SOLUTIONS FOR NONLINEAR VARIATIONAL INEQUALITIES WITH A NONSMOOTH POTENTIAL

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First we examine a resonant variational inequality driven by the p-Laplacian and with a nonsmooth potential. We prove the existence of a nontrivial solution. Then we use this existence theorem to obtain nontrivial positive solutions for a class of resonant elliptic equations involving the p-Laplacian and a nonsmooth potential. Our approach is variational based on the nonsmooth critical point theory for functionals of the form $\varphi = \varphi_1 + \varphi_2$ with φ_1 locally Lipschitz and φ_2 proper, convex, lower semicontinuous.

1. Introduction

In this paper, we consider the following nonlinear variational inequality at resonance with a nonsmooth potential function ($Z \subseteq R^N$ is a bounded domain with a C^2-boundary ∂Z):

$$\int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z) - Dx(z))_{R^N} dz - \lambda_1 \int_Z |x(z)|^{p-2} x(z)(y-x)(z)dz$$
$$\geq \int_Z u(z)(y-x)(z)dz \quad \forall y \in C, \tag{1.1}$$

where $C = \{x \in W_0^{1,p}(Z) : x(z) \geq g(z) \text{ a.e on } Z\}$, with $g \in W^{1,p}(Z)$, $g(z) \leq 0$ a.e. on Z, and $u \in L^q(Z), (1/p + 1/q = 1, 1 < p < \infty), u(z) \in \partial j(z, x(z))$ a.e. on Z. Here, the potential function $z \to j(z, x)$ is only locally Lipschitz, not necessarily C^1, and $\partial j(z, \cdot)$ denotes the generalized (Clarke) subdifferential (see Section 2). In addition, $\lambda_1 > 0$ stands for the first eigenvalue of the negative p-Laplacian with Dirichlet boundary condition (denoted by $(-\triangle_p, W_0^{1,p}(Z))$).

Also, we study the following nonlinear elliptic problem at resonance with nonsmooth potential:

$$-\text{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) - \lambda_1 |x(z)|^{p-2} x(z) \in \partial j(z, x(z)) \quad \text{a.e on } Z,$$
$$x|_{\partial Z} = 0, \quad 1 < p < \infty, \ x \geq 0. \tag{1.2}$$

Having as a starting point a solution of problem (1.1) when $g \equiv 0$, we show that problem (1.2) has at least one nontrivial positive solution. Moreover, by strengthening a little our hypotheses, we show that the solution is in fact strictly positive and smooth.

Problems similar to (1.1) were investigated by Szulkin [19] and more recently by Le [13, 14] and Zhou and Huang [22]. In [19], the problem under consideration is semilinear (i.e., $p = 2$), not in resonance, $g \equiv 0$, and the potential function is smooth (i.e., a C^1 function). His approach is variational. Le [13, 14] considers nonlinear variational inequalities not in resonance, which though, are driven by general nonlinear differential operators, which include as a special case the p-Laplacian. The right-hand side nonlinearity is Carathéodory (thus the corresponding potential function is C^1) and it may also depend on the gradient of the unknown function (see Le [13]). His approach is based on the method of upper and lower solutions. Moreover, we mention that in Le [14] the interested reader can find a rich bibliography on the subject of variational inequalities. In Zhou and Huang [22], the problem is nonlinear, not in resonance, with smooth potential. The approach in that paper is based on the theory of nonlinear complementarity problems.

Problems like (1.2) were studied in the context of semilinear equations (i.e., $p = 2$) which are either nonresonant (see Alves and Miyagaki [1], Zhou [21]) or are near-resonance (see Mawhin and Schmitt [16] (ordinary differential equations) and Chiappinelli et al. [3] (elliptic equations)). For the near resonance problems, the authors obtain multiplicity results. For nonlinear problems driven by the p-Laplacian, the investigation is lagging behind and only recently Kyritsi and Papageorgiou [11] extended the aforementioned works on nearly resonant problems.

Unilateral problems with nonsmooth potential are also known in the literature as "variational-hemivariational inequalities" and arise in mechanics and engineering when one wants to consider more realistic nonmonotone and multivalued laws. We refer to the book of Naniewicz and Panagiotopoulos [17] and the paper of Goeleven et al. [7]. We should mention that our work here generalizes in many respects the semilinear eigenvalue problems studied by Goeleven et al. [7].

Our approach is variational, based on the nonsmooth extension of the theory of Szulkin [19], due to Kourogenis et al. [9]. For the convenience of the reader, in the next section we recall the basic definitions and facts from this theory, as well as relevant notions from convex analysis and the subdifferential theory for locally Lipschitz functions. For details, we refer to the books of Clarke [4] and Denkowski et al. [6].

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for all $x \in X$, we can find a neighborhood U of x and a constant $k_U > 0$ such that $|\varphi(y) - \varphi(u)| \leq k_U \|y - u\|$, for all $y, u \in U$. From convex analysis, we know that if $\psi : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is convex, lower semicontinuous, and proper (i.e., ψ is not identically $+\infty$), then ψ is locally Lipchitz in the interior of its effective domain $\text{dom}\,\psi = \{x \in X : \psi(x) < +\infty\}$. In particular, then a continuous, convex function $\psi : X \to \mathbb{R}$ is locally Lipschitz. In analogy to the directional derivative of a convex function, for a locally Lipschitz function $\varphi : X \to R$,

we define the generalized directional derivative of φ at x in the direction $h \in X$ by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}. \qquad (2.1)$$

It is easy to check that $\varphi^0(x; \cdot)$ is sublinear continuous, so it is the support function of a nonempty, w^*-compact, and convex set $\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \le \varphi^0(x;h) \text{ for all } h \in X\}$. The multifunction $\partial\varphi : X \to 2^{X^*} \setminus \{\varnothing\}$ is called the generalized (or Clarke) subdifferential of φ. If φ is also convex, then the generalized subdifferential coincides with the subdifferential in the sense of convex analysis, given by $\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \le \varphi(y) - \varphi(x) \text{ for all } y \in X\}$. Moreover, if $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. If $\varphi, \psi : X \to \mathbb{R}$ are locally Lipschitz functions and $\lambda \in R$, then for all $x \in X$, we have

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x), \qquad \partial(\lambda\varphi)(x) = \lambda\partial\varphi(x). \qquad (2.2)$$

Let $\Gamma_0(X)$ be the cone of convex, lower semicontinuous, proper functions. Let $\varphi_1 : X \to \mathbb{R}$ be locally Lipschitz, $\varphi_2 \in \Gamma_0(X)$, and set $\varphi = \varphi_1 + \varphi_2$. For such functions exists a nonsmooth critical point theory (see Kourogenis et al. [9]), which extends the theory of Szulkin [19], where $\varphi_1 \in C^1(X)$. We say that $x \in X$ is a critical point of $\varphi = \varphi_1 + \varphi_2$ if

$$\varphi_1^0(x; y - x) + \varphi_2(y) - \varphi_2(x) \ge 0 \quad \forall y \in X. \qquad (2.3)$$

We say that $\varphi = \varphi_1 + \varphi_2$ satisfies the generalized nonsmooth Palais-Smale condition (generalized nonsmooth PS-condition for short), if every sequence $\{x_n\}_{n \ge 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n \ge 1}$ is bounded and there exists $\{\varepsilon_n\}_{n \ge 1} \subseteq R_+, \varepsilon_n \downarrow 0$ such that

$$\varphi_1^0(x_n; y - x_n) + \varphi_2(y) - \varphi_2(x_n) \ge -\varepsilon_n \|y - x_n\| \quad \forall y \in X, \qquad (2.4)$$

has a strongly convergent subsequence. Remark that if $\varphi_2 \equiv 0$, then using Szulkin [19, Lemma 3], we can find $x_n^* \in X^*, \|x_n^*\| \le 1$, such that

$$\begin{aligned} \varphi_1^0(x_n; h) &\ge \varepsilon_n \langle x_n^*, h \rangle \quad \forall h \in X, \\ &\Longrightarrow \varepsilon_n x_n^* \in \partial\varphi_1(x_n) \quad \forall n \ge 1. \end{aligned} \qquad (2.5)$$

So it follows that $m(x_n) = \inf[\|u^*\| : u^* \in \partial\varphi_1(x_n)] \le \varepsilon_n \to 0$ as $n \to \infty$ and we recover the nonsmooth PS-condition introduced by Chang [2] (see also Kourogenis and Papageorgiou [10] for extensions).

Moreover, if $\varphi_1 \in C^1(X)$ and $\varphi_2 \equiv 0$, we recover the classical PS-condition (see Denkowski et al. [5, page 171] and Struwe [18, page 70]). The following generalized nonsmooth mountain pass theorem can be found in Kourogenis et al. [9].

THEOREM 2.1. *If X is a Banach space, $\varphi : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 : X \to \mathbb{R}$ locally Lipschitz and $\varphi_2 \in \Gamma_0(X)$, φ satisfies the generalized nonsmooth PS-condition, and there exist $r > 0$ and $x_0 \in X$ with $\|x_0\| > r$ such that*

$$\inf[\varphi(x) : \|x\| = r] \ge \beta > \max\{\varphi(0), \varphi(x_0)\}, \qquad (2.6)$$

then φ has a critical point $x \in X$ with corresponding critical value $c = \varphi(x)$ given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \tag{2.7}$$

where $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = x_0\}$. Moreover, $c \geq \beta$.

Since our problems are at resonance, they involve the principal eigenvalue λ_1 of $(-\triangle_p, W_0^{1,p}(Z))$. So, very briefly, we recall what is known about λ_1. For details, we refer to Denkowski et al. [5]. So, on Z, we consider the following nonlinear eigenvalue problem:

$$-\operatorname{div}\left(\|Dx(z)\|^{p-2} Dx(z)\right) = \lambda |x(z)|^{p-2} x(z) \quad \text{a.e on } Z,$$
$$x|_{\partial Z} = 0, \quad 1 < p < \infty, \quad \lambda \in R. \tag{2.8}$$

Every $\lambda \in R$ for which (2.8) has a nontrivial solution is said to be an eigenvalue of $(-\triangle_p, W_0^{1,p}(Z))$ and the nontrivial solution $x \in W_0^{1,p}(Z)$ is a corresponding eigenfunction. Using as a test function $x \in W_0^{1,p}(Z)$, we see that every eigenvalue λ is nonnegative. In fact, the first eigenvalue λ_1 is strictly positive, isolated, and simple (i.e., the corresponding eigenspace is one-dimensional). Moreover, there is a variational characterization of $\lambda_1 > 0$ via a Rayleigh quotient,

$$\lambda_1 = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z), \, x \neq 0\right]. \tag{2.9}$$

This infimum is realized at the corresponding normalized eigenfunction u_1. Note that in (2.9), we can replace x by $|x|$, and so we infer that u_1 does not change sign on Z. Moreover, from nonlinear regularity theory, we know that $u_1 \in C^1(\bar{Z})$ and we can say that $u_1(z) > 0$ for all $z \in Z$.

3. Generalized variational inequalities

In this section, we deal with problem (1.1). Our hypotheses on the nonsmooth potential $j(z,x)$ are the following:

H(j) $j : Z \times R \to R$ is a function such that $j(z,0) = 0$ a.e. on Z and
    (i) for all $x \in R$, $z \to j(z,x)$ is measurable;
    (ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;
    (iii) for every $M > 0$, there exists $\alpha_M \in L^\infty(Z)$ such that for almost all $z \in Z$, all $|x| \leq M$, and all $u \in \partial j(z,x)$, we have $|u| \leq \alpha_M(z)$;
    (iv) $\lim_{x \to +\infty}(u/x^{p-1}) = 0$ uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$;
    (v) let $g_1(z,x) = \max[u : u \in \partial j(z,x)]$, $g_0(z,x) = \min[u : u \in \partial j(z,x)]$, and

$$G_i(z,x) = \begin{cases} \dfrac{pj(z,x)}{x} - g_i(z,x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \quad i = 0,1. \tag{3.1}$$

We assume that there exists a function $G_+ \in L^1(Z)$ such that $G_+(z) = \liminf_{x \to +\infty} G_1(z,x)$ uniformly for almost all $z \in Z$, $\int_Z G_+(z) u_1(z) dz > 0$, and $G_0(z,x) \leq \theta(z), \theta \in L^1(Z)$;

(vi) $\limsup_{x \to 0^+} (pj(z,x)/x^p) < 0$ uniformly for almost all $z \in Z$.

Remark 3.1. Hypothesis H(j)(v) is an extension of the classical Landesman-Lazer condition.

Let $\varphi_1 : W_0^{1,p}(Z) \to \mathbb{R}$ be the function defined by

$$\varphi_1(x) = \frac{1}{p} \|Dx\|_p^p - \frac{\lambda_1}{p} \|x\|_p^p - \int_Z j(z,x(z)) dz. \tag{3.2}$$

We know that φ_1 is locally Lipschitz (see Hu and Papageorgiou [8, page 313]). Also, we set

$$\varphi_2(x) = i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases} \tag{3.3}$$

Since $C \subseteq W_0^{1,p}(Z)$ is nonempty, closed, and convex, we have that $i_C \in \Gamma_0(W_0^{1,p}(Z))$. We define

$$\varphi = \varphi_1 + \varphi_2. \tag{3.4}$$

PROPOSITION 3.2. *If hypotheses H(j) hold, then φ satisfies the generalized nonsmooth PS-condition.*

Proof. Let $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that $|\varphi(x_n)| \leq M_1$ for some $M_1 > 0$, all $n \geq 1$, and $\varphi_1^0(x_n; y - x_n) + \varphi_2(y) - \varphi_2(x_n) \geq -\varepsilon_n \|y - x_n\|$ for all $y \in W_0^{1,p}(Z)$.

Evidently, $\{x_n\}_{n \geq 1} \subseteq C$. We can find $x_n^* \in \partial \varphi_1(x_n)$ such that $\varphi_1^0(x_n; y - x_n) = \langle x_n^*, y - x_n \rangle$. This is a consequence of the fact that $\varphi_1^0(x_n; \cdot)$ is the support function of $\partial \varphi_1(x_n)$ and the latter is w-compact in $W^{-1,q}(Z) = W_0^{1,p}(Z)^*$. For every $n \geq 1$, we have

$$x_n^* = A(x_n) - \lambda_1 |x_n|^{p-2} x_n - u_n. \tag{3.5}$$

Here, $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is the nonlinear operator defined by

$$\langle A(x), w \rangle = \int_Z \|Dx(z)\|^{p-2} (Dx(z), Dw(z))_{\mathbb{R}^N} dz, \quad \forall x, w \in W_0^{1,p}(Z), \tag{3.6}$$

and $u_n \in L^q(Z)$, $u_n(z) \in \partial j(z, x_n(z))$ a.e. on Z (see Clarke [4, page 83] and Denkowski et al. [6, page 617]). It is easy to check that A is monotone, demicontinuous, hence it is maximal monotone.

We claim that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose that this is not the case. By passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Set $v_n = x_n/\|x_n\|$, $n \geq 1$. Evidently, $\|v_n\| = 1$, $n \geq 1$, and so we may assume that

  (i) $v_n \xrightarrow{w} v$ in $W_0^{1,p}(Z)$, $v_n \to v$ in $L^p(Z)$ (from the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$),

  (ii) $v_n(z) \leq v(z)$ a.e on Z and $|v_n(z)| \leq k(z)$ a.e on Z, $n \geq 1$, with $k \in L^p(Z)$.

(See Denkowski et al. [6, page 147]).

By virtue of hypotheses H(j)(iii) and (iv), given $\varepsilon > 0$, we can find $\alpha_\varepsilon \in L^\infty(Z)$ such that for almost all $z \in Z$, all $x \geq 0$, and all $u \in \partial j(z,x)$, we have

$$|u| \leq \alpha_\varepsilon(z) + \varepsilon x^{p-1}. \tag{3.7}$$

Using the mean value theorem for locally Lipschitz functions (see Clarke [4, page 41] and Denkowski et al. [6, page 609]), for almost all $z \in Z$ and all $x \geq 0$, we have

$$|j(z,x)| \leq |j(z,0)| + \alpha_\varepsilon(z)|x| + \varepsilon|x|^p \leq \beta_\varepsilon(z) + 2\varepsilon|x|^p \quad (\text{with } \beta_\varepsilon \in L^1(Z))$$

$$\Longrightarrow \limsup_{n \to \infty} \left| \int_Z \frac{j(z,x_n(z))}{\|x_n\|^p} dz \right| \leq 2\varepsilon \tag{3.8}$$

(see (3.7)).

Let $\varepsilon \downarrow 0$ to conclude that $\int_Z (j(z,x_n(z))/\|x_n\|^p) dz \to 0$ as $n \to \infty$. From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$, we have

$$\frac{\varphi(x_n)}{\|x_n\|^p} = \frac{1}{p}\|Dv_n\|_p^p - \frac{\lambda_1}{p}\|v_n\|_p^p - \int_Z \frac{j(z,x_n(z))}{\|x_n\|^p} dz \leq \frac{M_1}{\|x_n\|^p} \tag{3.9}$$

(since $\varphi_2(x_n) = 0$, because $x_n \in C$).

Passing to the limit as $n \to \infty$ and using the weak lower semicontinuity of the norm functional, we obtain

$$\frac{1}{p}\|Dv\|_p^p \leq \frac{\lambda_1}{p}\|v\|_p^p$$

$$\Longrightarrow \frac{1}{p}\|Dv\|_p^p = \frac{\lambda_1}{p}\|v\|_p^p \tag{3.10}$$

$$\Longrightarrow v = \pm u_1 \quad \text{or} \quad v = 0$$

(see [4]).

If $v = 0$, then $\|Dv_n\|_p \to 0$, and so $v_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|v_n\| = 1$ for all $n \geq 1$. So, $v = \pm u_1$. Recall that $x_n(z) \geq g(z)$ a.e. on Z, $n \geq 1$. Hence, $v_n(z) \geq g(z)/\|x_n\|$ a.e. on Z, and so, in the limit, we have $v(z) \geq 0$ a.e on Z. Therefore, $v = u_1$.

Now we fix $y = 0 \in C$. From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq C$, we have

$$\|Dx_n\|_p^p - \lambda_1\|x_n\|_p^p - \int_Z v_n(z)x_n(z)dz \leq \varepsilon_n\|x_n\|. \tag{3.11}$$

Also, since $|\varphi(x_n)| \leq M_1$ for all $n \geq 1$, we have

$$-\|Dx_n\|_p^p + \lambda_1\|x_n\|_p^p + \int_Z pj(z,x_n(z))dz \leq pM_1. \tag{3.12}$$

Adding (3.11) and (3.12), we obtain

$$\int_Z (pj(z,x_n(z)) - u_n(z)x_n(z))dz \leq \varepsilon_n\|x_n\| + pM_1$$

$$\implies \int_Z \left(\frac{pj(z,x_n(z))}{\|x_n\|} - u_n(z)v_n(z)\right)dz \leq \varepsilon_n + \frac{pM_1}{\|x_n\|}. \tag{3.13}$$

Set

$$h_n(z) = \begin{cases} \dfrac{j(z,x_n(z))}{x_n(z)} & \text{if } x_n(z) \neq 0, \\ 0 & \text{if } x_n(z) = 0. \end{cases} \tag{3.14}$$

Then we have

$$\begin{aligned}
\varepsilon_n + \frac{pM_1}{\|x_n\|} &\geq \int_Z \frac{pj(z,x_n(z))}{\|x_n\|}dz - \int_Z u_n(z)v_n(z)dz \\
&\geq \int_Z ph_n(z)v_n(z)dz - \int_{\{v_n>0\}} g_1(z,x_n(z))v_n(z)dz \\
&\quad - \int_{\{v_n<0\}} g_0(z,x_n(z))v_n(z)dz \quad (\text{since } j(z,0) = 0 \text{ a.e. on } Z) \\
&\geq \int_{\{v_n>0\}} G_1(z,x_n(z))v_n(z)dz + \int_{\{v_n<0\}} \theta(z)v_n(z)dz
\end{aligned} \tag{3.15}$$

(see hypothesis H(j)(v) and recall that $v_n(z) \geq 0$ a.e. on Z).

Recall that $v = u_1$ and $u_1(z) > 0$ for all $z \in Z$. It follows that $x_n(z) \to +\infty$ a.e. on Z as $n \to \infty$, and if by χ_E we denote the characteristic function of a measurable set $E \subseteq Z$, we have

$$\chi_{\{v_n>0\}}(z) \longrightarrow 1, \quad \chi_{\{v_n<0\}}(z) \longrightarrow 0 \quad \text{a.e. on } Z. \tag{3.16}$$

Also, from the definition of the function $G_+(\cdot)$ (see hypothesis H(j)(v)), given $\varepsilon > 0$, we can find $M_2 > 0$ such that for almost all $z \in Z$ and all $x > M_2$, we have

$$G_1(z,x) \geq G_+(z) - \varepsilon. \tag{3.17}$$

On the other hand, for almost all $z \in Z$ and all $x \in [0,M_2]$, we have

$$G_1(z,x) \geq -\beta_{M_2}(z) \quad \text{with } \beta_{M_2} \in L^1(Z)_+. \tag{3.18}$$

Therefore, for almost all $z \in Z$ and all $x \in \mathbb{R}_+$, we have

$$G_1(z,x) \geq -\xi(z) \quad \text{with } \xi \in L^1(Z)_+. \tag{3.19}$$

Passing to the limit as $n \to \infty$ in (3.15) and using Fatou's lemma (remark that (3.19) permits the use of Fatou's lemma), we obtain

$$\int_Z G_+(z)u_1(z)dz \le 0, \tag{3.20}$$

which contradicts hypothesis H(j)(v). Therefore, it follows that $\{x_n\}_{n\ge 1} \subseteq C$ is bounded in $W_0^{1,p}(Z)$, and so we may assume that $x_n \overset{w}{\to} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$, with $x \in C$. We have

$$\langle A(x_n), x_n - y \rangle - \lambda_1 \int_Z |x_n|^{p-2} x_n(x_n - y)dz - \int_Z u_n(x_n - y)dz$$
$$\le \varepsilon_n \|y - x_n\| \quad \forall y \in C. \tag{3.21}$$

We fix $y = x \in C$. Hence we have

$$\langle A(x_n), x_n - x \rangle - \lambda_1 \int_Z |x_n|^{p-2} x_n(x_n - x)dz - \int_Z u_n(x_n - x)dz \le \varepsilon_n \|x - x_n\|. \tag{3.22}$$

Since $x_n \to x$ in $L^p(Z)$, we have $\int_Z |x_n|^{p-2} x_n(x_n - x)dz$, $\int_Z u_n(x_n - x)dz \to 0$ as $n \to \infty$. Hence,

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x \rangle \le 0. \tag{3.23}$$

Recall that A is maximal monotone, thus generalized pseudomonotone (see Denkowski et al. [5, page 37]). So we have

$$\|Dx_n\|_p^p = \langle A(x_n), x_n \rangle \longrightarrow \langle A(x), x \rangle = \|Dx\|_p^p. \tag{3.24}$$

Because $Dx_n \overset{w}{\to} Dx$ in $L^p(Z, \mathbb{R}^N)$ and $L^p(Z, \mathbb{R}^N)$ is uniformly convex, from the Kadec-Klee property, we have $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$, hence $x_n \to x$ in $W_0^{1,p}(Z)$. $\qquad \square$

PROPOSITION 3.3. If hypotheses H(j) hold, then $\varphi(tu_1) \to -\infty$ as $t \to +\infty$.

Proof. Note that for all $t \ge 0$, $tu_1 \in C$, and so

$$\varphi(tu_1) = -\int_Z j(z, tu_1(z))dz. \tag{3.25}$$

Recall that given $\varepsilon > 0$, we can find $M_2 = M_2(\varepsilon) > 0$ such that for all $z \in Z\backslash D$, $|D|_N = 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on R^N), and all $x > M_2$, we have

$$k_\varepsilon^+(z) = G_+(z) - \varepsilon \le G_1(z, x)$$
$$\implies \frac{G_1(z, x)}{x^p} \ge \frac{k_\varepsilon^+(z)}{x^p} = \frac{d}{dx}\left(-\frac{1}{p-1}\frac{k_\varepsilon^+(z)}{x^{p-1}}\right). \tag{3.26}$$

For all $z \in Z\backslash D$, all $x > M_2$, and all $u \in \partial j(z, x)$, we have

$$\frac{G_1(z, x)}{x^p} = \frac{pj(z, x)}{x^{p+1}} - \frac{g_1(z, x)}{x^p} \le \frac{pj(z, x)}{x^{p+1}} - \frac{u}{x^p}. \tag{3.27}$$

For all $z \in Z \setminus D$, the function $x \to pj(z,x)/x^p$ is locally Lipschitz on $(M_2, +\infty)$ and

$$\partial \left(\frac{j(z,x)}{x^p} \right) \subseteq \frac{\partial j(z,x) x^p - pj(z,x) x^{p-1}}{x^{2p}} = \frac{\partial j(z,x)}{x^p} - \frac{pj(z,x)}{x^{p+1}}$$

$$\implies \max \left[u' : u' \in \partial \left(\frac{j(z,x)}{x^p} \right) \right] \leq \frac{g_1(z,x)}{x^p} - \frac{pj(z,x)}{x^{p+1}} = -\frac{1}{x^p} G_1(z,x) \tag{3.28}$$

(see Clarke [4, page 48] and Denkowski et al. [6, page 612]). So, for all $z \in Z \setminus D$, $|D|_N = 0$, all $x > M_2$, and all $u' \in \partial(j(z,x)/x^p)$, we have

$$u' \leq \frac{d}{dx} \left(\frac{1}{p-1} \frac{k_\varepsilon^+(z)}{x^{p-1}} \right) \tag{3.29}$$

(see (3.26)).

For fixed $z \in Z \setminus D$, $|D|_N = 0$, the function $x \to j(z,x)/x^p$ is locally Lipschitz on $(M_2, +\infty)$, hence it is differentiable at all $x \in (M_2, +\infty) \setminus E_1(z)$, with $|E_1(z)|_1 = 0$. We set

$$u_0'(z,x) = \begin{cases} \dfrac{d}{dx} \left(\dfrac{j(z,x)}{x^p} \right) & \text{if } x \in (M_2, +\infty) \setminus E(z), \\ 0 & \text{otherwise}, \end{cases}$$

$$\implies u_0'(z,x) \leq \frac{d}{dx} \left(\frac{1}{p-1} \frac{k_\varepsilon^+(z)}{x^{p-1}} \right) \quad \text{for almost all } x \in (M_2, +\infty). \tag{3.30}$$

Consider $w, v > M_2$, $v > w$, and integrate the last inequality on the interval $[w, v]$. We obtain

$$\frac{j(z,v)}{v^p} - \frac{j(z,w)}{w^p} \leq \frac{k_\varepsilon^+(z)}{p-1} \left(\frac{1}{v^{p-1}} - \frac{1}{w^{p-1}} \right). \tag{3.31}$$

We have seen in the proof of Proposition 3.2 that given $\varepsilon > 0$, we can find $\beta_\varepsilon \in L^1(Z)_+$ such that for all $z \in Z \setminus D$, $|D|_N = 0$, and all $x \in R$, we have

$$|j(z,x)| \leq \beta_\varepsilon(z) + 2\varepsilon |x|^p$$

$$\implies \lim_{x \to +\infty} \frac{j(z,x)}{x^p} = 0 \quad \text{(since } \varepsilon > 0 \text{ was arbitrary).} \tag{3.32}$$

So, returning to inequality (3.31) and passing to the limit as $v \to +\infty$, we obtain

$$\frac{j(z,w)}{w^p} \geq \frac{k_\varepsilon^+(z)}{p-1} \frac{1}{w^{p-1}}$$

$$\implies \frac{j(z,w)}{w} \geq \frac{k_\varepsilon^+(z)}{p-1}$$

$$\implies \liminf_{w \to +\infty} \frac{j(z,w)}{w} \geq \frac{1}{p-1} G_+(z) \quad \text{(since } \varepsilon > 0 \text{ was arbitrary).} \tag{3.33}$$

Suppose that the conclusion of the proposition was not true. This means that we can find $M_3 > 0$ and a sequence $\{t_n\}_{n \geq 1} \subseteq \mathbb{R}_+$ with $t_n \to +\infty$, such that

$$
\begin{aligned}
&\varphi(t_n u_1) = -\int_Z j(z, t_n u_1(z)) dz \geq -M_3, \quad \forall n \geq 1, \\
&\implies \frac{\varphi(t_n u_1)}{t_n} = -\int_Z \frac{j(z, t_n u_1(z))}{t_n u_1(z)} u_1(z) dz \geq \frac{-M_3}{t_n} \quad (\text{recall } u_1(z) > 0, \ \forall z \in Z).
\end{aligned}
\tag{3.34}
$$

Via Fatou's lemma and using (3.33), we obtain

$$
\int_Z G_+(z) u_1(z) dz \leq 0,
\tag{3.35}
$$

a contradiction to hypothesis H(j)(v). So, $\varphi(t u_1) \to -\infty$ as $t \to +\infty$.   □

PROPOSITION 3.4. *If hypotheses H(j) hold, then* $\varphi(x) \geq \beta_1 \|x\|^p - \beta_2 \|x\|^\eta$ *for some* $\beta_1, \beta_2 > 0$,

$$
p < \eta < p^* =
\begin{cases}
\dfrac{N}{N-p} & \text{if } N > p, \\
+\infty & \text{if } N \leq p,
\end{cases}
\tag{3.36}
$$

and all $x \in W_0^{1,p}(Z)_+$.

Proof. By virtue of hypothesis H(j)(vi), we can find $\delta > 0$ and $M_4 > 0$ such that for almost all $z \in Z$ and all $x > M_4$, we have

$$
j(z, x) \leq -\frac{\delta}{p} x^p.
\tag{3.37}
$$

If we combine this with hypothesis H(j)(iii) and the mean value theorem of locally Lipschitz functions (see Clarke [4, page 41] and Denkowski et al. [6, page 609]), for almost all $z \in Z$, all $x > 0$, and for $\eta \in (p, p^*]$, we have

$$
j(z, x) \leq -\frac{\delta}{p} x^p + \xi_1 x^\eta \quad \text{for some } \xi_1 > 0.
\tag{3.38}
$$

So, for all $x \in W_0^{1,p}(Z)_+$ (i.e., $x \in W_0^{1,p}(Z)$ and $x(z) \geq 0$ a.e. on Z), we have

$$
\begin{aligned}
\varphi(x) &= \frac{1}{p} \|Dx\|_p^p - \frac{\lambda_1}{p} \|x\|_p^p - \int_Z j(z, x(z)) dz \\
&\geq \frac{1}{p} \|Dx\|_p^p - \frac{\lambda_1}{p} \|x\|_p^p + \frac{\delta}{p} \|x\|_p^p - \xi_1 \|x\|_\eta^\eta
\end{aligned}
\tag{3.39}
$$

(see (3.38)).

Using Poincaré's inequality and the fact that $W_0^{1,p}(Z)$ is embedded continuously in $L^\eta(Z)$ (recall $\eta \leq p^*$), we see that we can find $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$
\varphi(x) \geq \beta_1 \|x\|^p - \beta_2 \|x\|^\eta.
\tag{3.40}
$$

□

Propositions [3.3](#) and [3.4](#) imply that the energy functional φ has the generalized nonsmooth mountain pass geometry. So, applying [Theorem 2.1](#), we will be able to prove the following existence theorem for problem [(1.1)](#).

Theorem 3.5. If *hypotheses H(j) hold,* then *problem (1.1) has a nontrivial solution $x \in W_0^{1,p}(Z)$.*

Proof. By virtue of [Proposition 3.4](#), if we choose $r > 0$ small, then for all $x \in W_0^{1,p}(Z)$ with $\|x\| = r$, we have

$$\varphi(x) \geq \xi_2 > 0. \tag{3.41}$$

On the other hand, because of [Proposition 3.3](#), we can find $t > r$ such that $\varphi(tu_1) \leq \varphi(0) \leq 0$. These facts in conjunction with [Proposition 3.2](#) permit the use of [Theorem 2.1](#), which gives $x \in C$ such that

$$\varphi(x) \geq \xi_2 > 0 \geq \varphi(0) \quad (\text{i.e., } x \neq 0), \qquad \varphi_1^0(x;h) + \varphi_2(x+h) - \varphi_2(x) \geq 0, \quad \forall h \in W_0^{1,p}(Z). \tag{3.42}$$

Set $\psi_1(h) = \varphi_1^0(x;h)$ and $\psi_2(h) = \varphi_2(x+h) - \varphi_2(x)$. Then $\psi_1 : W_0^{1,p}(Z) \to \mathbb{R}$ is continuous, convex, while $\psi_2 \in \Gamma_0(W_0^{1,p}(Z))$. Remark that $\partial_c\psi_1(0) = \partial\varphi_1(x)$ and $\partial_c\psi_2(0) = \partial_c\varphi_2(x)$. Moreover, from the second inequality in [(3.42)](#), we have

$$\psi_1(h) + \psi_2(h) \geq 0 \implies 0 \in \partial_c(\psi_1 + \psi_2)(0) \quad (\text{since } \psi_1(0) = \psi_2(0) = 0). \tag{3.43}$$

But from convex analysis (see Denkowski et al. [6, page 549]), we have that

$$\partial_c(\psi_1 + \psi_2)(0) = \partial_c\psi_1(0) + \partial_c\psi_2(0) = \partial\varphi_1(x) + \partial_c\varphi_2(x). \tag{3.44}$$

We know that $\partial_c\varphi_2(x) = N_C(x) =$ the normal cone to C at x (see Denkowski et al. [6, page 622]). From [(3.43)](#) and [(3.44)](#), we see that we can find $x^* \in \partial\varphi_1(x)$ and $v^* \in \partial_c\varphi_2(x) = N_C(x)$ such that $x^* + v^* = 0$. We have

$$x^* = A(x) - \lambda_1|x|^{p-2}x - u, \tag{3.45}$$

with $u \in L^q(Z)$, $u(z) \in \partial j(z,x(z))$ a.e on Z, and $\langle v^*, x - y \rangle \geq 0$ for all $y \in C$.

It follows that $\langle x^*, y - x \rangle \geq 0$ for all $y \in C$. Therefore, we have

$$\begin{aligned}
&\langle A(x) - \lambda_1|x|^{p-2}x - u, y - x \rangle \geq 0 \quad \forall y \in C, \\
&\implies \int_Z \|Dx\|^{p-2}(Dx, D(y-x))_{\mathbb{R}^N}\,dz - \lambda_1 \int_Z |x|^{p-2}x(y-x)\,dz \\
&\quad \geq \int_Z u(y-x)\,dz \quad \forall y \in C, \\
&\implies x \in W_0^{1,p}(Z) \text{ is a nontrivial solution of problem (1.1).} \qquad \square
\end{aligned} \tag{3.46}$$

4. Positive solutions

In this section, we deal with problem (1.2). Having as our starting point the existence theorem of the previous section, we establish that problem (1.2) has a nontrivial positive solution.

THEOREM 4.1. *If hypotheses H(j) hold, there exists $M_0 > 0$ such that for almost all $z \in Z$, all $x \geq M_0$, and all $u \in \partial j(z,x)$, $u \geq 0$ or $u \leq 0$ and $\partial j(z,0) \subseteq R_+$ a.e. on Z, then problem (1.2) has a nontrivial solution $x \in W_0^{1,p}(Z)_+$ (i.e., $x(z) \geq 0$ a.e. on Z).*

Proof. Let $C = W_0^{1,p}(Z)_+ = \{x \in W_0^{1,p}(Z) : x(z) \geq 0 \text{ a.e. on } Z\}$ (so the obstacle now is $g \equiv 0$) and let φ_1, φ_2, and φ be as before. From Theorem 3.5, we know that there exists $x \in C$, $x \neq 0$, such that x is a critical point of $\varphi = \varphi_1 + \varphi_2$. So we can find $x^* \in \partial \varphi_1(x)$ such that

$$x^* = A(x) - \lambda_1 |x|^{p-2} x - u, \quad \text{with } u \in L^q(Z), \ u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \qquad (4.1)$$

and $\langle x^*, y - x \rangle \geq 0$ for all $y \in C$.

First we assume that for almost all $z \in Z$, all $x \geq M_0$, and all $u \in \partial j(z,x)$, we have $u \geq 0$. Let $\theta \in W_0^{1,p}(Z)$, $\varepsilon > 0$, and $y = (x + \varepsilon \theta)^+ = x + \varepsilon \theta + (x + \varepsilon \theta)^-$. We obtain

$$
\begin{aligned}
\langle x^*, \varepsilon \theta \rangle &\geq -\langle x^*, (x + \varepsilon \theta)^- \rangle \\
&= -\langle A(x), (x + \varepsilon \theta)^- \rangle + \lambda_1 \int_Z |x|^{p-2} x (x + \varepsilon \theta)^- dz + \int_Z u(x + \varepsilon \theta)^- dz.
\end{aligned}
\qquad (4.2)
$$

Set $Z_-^\varepsilon = \{z \in Z : (x + \varepsilon \theta)(z) < 0\}$. We have

$$
\begin{aligned}
-\langle A(x), (x + \varepsilon \theta)^- \rangle &= -\int_Z \|Dx\|^{p-2} (Dx, D(x + \varepsilon \theta)^-)_{R^N} dz \\
&= \int_{Z_-^\varepsilon} \|Dx\|^{p-2} (Dx, D(x + \varepsilon \theta))_{R^N} dz \\
&\geq \varepsilon \int_{Z_-^\varepsilon} \|Dx\|^{p-2} (Dx, D\theta)_{R^N} dz
\end{aligned}
\qquad (4.3)
$$

(see [6, page 348]).

Also, we have

$$
\begin{aligned}
\lambda_1 \int_Z |x|^{p-2} x (x + \varepsilon \theta)^- dz \\
= -\lambda_1 \int_{Z_-^\varepsilon} |x|^{p-2} x (x + \varepsilon \theta) dz \geq 0 \quad (\text{since } x \in C \text{ and } (x + \varepsilon \theta)|_{Z_-^\varepsilon} < 0).
\end{aligned}
\qquad (4.4)
$$

In addition, we have

$$
\begin{aligned}
\int_Z u(x + \varepsilon \theta)^- dz &= -\int_{Z_-^\varepsilon} u(x + \varepsilon \theta) dz \\
&= -\int_{Z_-^\varepsilon \cap \{x < M_0\}} u(x + \varepsilon \theta) dz - \int_{Z_-^\varepsilon \cap \{x \geq M_0\}} u(x + \varepsilon \theta) dz.
\end{aligned}
\qquad (4.5)
$$

By hypothesis, we have

$$-\int_{Z_-^\varepsilon \cap \{x \geq M_0\}} u(x + \varepsilon\theta)dz \geq 0. \tag{4.6}$$

Also, since by hypothesis, $\partial j(z,0) \subseteq R_+$ a.e. on Z, we see that $u(z) \geq 0$ a.e. on $Z_-^\varepsilon \cap \{x = 0\}$. Moreover, remark that since $x(z) \geq 0$ a.e. on Z, we have $\theta(z) < 0$ a.e. on Z_-^ε. So we obtain

$$-\int_{Z_-^\varepsilon \cap \{x=0\}} u\varepsilon\theta\, dz \geq 0$$
$$\implies -\int_{Z_-^\varepsilon \cap \{x<M_0\}} u(x + \varepsilon\theta)dz$$
$$\geq -\int_{Z_-^\varepsilon \cap \{0<x<M_0\}} u(x + \varepsilon\theta)dz \tag{4.7}$$
$$\geq \xi_3 \int_{Z_-^\varepsilon \cap \{0<x<M_0\}} (x + \varepsilon\theta)dz \quad \text{for some } \xi_3 > 0$$
$$\geq \xi_3\varepsilon \int_{Z_-^\varepsilon \cap \{0<x<M_0\}} \theta\, dz \quad (\text{since } x(z) \geq 0 \text{ a.e. on } Z)$$

(see hypothesis $H(j)(iii)$).

Using (4.6) and (4.7) in (4.5), we have

$$\int_Z u(x + \varepsilon\theta)^- dz \geq \xi_3\varepsilon \int_{Z_-^\varepsilon \cap \{0<x<M_0\}} \theta\, dz. \tag{4.8}$$

We return to (4.2) and we use (4.3), (4.4), and (4.8). We obtain

$$\langle x^*, \theta \rangle \geq \int_{Z_-^\varepsilon} \|Dx\|^{p-2}(Dx, D\theta)_{R^N}\, dz + \xi_3 \int_{Z_-^\varepsilon \cap \{0<x<M_0\}} \theta\, dz. \tag{4.9}$$

Recall that $Dx(z) = 0$ a.e. on $\{x = 0\}$ (see Denkowski et al. [6, page 349]). Also note that $|Z_-^\varepsilon \cap \{0 < x < M_0\}|_N \to 0$ as $\varepsilon \downarrow 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on R^N). So, if we pass to the limit as $\varepsilon \downarrow 0$, we obtain

$$\langle x^*, \theta \rangle \geq 0, \quad \forall \theta \in W_0^{1,p}(Z),$$
$$\implies x^* = A(x) - \lambda_1 |x|^{p-2}x - u = 0,$$
$$-\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) - \lambda_1 |x(z)|^{p-2}x(z) = u(z) \quad \text{a.e on } Z, \; x|_{\partial Z} = 0, \tag{4.10}$$
$$\implies x \in W_0^{1,p}(Z)_+ \text{ is a nontrivial solution of (1.2).}$$

Next assume that for almost all $z \in Z$, all $x \geq M_0$, and all $u \in \partial j(z,x)$, we have $u \leq 0$. Then we have (see hypothesis H(j)(iii) and recall that $\partial j(z,0) \subseteq R_+$ a.e. on Z)

$$-\int_{Z_-^\varepsilon \cap \{x<M_0\}} u(x + \varepsilon\theta)dz \geq \xi_3\varepsilon \int_{Z_-^\varepsilon \cap \{0<x<M_0\}} \theta\, dz. \tag{4.11}$$

Also we have

$$-\int_{Z^\varepsilon_-\cap\{x\geq M_0\}} u(x+\varepsilon\theta)\,dz \geq -\varepsilon\int_{Z^\varepsilon_-\cap\{x\geq M_0\}} u\theta\,dz \quad \text{(since } -ux\geq 0\text{).} \tag{4.12}$$

Remark that $|Z^\varepsilon_-\cap\{x\geq M_0\}|_N \to 0$ as $\varepsilon\downarrow 0$. Using (4.3), (4.4), (4.11), and (4.12) in (4.2), we obtain

$$\langle x^*,\theta\rangle \geq \int_{Z^\varepsilon_-} \|Dx\|^{p-2}(Dx,D\theta)_{R^N}\,dz + \xi_3\int_{Z^\varepsilon_-\cap\{0<x<M_0\}} \theta\,dz - \int_{Z^\varepsilon_-\cap\{x\geq M_0\}} u\theta\,dz$$

$$\implies \langle x^*,\theta\rangle \geq 0 \quad \forall\theta\in W^{1,p}_0(Z)\text{ (by passing to the limit as }\varepsilon\downarrow 0\text{)}$$

$$\implies x^* = A(x) - \lambda_1|x|^{p-2}x - u = 0$$

$$\implies x = \text{solution of (1.2).} \qquad\qquad\square \tag{4.13}$$

If we strengthen our hypotheses on j, we can have that x is smooth and strictly positive.

Theorem 4.2. *If hypotheses $H(j)$ hold, there exists $M_0>0$ such that for almost all $z\in Z$, all $x\geq M_0$, and all $u\in\partial j(z,x)$, $u\geq 0$ or $u\leq 0$, $\partial j(z,0)\subseteq\mathbb{R}_+$ a.e. on Z, and for almost all $z\in Z$, all $x\in\mathbb{R}_+$, and all $u\in\partial j(z,x)$, $u\geq -\xi_4 x^{p-1}$ for some $\xi_4\geq\lambda_1$, then problem (1.2) has a solution $x\in C^1(\bar{Z})$ such that $x(z)>0$ for all $z\in Z$ and $\partial x/\partial n<0$ on ∂Z.*

Proof. Let $x\in W^{1,p}_0(Z)_+$ be the nontrivial solution of (1.2) obtained in Theorem 4.1. From Ladyzhenskaya and Ural'tseva [12, page 286], we know that $x\in L^\infty(Z)_+$. Invoking [15, Theorem 1], we infer that $x\in C^1(\bar{Z})_+$.

Moreover, since $-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = u(z)+\lambda_1|x(z)|^{p-2}x(z) \geq (-\xi_4+\lambda_1)|x(z)|^{p-1}$ a.e. on Z, we can apply [20, Theorem 5] to conclude that $x(z)>0$ for all $z\in Z$ and $\partial x/\partial n<0$ on ∂Z. $\qquad\square$

Remark 4.3. If, for simplicity, we drop the z-dependence, the following function satisfies the hypotheses in all the theorems of this paper:

$$j(x) = \begin{cases} \ln\left(1+x^2\right) & \text{if } x<0, \\ -x^r & \text{if } x\in[0,1]\text{, with } r<p, \\ x-2 & \text{if } x\geq 1. \end{cases} \tag{4.14}$$

Acknowledgment

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