solution in $D(A) \subseteq H_0(\Omega)$ for the following nonlinear Dirichlet problem:

$$\begin{gathered} -Lu(x) + g\big(u(x)\big)\omega(x) = f_0(x) - \sum_{j=1}^{n} D_j f_j(x) \quad \text{on } \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where L is an elliptic operator in divergence form

$$Lu(x) = \sum_{i,j=1}^{n} D_j\big(a_{ij}(x)D_iu(x)\big), \quad \text{with } D_j = \frac{\partial}{\partial x_j} \tag{1.2}$$

and the coefficients a_{ij} are measurable, real-valued functions whose coefficient matrix $(a_{ij}(x))$ is symmetric and satisfies the degenerate ellipticity condition

$$|\xi|^2\omega(x) \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le |\xi|^2\nu(x) \tag{1.3}$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega \subset \mathbb{R}^n$ a bounded open set with piecewise smooth boundary (i.e., $\partial\Omega \in C^{0,1}$), and ω and ν two weight functions (i.e., locally integrable nonnegative functions).

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The basic idea is to reduce (1.1) to an operator equation

$$Au = T, \quad u \in D(A), \tag{1.4}$$

where $D(A) = \{u \in H_0(\Omega) : u(x)g(u(x)) \in L^1(\Omega,\omega)\}$, and apply the theorem below.

Theorem 1.1. *Suppose that the following assumptions are satisfied.*

(H1) Dual pairs. *Let the dual pairs $\{X, X^+\}$ and $\{Y, Y^+\}$ be given, where X, X^+, Y, and Y^+ are Banach spaces with corresponding bilinear forms $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$ and the continuous embeddings $Y \subseteq X$ and $X^+ \subseteq Y^+$.*

The dual pairs are compatible, that is,

$$\langle T,u\rangle_X = \langle T,u\rangle_Y, \quad \forall T \in X^+,\ u \in Y. \tag{1.5}$$

Moreover, the Banach spaces X and Y are separable and X is reflexive.

(H2) Operator A. *Let the operator $A : D(A) \subseteq X \to Y^+$ be given, and let K be a bounded closed convex set in X containing the zero point as an interior point and $K \cap Y \subseteq D(A)$.*

(H3) Local coerciveness. *There exists a number $\alpha \geq 0$ such that $\langle Av, v\rangle_Y \geq \alpha$ for all $v \in Y \cap \partial K$, where ∂K denotes the boundary of K in the Banach space X.*

(H4) Continuity. *For each finite-dimensional subspace Y_0 of the Banach space Y, the mapping $u \mapsto \langle Au, v\rangle_Y$ is continuous on $K \cap Y_0$ for all $v \in Y_0$.*

(H5) Generalized condition (M). *Let $\{u_n\}$ be a sequence in $Y \cap K$ and let $T \in X^+$. Then, from*

$$u_n \rightharpoonup u \quad \text{in } X \text{ as } n \longrightarrow \infty, \tag{1.6}$$

$$\begin{gathered} \langle Au_n, v\rangle_Y \longrightarrow \langle T, v\rangle_X \quad \text{as } n \longrightarrow \infty,\ \forall v \in Y, \\ \overline{\lim_{n\to\infty}} \langle Au_n, u_n\rangle_Y \leq \langle T,u\rangle_X, \end{gathered} \tag{1.7}$$

it follows that $Au = T$.

(H6) Quasiboundedness. *Let $\{u_n\}$ be a sequence in $Y \cap K$. Then, from (1.6) and $\langle Au_n, u_n\rangle_Y \leq C\|u\|_X$ for all n, it follows that the sequence $\{Au_n\}$ is bounded in Y^+.*

(H7) *The operator A is coercive, that is, $\langle Av, v\rangle_Y / \|v\|_X \to \infty$ as $\|v\|_X \to \infty$, $v \in Y$.*

Then $X^+ \subseteq R(A)$, that is, the equation $Au = T$ has a solution u for each $T \in X^+$.

Proof. See [7, Theorem 27.B and Corollary 27.19]. □

We will apply this theorem to a sufficiently large ball K in the Banach spaces $X = H_0(\Omega)$, $X^+ = (H_0(\Omega))^*$, and $Y^+ = Y^*$.

We make the following basic assumption on the weights ω and v.

The weighted Sobolev inequality (WSI). Let Ω be an open bounded set in $\mathbb{R}^n$. There is an index $q = 2\sigma$, $\sigma > 1$, such that for every ball B and every $f \in \mathrm{Lip}_0(B)$ (i.e, $f \in \mathrm{Lip}(B)$ whose support is contained in the interior of B),

$$\left(\frac{1}{v(B)}\int_B |f|^q v\,dx\right)^{1/q} \leq CR_B\left(\frac{1}{\omega(B)}\int_B |\nabla f|^2 \omega\,dx\right)^{1/2}, \tag{1.8}$$

with the constant C independent of f and B, R_B the radius of B, and the symbol ∇ indicating the gradient, $v(B) = \int_B v(x)dx$, and $\omega(B) = \int_B \omega(x)dx$.

Thus, we can write

$$\left(\int_B |f|^q v\,dx\right)^{1/q} \leq C_{B,\omega,v}\left(|\nabla f|^2 \omega\,dx\right)^{1/2}, \tag{1.9}$$

where $C_{B,\omega,v}$ is called the Sobolev constant and

$$C_{B,\omega,v} = \frac{C[v(B)]^{1/q} R_B}{[\omega(B)]^{1/2}}. \tag{1.10}$$

For instance, the WSI holds if ω and v are as in [6, Chapter X, Theorem 4.8], or if ω and v are as in [1, Theorem 1.5].

The following theorem will be proved in Section 3.

Theorem 1.2. *Let L be the operator (1.2) and satisfy (1.3). Suppose that the following assumptions are satisfied:*

(i) $(v,\omega) \in A_2$;
(ii) *the function $g : \mathbb{R} \to \mathbb{R}$ is continuous with $xg(x) \geq 0$ for all $x \in \mathbb{R}$;*
(iii) $f_0/v \in L^{q'}(\Omega, v)$ *and* $f_j/\omega \in L^2(\Omega,\omega)$, $j = 1,2,\dots,n$ *(where q is as in WSI). Then problem (1.1) has solution $u \in D(A) \subseteq H_0(\Omega)$;*
(iv) *if the function $g : \mathbb{R} \to \mathbb{R}$ is monotone increasing, then the solution is unique.*

Example 1.3. Consider the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : |x| < 1 \text{ and } |y| < 1\}$. By Theorem 1.2, the problem

$$\begin{gathered} -Lu(x) + u(x,y)e^{u^2(x,y)}|x|^{1/2} = 1 - \frac{\partial}{\partial x}\left(x^2|y|\right) - \frac{\partial}{\partial y}\left(y^2|x|\right) \quad \text{on } \Omega, \\ u(x,y) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.11}$$

where

$$Lu(x) = \left[\frac{\partial}{\partial x}\left(|x|^{1/2}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(|x|^{-1/2}\frac{\partial u}{\partial y}\right)\right] \tag{1.12}$$

has a unique solution $u \in D(A) = \{u \in H_0(\Omega) : g(u(x,y))u(x,y) \in L^1(\Omega,\omega)\}$, where $g(t) = te^{t^2}$, $\omega(x,y) = |x|^{1/2}$, $v(x,y) = |x|^{-1/2}$, $f_0(x,y) = 1$, $f_1(x,y) = x^2|y|$, and $f_2(x,y) = y^2|x|$.

2. Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight if there is a constant $C_1 = C(p,\omega)$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \leq C_1, \tag{2.1}$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \le p$, then $A_q \subset A_p$ (see [4, 5] for more information about A_p-weights). The weight ω satisfies the doubling condition if $\omega(2B) \le C\omega(B)$, for all balls $B \subset \mathbb{R}^n$, where $\omega(B) = \int_B \omega(x)dx$ and $2B$ denotes the ball with the same center as B which is twice as large. If $\omega \in A_p$, then ω is doubling (see [5, Corollary 15.7]).

We say that the pair of weights (v,ω) satisfies the condition A_p ($1 < p < \infty$ and $(v,\omega) \in A_p$) if and only if there is a constant C_2 such that

$$\left(\frac{1}{|B|}\int_B v(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \le C_2, \tag{2.2}$$

for every ball $B\subset\mathbb{R}^n$.

Remark 2.1. If $(v,\omega) \in A_p$ and $\omega \le v$, then $\omega \in A_p$ and $v \in A_p$.

Given a measurable subset Ω of $\mathbb{R}^n$, we will denote by $L^p(\Omega,\omega)$, $1 \le p < \infty$, the Banach space of all measurable functions f defined on Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p} < \infty. \tag{2.3}$$

We will denote by $W^{k,p}(\Omega,\omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega,\omega)$ such that the weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \le |\alpha| \le k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)dx + \sum_{1\le|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^p \omega(x)dx\right)^{1/p}. \tag{2.4}$$

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\bar{\Omega})$ with respect to the norm (2.4) (see [2, Proposition 3.5]). The space $W_0^{k,p}(\Omega,\omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega,\omega)} = \left(\sum_{1\le|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^p \omega(x)dx\right)^{1/p}. \tag{2.5}$$

When $k = 1$ and $p = 2$, the spaces $W^{1,2}(\Omega,\omega)$ and $W_0^{1,2}(\Omega,\omega)$ are Hilbert spaces. We will denote by $H_0(\Omega)$ the closure of $C_0^\infty(\bar{\Omega})$ with respect to the norm

$$\|u\|_{H_0(\Omega)} = \left(\int_\Omega \langle \mathcal{A}(x)\nabla u(x), \nabla u(x)\rangle dx\right)^{1/2}, \tag{2.6}$$

where $\mathcal{A}(x) = [a_{ij}(x)]$ (the coefficient matrix) and the symbol ∇ indicates the gradient.

Remark 2.2. Using the condition (1.3), we have

$$\|u\|_{W_0^{1,2}(\Omega,\omega)} \leq \|u\|_{H_0(\Omega)} \leq \|u\|_{W_0^{1,2}(\Omega,\nu)}, \tag{2.7}$$

$$W_0^{1,2}(\Omega,\nu) \subset H_0(\Omega) \subset W_0^{1,2}(\Omega,\omega). \tag{2.8}$$

LEMMA 2.3. *If $\omega \in A_2$, then $W_0^{1,2}(\Omega,\omega) \hookrightarrow L^2(\Omega,\omega)$ is compact and*

$$\|u\|_{L^2(\Omega,\omega)} \leq C_3\|u\|_{W_0^{1,2}(\Omega,\omega)}. \tag{2.9}$$

Proof. The proof follows the lines of [3, Theorem 4.6]. □

We introduce the following definition of (weak) solutions for problem (1.1).

Definition 2.4. A function $u \in D(A) \subseteq H_0(\Omega)$ is (weak) solution to the problem (1.1) if

$$\begin{aligned}\int_\Omega a_{ij}(x)D_iu(x)D_j\varphi(x)dx + \int_\Omega g(u(x))\varphi(x)\omega(x)dx \\ = \int_\Omega f_0(x)\varphi(x)dx + \int_\Omega f_j(x)D_j\varphi(x)dx,\end{aligned} \tag{2.10}$$

for all $\varphi \in Y = H_0(\Omega) \cap W^{k,p}(\Omega,\nu)$, where $p > 4$, $k > n/2$, and $\|\varphi\|_Y = \|\varphi\|_{W^{k,p}(\Omega,\nu)}$, with $D(A) = \{u \in H_0(\Omega) : g(u(x))u(x) \in L^1(\Omega,\omega)\}$.

Remark 2.5. Using that $p > 4$, we have that $\nu \in A_2 \subset A_{p/2}$ and

$$\|\cdot\|_{L^2(\Omega)} \leq \left[\nu^{1/(1-p/2)}(\Omega)\right]^{(p-2)/2p}\|\cdot\|_{L^p(\Omega,\nu)}. \tag{2.11}$$

Thus, $W^{k,p}(\Omega,\nu) \subset W^{k,2}(\Omega) \subset C(\bar{\Omega})$ (by the Sobolev embedding theorem).

Therefore $\|\cdot\|_{C(\bar{\Omega})} \leq C\|\cdot\|_Y$ and the embedding $Y \subset C(\bar{\Omega})$ is continuous.

3. Proof of Theorem 1.2

(I) *Existence.* For $u \in D(A)$ and $\varphi \in Y$, we define

$$\begin{aligned}B_1(u,\varphi) &= \int_\Omega a_{ij}(x)D_iu(x)D_j\varphi(x)dx, \\ B_2(u,\varphi) &= \int_\Omega g(u(x))\varphi(x)\omega(x)dx, \\ T(\varphi) &= \int_\Omega f_0(x)\varphi(x)dx + \sum_{j=1}^n \int_\Omega f_j(x)D_j\varphi(x)dx.\end{aligned} \tag{3.1}$$

Then $u \in D(A) \subseteq H_0(\Omega)$ is solution to problem (1.1) if

$$B_1(u,\varphi) + B_2(u,\varphi) = T(\varphi), \quad \forall\varphi \in Y. \tag{3.2}$$

Step 1 ($T \in (H_0(\Omega))^*$). In fact, using hypothesis (iii), Lemma 2.3, the Hölder inequality, the WSI, and (2.7), we obtain

$$\begin{aligned}
|T(\varphi)| &\leq \int_\Omega |f_0||\varphi|dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|dx \\
&= \int_\Omega \left(\frac{|f_0|}{\nu}\right)\nu^{1/q'}|\varphi|\nu^{1/q}dx + \sum_{j=1}^n \int_\Omega \left(\frac{|f_j|}{\omega}\right)\omega^{1/2}|D_j\varphi|\omega^{1/2}dx \\
&\leq \left\|\frac{f_0}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\varphi\|_{L^q(\Omega,\nu)} + \sum_{j=1}^n \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)} \|D_j\varphi\|_{L^2(\Omega,\omega)} \\
&\leq C_{B,\omega,\nu}\left\|\frac{f_0}{\nu}\right\|_{L^{q'}(\Omega,\nu)} \|\nabla\varphi\|_{L^2(\Omega,\omega)} + \sum_{j=1}^n \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)} \|\nabla\varphi\|_{L^2(\Omega,\omega)} \\
&\leq C\left(\left\|\frac{f_0}{\nu}\right\|_{L^{q'}(\Omega,\nu)} + \sum_{j=1}^n \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)}\right)\|\varphi\|_{H_0(\Omega)}, \quad \forall \varphi \in H_0(\Omega).
\end{aligned} \tag{3.3}$$

Step 2. By condition (1.3) and the hypothesis that the matrix $\mathcal{A}$ is symmetric, we obtain

$$\begin{aligned}
|B_1(u,\varphi)| &\leq \int_\Omega big|\langle \mathcal{A}\nabla u, \nabla\varphi\rangle|dx \\
&\leq \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle^{1/2}\langle \mathcal{A}\nabla\varphi, \nabla\varphi\rangle^{1/2}dx \\
&\leq \|u\|_{H_0(\Omega)}\|\varphi\|_{H_0(\Omega)} \\
&\leq \|u\|_{H_0(\Omega)}\|\varphi\|_{W_0^{1,2}(\Omega,\nu)} \\
&\leq \|u\|_{H_0(\Omega)}\|\varphi\|_Y,
\end{aligned} \tag{3.4}$$

for all $u \in H_0(\Omega)$, $\varphi \in Y$.

Hence there exists exactly one linear continuous operator

$$A_1 : H_0(\Omega) \longrightarrow Y^*, \tag{3.5}$$

with

$$\langle A_1 u, \varphi\rangle_Y = B_1(u,\varphi), \quad \forall u \in H_0(\Omega),\ \varphi \in Y. \tag{3.6}$$

Step 3. Note that $|g(x)| \leq xg(x) + C_4$, for all $x \in \mathbb{R}$. Therefore, if $u \in D(A)$, we have that $g(u(x)) \in L^1(\Omega,\omega)$. By using hypothesis (ii), Lemma 2.3, and Remark 2.5, we obtain for $u \in D(A)$ fixed

$$\begin{aligned}
|B_2(u,\varphi)| &\leq \int_\Omega |g(u(x))||\varphi(x)|\omega(x)dx \\
&\leq \|\varphi\|_{C(\bar{\Omega})}\int_\Omega |g(u(x))|\omega(x)dx \\
&\leq C\|\varphi\|_Y.
\end{aligned} \tag{3.7}$$

Thus, there exists a unique operator

$$A_2 : D(A) \subseteq H_0(\Omega) \longrightarrow Y^*, \tag{3.8}$$

with

$$\langle A_2 u, \varphi \rangle_Y = B_2(u, \varphi), \quad \forall u \in D(A),\ \varphi \in Y. \tag{3.9}$$

Step 4. We define the operator

$$A : D(A) \subseteq H_0(\Omega) \longrightarrow Y^*, \quad A = A_1 + A_2. \tag{3.10}$$

We have

$$\langle Au, \varphi \rangle_Y = \langle A_1 u, \varphi \rangle_Y + \langle A_2 u, \varphi \rangle_Y = B_1(u, \varphi) + B_2(u, \varphi). \tag{3.11}$$

Thus, $u \in D(A)$ is a solution to problem (1.1) if

$$\langle Au, \varphi \rangle_Y = T(\varphi), \quad \forall \varphi \in Y. \tag{3.12}$$

Then, the problem (1.1) corresponds to the operator equation (1.4).
Step 5. Global coerciveness of operator A. Using the condition (1.3) and hypothesis (ii), we obtain

$$\begin{aligned}\langle A\varphi, \varphi \rangle_Y &= B_1(\varphi, \varphi) + B_2(\varphi, \varphi) \\ &= \int_\Omega a_{ij}(x) D_i\varphi(x) D_j\varphi(x)\,dx + \int_\Omega g(\varphi(x))\varphi(x)\omega(x)\,dx \\ &\geq \int_\Omega \langle \mathcal{A}\nabla\varphi, \nabla\varphi \rangle\,dx \\ &= \|\varphi\|^2_{H_0(\Omega)}.\end{aligned} \tag{3.13}$$

Thus

$$\lim_{\|\varphi\|_{H_0(\Omega)} \to \infty} \frac{\langle A\varphi, \varphi \rangle_Y}{\|\varphi\|_{H_0(\Omega)}} = +\infty. \tag{3.14}$$

Step 6. Generalized condition (M). Let $T \in (H_0(\Omega))^*$ and let $\{u_n\}$ be a sequence in Y with

$$u_n \rightharpoonup u \quad \text{in } H_0(\Omega), \tag{3.15}$$

$$\langle Au_n, \varphi \rangle_Y \longrightarrow T(\varphi) \quad \text{as } n \longrightarrow \infty,\ \forall \varphi \in Y, \tag{3.16}$$

$$\overline{\lim_{n\to\infty}} \langle Au_n, u_n \rangle \leq T(u). \tag{3.17}$$

We want to show that this implies that $Au = T$.

Using that the operator A_1 is linear and continuous, we obtain

$$\langle A_1 u_n, \varphi \rangle_Y \longrightarrow \langle A_1 u, \varphi \rangle_Y, \quad \forall \varphi \in Y. \tag{3.18}$$

Because of (3.16), it is sufficient to prove that $u \in D(A)$ and

$$\langle A_2 u_n, \varphi\rangle_Y \longrightarrow \langle A_2 u, \varphi\rangle_Y, \quad \forall \varphi \in Y. \tag{3.19}$$

Therefore, it is sufficient to show that

$$\int_\Omega [g(u_n(x)) - g(u(x))]\varphi(x)\omega(x)dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.20}$$

Using the same argument in Step 3, we obtain

$$\begin{aligned} \left| \int_\Omega (g(u_n(x)) - g(u(x)))\varphi(x)\omega(x)dx \right| &\le \int_\Omega |g(u_n(x)) - g(u(x))|\,|\varphi(x)|\,\omega(x)dx \\ &\le \|\varphi\|_{C(\bar{\Omega})} \int_\Omega |g(u_n(x)) - g(u(x))|\,\omega(x)dx \\ &\le C\|\varphi\|_Y \int_\Omega |g(u_n(x)) - g(u(x))|\,\omega(x)dx. \end{aligned} \tag{3.21}$$

Therefore, it is sufficient to show that

$$g(u_n(x)) \longrightarrow g(u(x)) \quad \text{in } L^1(\Omega, \omega). \tag{3.22}$$

Note that it is sufficient to prove (3.22) for a subsequence of $\{u_n\}$.

If $(v, \omega) \in A_2$ and $\omega \le v$, then $\omega \in A_2$ (see Remark 2.1). By Lemma 2.3,

$$W_0^{1,2}(\Omega, \omega) \hookrightarrow L^2(\Omega, \omega) \tag{3.23}$$

is compact and $\|u\|_{L_2(\Omega,\omega)} \le C_2 \|u\|_{W_0^{1,2}(\Omega,\omega)}$. Using (2.7), we also have that

$$H_0(\Omega) \hookrightarrow L^2(\Omega, \omega) \tag{3.24}$$

is compact. This implies $u_n \to u$ in $L^2(\Omega, \omega)$. Using again that $\omega \in A_2$, we have $u_n \to u$ in $L^1(\Omega)$. Thus, there exists a subsequence, again denoted by $\{u_n\}$, such that $u_n(x) \to u(x)$ for almost all $x \in \Omega$. The continuity of g implies that $g(u_n(x)) \to g(u(x))$ for almost all $x \in \Omega$. Moreover, since $u_n \rightharpoonup u$ in $H_0(\Omega)$, it follows that

$$\sup ||u_n||_{H_0(\Omega)} \le C, \quad \text{independent of } n. \tag{3.25}$$

Hence, using (1.2), we obtain

$$\langle A_1 u_n, u_n\rangle_Y \le \Lambda ||u_n||^2_{H_0(\Omega)} \le \Lambda C^2, \quad \text{with } C \text{ independent of } n. \tag{3.26}$$

Therefore, using (3.16), we obtain

$$\overline{\lim_{n\to\infty}} \langle A_2 u_n, u_n\rangle_Y = \overline{\lim_{n\to\infty}} \int_\Omega g(u_n(x))u_n(x)\omega(x)dx \le C, \tag{3.27}$$

with C independent of n.

The continuity of g implies that $g(u_n(x))u_n(x)\omega(x) \to g(u(x))u(x)\omega(x)$ for almost all $x \in \Omega$. Therefore, by Fatou lemma, we have

$$\int_\Omega g(u(x))u(x)\omega(x)dx < \infty, \tag{3.28}$$

that is, $u \in D(A)$.

Now we want to show that $g(u_n(x)) \to g(u(x))$ in $L^1(\Omega,\omega)$.

Let $a > 0$ be fixed. For each $x \in \Omega$, we have either

$$|u_n(x)| \le a \quad \text{or} \quad |g(u_n(x))| \le a^{-1}g(u_n(x))u_n(x) \tag{3.29}$$

(if $x \neq 0$, we can write $g(x) = x^{-1}[g(x)x]$). We get $|g(x)| \le c(a)$ if $|x| \le a$ (because g is continuous).

Let X be a measurable subset of Ω. Then

$$\begin{aligned}\int_X |g(u_n(x))|\omega(x)dx &= \int_{X\cap\{x:|u_n(x)|\le a\}} |g(u_n(x))|\omega(x)dx \\ &\quad + \int_{X\cap\{x:|u_n(x)|> a\}} |g(u_n(x))|\omega(x)dx \\ &\le c(a)\omega(X) + a^{-1}\int_X g(u_n(x))u_n(x)\omega(x)dx \\ &\le c(a)\omega(X) + a^{-1}C \quad (\text{by } (3.27)).\end{aligned} \tag{3.30}$$

Hence, for all $\varepsilon > 0$, we have

$$\int_X |g(u_n(x))|\omega(x)dx \le \frac{\varepsilon}{2} \tag{3.31}$$

if a is sufficiently large and $\omega(X)$ is sufficiently small. Therefore, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that

$$\begin{aligned}&\int_X |g(u_n(x)) - g(u(x))|\omega(x)dx \\ &\quad \le \int_X |g(u_n(x))|\omega(x)dx + \int_X |g(u(x))|\omega(x)dx \le \varepsilon,\end{aligned} \tag{3.32}$$

with $\omega(X) < \delta$. Thus, the Vitali convergence theorem tells us that (3.22) holds.

Step 7. Quasiboundedness of the operator A. Let $\{u_n\}$ be a sequence in Y with $u_n \rightharpoonup u$ in $H_0(\Omega)$ and suppose that

$$\langle Au_n, u_n\rangle_Y \le C\|u_n\|_{H_0(\Omega)}, \quad \forall n. \tag{3.33}$$

We want to show that the sequence $\{Au_n\}$ is bounded in Y^*. In fact, the boundedness of $\{u_n\}$ in $H_0(\Omega)$ implies that

$$\overline{\lim_{n\to\infty}}\langle Au_n, u_n\rangle_Y \le C. \tag{3.34}$$

Suppose by contradiction that the sequence $\{Au_n\}$ is unbounded in Y^*. Then there exists a subsequence, again denoted by $\{u_n\}$, such that

$$||Au_n||_{Y^*} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{3.35}$$

By the same arguments as in Step 6, we obtain that

$$\langle Au_n, \varphi \rangle_Y \longrightarrow \langle Au, \varphi \rangle_Y \quad \text{as } n \longrightarrow \infty, \ \forall \varphi \in Y. \tag{3.36}$$

The uniform boundedness principle tells us that the sequence $\{Au_n\}$ is bounded (which is a contradiction with (3.35)).

Therefore, by Theorem 1.1, the equation $Au = T$, with $T \in (H_0(\Omega))^*$, has a solution $u \in D(A) \subseteq H_0(\Omega)$, and it is the solution for problem (1.1).

(II) *Uniqueness.* If the function $g : \mathbb{R} \to \mathbb{R}$ is monotone increasing, we have that $(g(a) - g(b))(a - b) \geq 0$, for all $a, b \in \mathbb{R}$. Then

$$\begin{aligned} \langle Au - Av, u - v \rangle_Y &= \int_\Omega \langle \mathcal{A} \nabla (u - v), \nabla (u - v) \rangle dx \\ &\quad + \int_\Omega \big(g(u(x)) - g(v(x))\big)\big(u(x) - v(x)\big)\omega(x) dx \\ &\geq \int_\Omega \langle \mathcal{A} \nabla (u - v), \nabla (u - v) \rangle dx = \|u - v\|^2_{H_0(\Omega)}, \end{aligned} \tag{3.37}$$

for all $u, v \in D(A)$.

Therefore, if $u, v \in D(A)$ and $Au = Av = T$, we obtain that $u = v$.

References

[1] S. Chanillo and R. L. Wheeden, *Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions*, Amer. J. Math. **107** (1985), no. 5, 1191–1226.

[2] V. Chiadò Piat and F. Serra Cassano, *Relaxation of degenerate variational integrals*, Nonlinear Anal. **22** (1994), no. 4, 409–424.

[3] B. Franchi and R. Serapioni, *Pointwise estimates for a class of strongly degenerate elliptic operators: a geometrical approach*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), no. 4, 527–568.

[4] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing, Amsterdam, 1985.

[5] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1993.

[6] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Pure and Applied Mathematics, vol. 123, Academic Press, Florida, 1986.

[7] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. II/B*, Springer-Verlag, New York, 1990.

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