CRITICAL VALUES LIE ON A LINE

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We prove that critical values set of a differentiable map lies on a line of certain smoothness class.

1. Introduction

For those familiar with the "space-filling curves" topic, the headline of the paper is no surprise. G. Peano in 1890 constructed the first such continuous function $f_p : [0,1] \xrightarrow{\text{onto}} [0,1]^2$. Nowadays, the topic is well developed by a number of mathematicians (see [9]).

A further question is how smooth can the line be? Or how far from rectifiable is the line? In 1935, Whitney [10] published his example of a C^1-function $f_W : [0,1]^2 \xrightarrow{\text{onto}} [0,1]$ not constant on a connected set of critical points. The author in [2] constructed Whitney-type examples of maps $f \in C^k(\mathbb{R}^n, \mathbb{R}^m)$ for maximal possible k.

Theorem 1.1 [2]. *For any $n, m \in \mathbb{N}$, there exist a map $f : [0,1]^n \to [0,1]^m$, contained in C^k for all real $k < n/m$, and a connected set $E \subseteq [0,1]^n$ such that every partial derivative of f of order $< n/m$ vanishes on E and $f(E) = [0,1]^m$.*

Theorem 1.2 [2]. *Let n, m, p be nonnegative integer numbers, $n > m > p$; then there exists a map $f : \mathbb{R}^n \to \mathbb{R}^m$, contained in C^k for all real $k < (n-p)/(m-p)$, and a connected subset E of points of rank p such that $f(E)$ contains an open set.*

The first theorem holds important information that $[0,1]^m$ can be covered by a line of smoothness class $C^{<1/m}$ (i.e., we write $f \in C^{<k_0}$ if $f \in C^k$ for every $k < k_0$). In this paper, the author determines the smoothness class of a line that can cover a critical values set of a differentiable map.

Main theorem1. *Let $F : \mathbb{R}^n \xrightarrow{C^{k\cdot\lambda}} \mathbb{R}^m$, $k \in \mathbb{N}$, $\lambda \in [0,1)$; then $F(C_p(F)) \subseteq f(\Sigma_\mu f)$ for some $C^{<\mu}$- function $f : \mathbb{R} \to \mathbb{R}^m$, where $\mu = \max\{1/(p + ((n-p)/(k+\lambda))), 1/m\}$ and $\Sigma_\mu f := \{x \in \mathbb{R}$: any partial derivative of f of order $< \mu$ vanishes at $x\}$.*

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This is a Sard-type theorem, and sharpness of the μ can be seen in the results, where necessary and sufficient conditions for the Morse-Sard theorem are establi shed, that are in [11, 3] for the case $C^k(\mathbb{R}^1,\mathbb{R}^1)$, and [1] for the case $C^k(\mathbb{R}^n,\mathbb{R}^1)$.

2. Notations and preliminary lemmas

Definition 2.1. Let $f:\mathbb{R}^m\to\mathbb{R}^n$ be a continuous function and $\lambda\in[0,1)$. It is said that $f\in C^{0\cdot\lambda}$ if f satisfies a λ-Hölder condition: for every compact neighborhood U, there exists $M>0$ such that

$$\left|f(x)-f(y)\right|\leqslant M\cdot|x-y|^{\lambda}\quad \forall x,y\in U. \tag{2.1}$$

Definition 2.2. For $k\in\mathbb{N}$, $\lambda\in[0,1)$, a function $f:\mathbb{R}^m\to\mathbb{R}^n$ is a $C^{k\cdot\lambda}$-function (or $f\in C^{k\cdot\lambda}$) if $f\in C^k$ and every kth partial derivative of f is a $C^{0\cdot\lambda}$-function. If $f\in C^{p\cdot\beta}$ for all $p+\beta<k+\lambda$, $f\in C^{k\cdot\lambda}$.

Definition 2.3. For $f:\mathbb{R}^m\xrightarrow{C^{0\cdot\lambda}}\mathbb{R}^n$, define partial derivatives of order λ: $f_1^{(\lambda)},\dots,f_m^{(\lambda)}$ by the formula

$$f_i^{(\lambda)}(a)=\lim_{t\to 0}\operatorname{sign}(t)\frac{f(a_1,\dots,a_{i-1},a_i+t,a_{i+1},\dots,a_n)-f(a)}{|t|^{\lambda}} \tag{2.2}$$

for $a=(a_1,\dots,a_m)\in\mathbb{R}^m$. If all partial derivatives of order λ are continuous, $f\in C^{\lambda}$.

Definition 2.4. For $k\in\mathbb{R}^+$, a function $f:\mathbb{R}^m\xrightarrow{C^{[k]\cdot k-[k]}}\mathbb{R}^n$ is a C^k-function (or $f\in C^k$) if $f\in C^{[k]}$ and every $[k]$th partial derivative of f is a $C^{k-[k]}$-function, where $[k]$ is the integer part of k. If $f\in C^k$ for every $k<k_0$, $f\in C^{<k_0}$.

We begin by setting $K_0^n=\{Q_{i_0},\ i_0\in\mathbb{N}\}$, where Q_{i_0} is a closed cube in $\mathbb{R}^n$ with side length 1 and every coordinate of any vertex of Q_{i_0} is an integer. In general, having constructed the cubes of K_{s-1}^n, divide each $Q_{i_0,i_1,i_2,\dots,i_{s-1}}\in K_{s-1}^n$ into 2^n closed cubes of side $1/2^s$, and let K_s^n be the set of all these cubes. More precisely, we will write

$$K_s^n=\{Q_{i_0,i_1,i_2,\dots,i_{s-1},i_s};\ Q_{i_0,i_1,i_2,\dots,i_{s-1},i_s}\subseteq Q_{i_0,i_1,i_2,\dots,i_{s-1}}\in K_{s-1}^n,\ 1\leqslant i_s\leqslant 2^n\}. \tag{2.3}$$

We also define

(i) $K^n=\bigcup_{s+1\in\mathbb{N}}K_s^n$ (note that K^n is defined for $\mathbb{R}^n$);
(ii) $S(\delta)$—the length of a side of $\delta\in K^n$.

LEMMA 2.5. *Let E_1, E_2 be copies of $\mathbb{R}$. For all $n,m\in\mathbb{N}$, there exists continuous $H_{n,m}:[0,1]\xrightarrow{\text{onto}}[0,1]^2\subseteq E_1\times E_2$ such that*

(1) *if $\alpha\in K^1_{(n+m)\cdot s}$, then $H_{n,m}(\alpha)=\alpha'\times\alpha''$, where $\alpha'\in K^1_{n\cdot s}$, $\alpha''\in K^1_{m\cdot s}$, $\alpha'\in E_1$, and $\alpha''\in E_2$,*
(2) *if $\alpha'\times\alpha''\subseteq[0,1]^2$ such that $\alpha'\subseteq E_1$, $\alpha''\subseteq E_2$, $\alpha'\in K^1_{n\cdot s}$, and $\alpha''\in K^1_{m\cdot s}$, then $H_{n,m}^{-1}(\operatorname{int}(\alpha'\times\alpha''))\subseteq\alpha\in K^1_{(n+m)\cdot s}$.*

Proof. We define for every $n,m\in\mathbb{N}$ a space-filling function $H_{n,m}$ as follows.

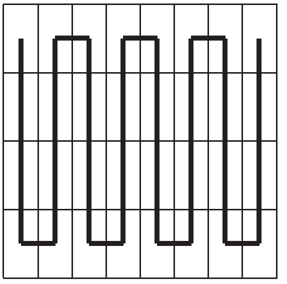

Figure 2.1. HU.

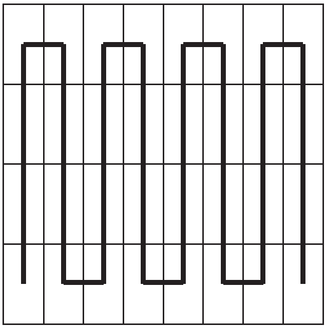

Figure 2.2. HD.

If the interval $[0,1]$ can be mapped continuously onto the square $[0,1]^2$, then after partitioning $[0,1]$ into 2^{n+m} congruent subintervals and $[0,1]^2$ into 2^{n+m} congruent subrectangles with sides $1/2^n$, $1/2^m$, each subinterval can be mapped continuously onto one of the subrectangles.

Next, each subinterval is, in turn, partitioned into 2^{n+m} congruent subintervals, and each subrectangle into 2^{n+m} congruent subrectangles with sides $1/2^{2n}$, $1/2^{2m}$ and the argument is repeated. If this is carried on indefinitely, $[0,1]$ and $[0,1]^2$ are partitioned into $2^{(n+m)s}$ congruent replicas, each with sides $1/2^{ns}$, $1/2^{ms}$ for $s \in \mathbb{N}$.

We need to demonstrate that the subsquares can be arranged so that adjacent subintervals correspond to adjacent subsquares with an edge in common, and so that the inclusion relationships are presented, that is, if a rectangle corresponds to an interval, then its subrectangles correspond to the subintervals of that interval.

We will use here combination of four different methods to construct these space-filling curves. These methods are based on an idea of Peano [9]. For future use, we designate them as VL(n,m), VR(n,m), HD(n,m), HU(n,m).

If we have a rectangle, then using any of those methods gives us 2^{n+m} equal subrectangles which are ordered according to the order assigned by the method used.

Figures 2.1, 2.2, 2.3, and 2.4 give us a basic idea of how these four methods work.

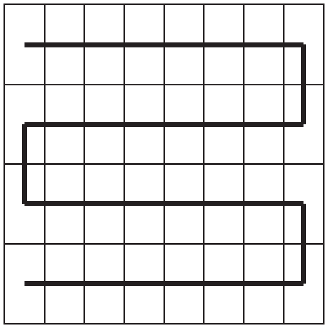

Figure 2.3. VL.

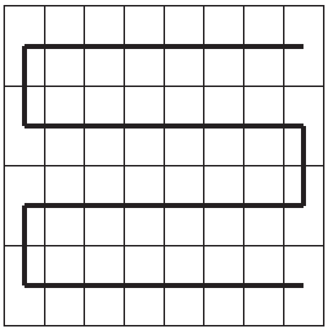

Figure 2.4. VR.

Note. As soon as the curves in all four methods are passing through all the subrectangles, the only essential difference among the four methods is the disposition of start and end points. That is denoted in abbreviations of the methods: V-vertical, H-horizontal, L-left, R-right, U-up, D-down.

Further, to create the next iteration curve, we will give the means of how to present each of the subrectangles from the previous iteration (see Figures 2.5, 2.6, 2.7, and 2.8).

Finally, in Figures 2.9 and 2.10, we indicate how this process is to be carried out for the next iteration.

Now we can define $H_{n,m}$ for any $n,m \in \mathbb{N}$.

Definition 2.6. Every $t \in [0,1]$ is uniquely determined by a sequence of nested closed intervals (that are generated by our successive partitioning), the lengths of which shrink to 0. With this sequence, there corresponds a unique sequence of nested closed squares, the diagonals of which shrink into a point, and which define a unique point in $[0,1]^2$, the image $H_{n,m}(t)$ of t.

The function $H_{n,m}$ satisfies the properties (1), (2) of Lemma 2.5 by its definition. □

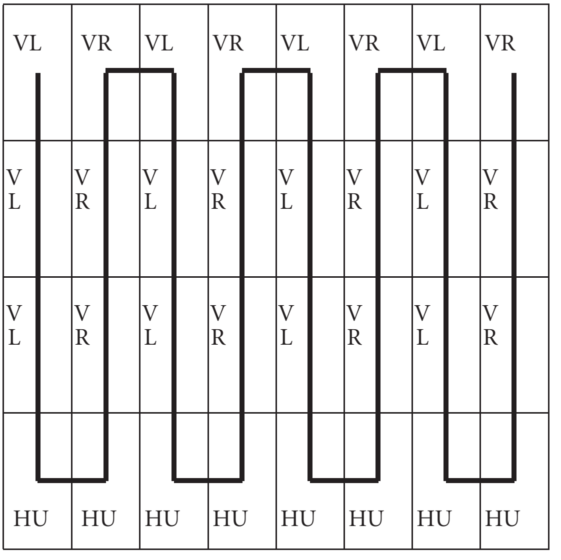

Figure 2.5. HU.

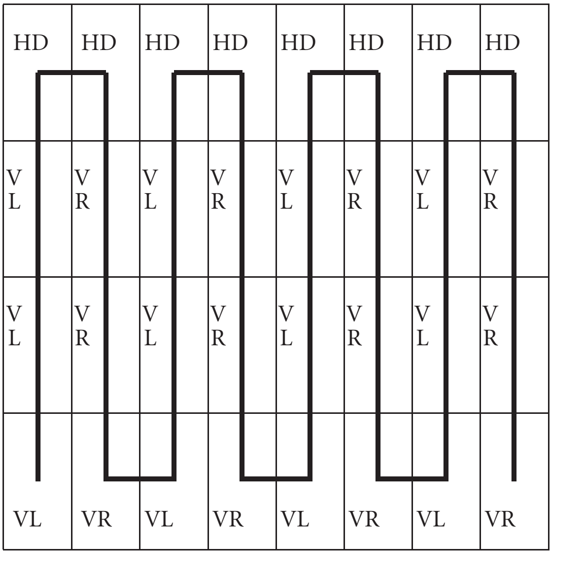

Figure 2.6. HD.

Lemma 2.7. *Let E_1, E_2 be copies of $\mathbb{R}$. For all $\tilde{n},\tilde{m}:\mathbb{N}\to\mathbb{N}$, there exists continuous function $H_{\tilde{n},\tilde{m}}:[0,1]\xrightarrow{\text{onto}}[0,1]^2\subseteq E_1\times E_2$ such that*

(1) *if $\alpha\in K^1_{\sum_{i=1}^{s}\tilde{n}(i)+\tilde{m}(i)}$ for some $s\in\mathbb{N}$, then $H_{\tilde{n},\tilde{m}}(\alpha)=\alpha'\times\alpha''$, where $\alpha'\in K^1_{\sum_{i=1}^{s}\tilde{n}(i)}$, $\alpha''\in K^1_{\sum_{i=1}^{s}\tilde{m}(i)}$,*

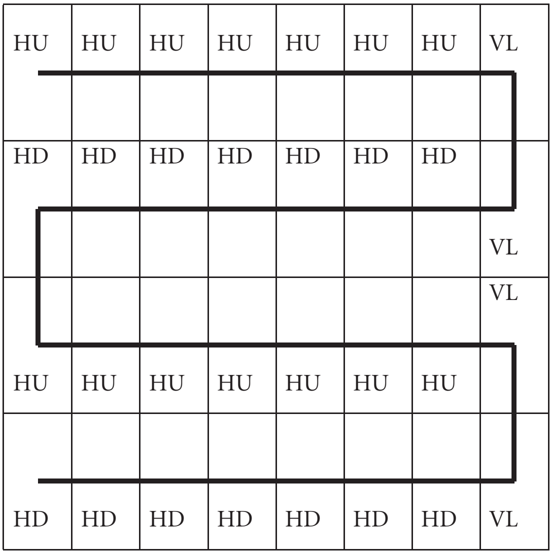

Figure 2.7. VL.

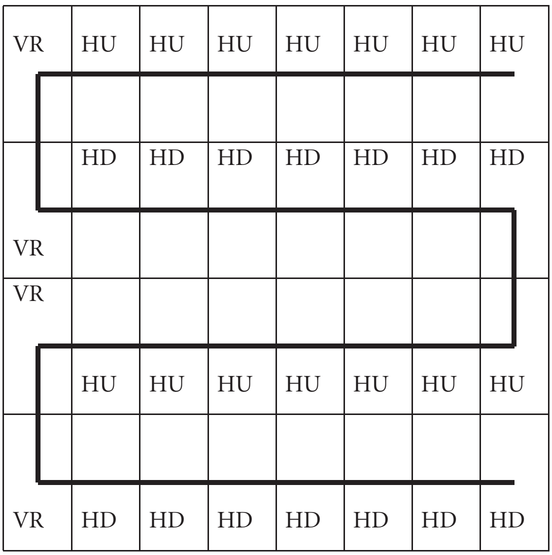

Figure 2.8. VR.

(2) *if* $\alpha' \times \alpha'' \subseteq [0,1]^2$ *such that* $\alpha' \subseteq E_1$, $\alpha'' \subseteq E_2$, $\alpha' \in K^1_{\sum_{i=1}^{s} \tilde{n}(i)}$, *and* $\alpha'' \in K^1_{\sum_{i=1}^{s} \tilde{m}(i)}$ *for some* $s \in \mathbb{N}$, *then* $H^{-1}_{\tilde{n},\tilde{m}}(\operatorname{int}(\alpha' \times \alpha'')) \subseteq \alpha \in K^1_{\sum_{i=1}^{s} \tilde{n}(i)+\tilde{m}(i)}$.

Proof. The proof is similar to the proof of Lemma 2.5, with the only difference that if we used, for instance, a method $\mathrm{VL}(n,m)$ to decompose a subrectangle on an iteration s in

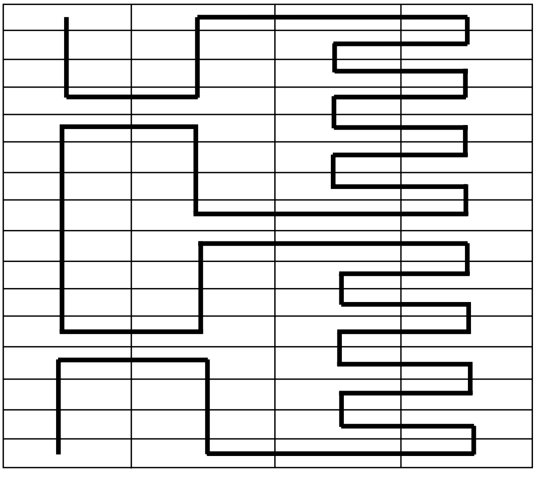

Figure 2.9. Next iteration when started with method VL.

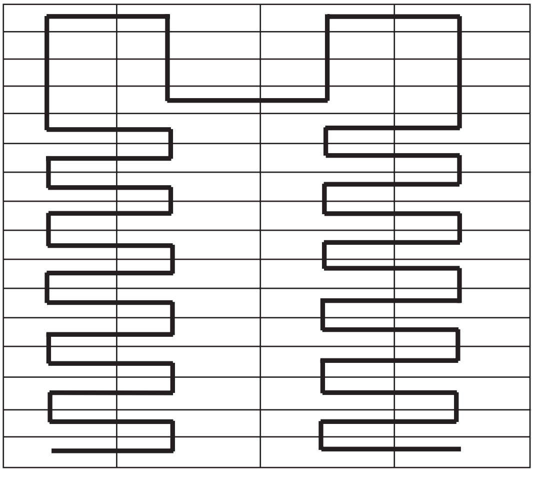

Figure 2.10. Next iteration when started with method HD.

Lemma 2.5, then here we use a corresponding method VL$(n(s), m(s))$ on the iteration s. □

Definition 2.8 [2]. Call a function $f_n : [0,1] \to [0,1]^n$ *cubes-preserving* if it has the following properties:

(i) if $\alpha \subseteq [0,1]$ and for some $s \in \mathbb{N}$, $\alpha \in K^1_{n\cdot s}$ implies $f_n(\alpha) \subseteq \delta$ for some $\delta \in K^n_s$,

(ii) if $\delta \subseteq [0,1]^n$ and for some $s \in \mathbb{N}$, $\delta \in K^n_s$ implies $f_n^{-1}(\mathrm{int}(\delta)) \subseteq \alpha$ for some $\alpha \in K^1_{n\cdot s}$,

where $\mathrm{int}(\delta)$ is the set of interior points of δ.

Note that a continuous cubes-preserving function f_n is a space-filling and measure-preserving function, that is, with the property that if $\alpha \subseteq [0,1]$ and for some $s \in \mathbb{N}$, $\alpha \in K^1_{n\cdot s}$, then $f_n(\alpha) = \delta$ for some $\delta \in K^n_s$.

Theorem 2.9 (space-filling function) [2, Theorem 1]. *For every $n \in \mathbb{N}$, there exists a continuous cubes-preserving function*

$$f_n : [0,1] \xrightarrow{\text{onto}} [0,1]^n. \tag{2.4}$$

Definition 2.10. For $m,n \in \mathbb{N}$, $k \in \mathbb{R}$, a function $\psi : B \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is D^k*-function* if there exist $K > 0$ such that for all $b, b' \in B$

$$\left|\psi(b) - \psi(b')\right|^k \leqslant K|b - b'|. \tag{2.5}$$

2.1. Properties of D^k-functions

Extension on closure property [1]. If $f : A \subseteq \mathbb{R}^m \xrightarrow{D^k} \mathbb{R}^n$ for some $k > 0$, and $\overline{A}$ is the closure of A, then there exists a unique function $\overline{f} : \overline{A} \subseteq \mathbb{R}^m \xrightarrow{C^0} \mathbb{R}^n$ such that $\overline{f} \restriction A = f$, and $\overline{f}$ is a D^k-function.

Composition property [1]. If $g \in D^k$ and $f \in D^p$, then $g \circ f \in D^{kp}$.

Subsets property [1]. If $f : A \subseteq \mathbb{R}^m \xrightarrow{D^k} \mathbb{R}^m$ for some $k > 0$, then $f \restriction B \in D^k$ for any $B \subseteq A$.

$C^{<k}$*-extension on $\mathbb{R}$ property.* If $F : B \subseteq \mathbb{R} \xrightarrow{D^{1/k}} \mathbb{R}^m$, $k > 0$, then $F = f \restriction B$ for some function $f : \mathbb{R} \xrightarrow{C^{<k}} \mathbb{R}^m$, with $\text{range}(F) \subseteq f(\Sigma_k f)$.

We prove this property as follows. Let $f \restriction \overline{B}$ be the $D^{1/k}$ extension of the function F on the closed set $\overline{B}$ the closure of B, that exists and is unique by the "extension on closure property." Then let T be a real number such that

$$\forall b, b' \in \overline{B} \quad \left|f(b) - f(b')\right|^{1/k} \leqslant T|b - b'|, \tag{2.6}$$

and let $A = \text{range}(f \restriction \overline{B})$; then $\text{range}(F) \subseteq A$.

We now define the function $f : \mathbb{R} \to \mathbb{R}^m$ as follows: if there exists a point $\overline{b} \in \mathbb{R}$ such that $\overline{b} = \max\{b \in \overline{B}\}$, then for all $x \geqslant \overline{b}$, $f(x) = f(\overline{b})$, respectively, if there exists a point $\underline{b} \in \mathbb{R}$ such that $\underline{b} = \min\{b \in \overline{B}\}$; then for all $x \leqslant \underline{b}$, $f(x) = f(\underline{b})$.

We designate $Z(\overline{B}) = \{(b,b') \subseteq \mathbb{R} \setminus \overline{B};\ b < b',\ b,b' \in \overline{B}\}$; this set is countable and we can write $Z(\overline{B}) = \{(b_n, b'_n);\ n \in \mathbb{N}\}$, where $b_n, b'_n \in \overline{B}$.

Let $f = (f_1, \dots, f_i, \dots, f_m)$, where $f_i : \overline{B} \to \mathbb{R}$, $1 \leqslant i \leqslant m$, are the component functions of the function $f \restriction \overline{B}$; then for all $n \in \mathbb{N}$, for all $x \in (b_n, b'_n)$, and for all i $(1 \leqslant i \leqslant m)$, we define

$$f_i(x) = \left(f_i(b'_n) - f_i(b_n)\right) \cdot g\left(\frac{x - b_n}{b'_n - b_n}\right) + f_i(b_n), \tag{2.7}$$

where (following [6, page 6]) $g:\mathbb{R}^1 \to [0,1]$ is a smooth map such that

$$\begin{gathered} g \upharpoonright (-\infty,0] = 0, \\ g \upharpoonright [1,\infty) = 1, \\ g'(x) > 0 \quad \text{for } 0 < x < 1. \end{gathered} \tag{2.8}$$

Then f is defined for all $x \in \mathbb{R}$, continuous, smooth on $\mathbb{R} \setminus \overline{B}$ and $A \subseteq \text{range}(f)$. To finish the proof of $C^{<k}$-extension on $\mathbb{R}$ property, it suffices to show that

$$f_i^{(t)} \upharpoonright \overline{B} = 0 \quad \forall i\, (1 \leqslant i \leqslant m),\ \forall t \in \{1,2,\dots,[k]\} \cup ([k],k). \tag{2.9}$$

It is evident for nonlimit points of $\overline{B}$. Let $B' \subseteq \overline{B}$ be the set of limit points of $\overline{B}$.

Case 1. If $k \leqslant 1$, then for all $b \in B'$ and some fixed $t : 0 \leqslant t < k$,

$$\left| f_i^{(t)}(b) \right| = \lim_{h\to 0} \frac{|f_i(b+h) - f_i(b)|}{|h|^t}. \tag{2.10}$$

Note that we may suppose without loss of generality that

$$\begin{gathered} h > 0, \\ b+h \in (b_n, b_n') \quad \text{for some } n \in \mathbb{N}, \\ \Delta_n := b+h-b_n. \end{gathered} \tag{2.11}$$

Then

$$\begin{aligned} \frac{|f_i(b+h) - f_i(b)|}{|h|^t} &\leqslant \frac{|f_i(b_n) - f_i(b)| + |f_i(b+h) - f_i(b_n)|}{\left(|b_n - b| + \Delta_n\right)^t} \\ &\leqslant \frac{|f_i(b_n) - f_i(b)|}{\left(|b_n - b| + \Delta_n\right)^t} + \frac{|f_i(b+h) - f_i(b_n)|}{\left(|b_n - b| + \Delta_n\right)^t} \\ &\leqslant \frac{T^k |b_n - b|^k}{(|b_n - b|)^t} + \frac{|f_i(b+h) - f_i(b_n)|}{\Delta_n^t}. \end{aligned} \tag{2.12}$$

We consider each summand of (2.12) separately:

$$\frac{T^k |b_n - b|^k}{(|b_n - b|)^t} = T^k |b_n - b|^{k-t}, \tag{2.13}$$

where $k - t > 0$;

$$\begin{aligned} \frac{|f_i(b+h) - f_i(b_n)|}{\Delta_n^t} &= \frac{|f_i(b_n') - f_i(b_n)|}{(b_n' - b_n)^k} \cdot \frac{|g(\Delta_n/(b_n' - b_n))|}{\Delta_n/(b_n' - b_n)} \cdot \frac{\Delta_n^{1-t}}{(b_n' - b_n)^{1-k}} \\ &\leqslant T^k \max(Dg) \left| \frac{\Delta_n}{b_n' - b_n} \right|^{1-t} (b_n' - b_n)^{k-t}, \end{aligned} \tag{2.14}$$

where $k - t > 0$, $1 - t > 0$, and $\Delta_n \leqslant b_n' - b_n$.

Turning back to (2.12), we see that

$$|f_i^{(t)}(b)| \leqslant \lim_{h\to 0}\left(T^k|b_n-b|^{k-t}+T^k\max(Dg)\left(\frac{\Delta_n}{b_n'-b_n}\right)^{1-t}(b_n'-b_n)^{k-t}\right)=0 \quad (2.15)$$

because either $\Delta_n/(b_n'-b_n)$ or $b_n'-b_n$ tends to 0 as h tends to 0. Let U_b be a compact neighborhood of b; then for the M required by Definition 2.1, we can take the number

$$\begin{aligned}&\max\left\{T^k|b_n-b|^{k-t}+T^k\max(Dg)(b_n'-b_n)^{k-t}:b_n,b_n',b\in U_b\right\}\\&\quad\leqslant T^k(\operatorname{diam}(U_b))^{k-t}(1+\max(Dg)).\end{aligned} \quad (2.16)$$

Case 2. If $k>1$ for every $t\in\mathbb{R}$, $1\leqslant t<k$, we can suppose by induction that

$$f_i^{(\tilde{t})}\restriction\overline{B}\equiv 0,\quad \tilde{t}=\begin{cases}[t] & \text{if } t\notin\mathbb{N},\\ t-1 & \text{if } t\in\mathbb{N}.\end{cases} \quad (2.17)$$

Then for all $b\in\overline{B}$,

$$\left|f_i^{(t)}(b)\right|=\lim_{h\to 0}\frac{\left|f_i^{(\tilde{t})}(b+h)-f_i^{(\tilde{t})}(b)\right|}{h^{t-\tilde{t}}} \quad (2.18)$$

and using (2.11),

$$\begin{aligned}\frac{\left|f_i^{(\tilde{t})}(b+h)-f_i^{(\tilde{t})}(b)\right|}{h^{t-\tilde{t}}}&=\frac{\left|f_i^{(\tilde{t})}(b+h)\right|}{(|b_n-b|+\Delta_n)^{t-\tilde{t}}}\\&\leqslant\frac{\left|f_i^{(\tilde{t})}(b+h)-f_i^{(\tilde{t})}(b_n)\right|}{\Delta_n^{t-\tilde{t}}}\\&=\left|f_i^{(\tilde{t}+1)}(\xi)\cdot\Delta_n^{1+\tilde{t}-t}\right|\end{aligned} \quad (2.19)$$

for some $\xi\in(b_n,b_n')$ (note that $f_i^{(\tilde{t})}(b)=f_i^{(\tilde{t})}(b_n)=0$ because $b,b_n\in\overline{B}$, and also that $f_i\in C^\infty$ on (b_n,b_n')).

From (2.7), it follows that

$$\begin{aligned}\left|f_i^{(\tilde{t}+1)}(\xi)\right|&=\frac{|f_i(b_n)-f_i(b_n')|}{(b_n'-b_n)^{\tilde{t}+1}}\cdot\left|g^{(\tilde{t}+1)}\left(\frac{\xi-b_n}{b_n'-b_n}\right)\right|\\&\leqslant\frac{T^k\cdot|b_n-b_n'|^k}{(b_n'-b_n)^{\tilde{t}+1}}\cdot r_{\tilde{t}+1}=\frac{T^k(b_n'-b_n)^k}{(b_n'-b_n)^{\tilde{t}+1}}\cdot r_{\tilde{t}+1},\end{aligned} \quad (2.20)$$

where $r_{\tilde{t}+1}=\max\{g^{(\tilde{t}+1)}(\alpha);\ \alpha\in[0,1]\}$.

Then for $f_i^{(t)}(b)$, we can write

$$\left| f_i^{(t)}(b) \right| \leqslant T^k \cdot \frac{\Delta_n^{1+\tilde{t}-t} r_{\tilde{t}+1}}{(b_n' - b_n)^{1+\tilde{t}-k}} = T^k r_{\tilde{t}+1} \cdot \left| \frac{\Delta_n}{b_n' - b_n} \right|^{1+\tilde{t}-t} \cdot (b_n' - b_n)^{k-t}; \tag{2.21}$$

it means that

$$\left| f_i^{(t)}(b) \right| \leqslant \lim_{h \to 0} \left(T^k r_{\tilde{t}+1} \cdot \left| \frac{\Delta_n}{b_n' - b_n} \right|^{1+\tilde{t}-t} \cdot (b_n' - b_n)^{k-t} \right), \tag{2.22}$$

where $k > t$, $\Delta_n/(b_n' - b_n) \leqslant 1$, $1 + \tilde{t} \geqslant t$. The limit is equal to 0 because either $\Delta_n/(b_n' - b_n)$ or $(b_n' - b_n)$ tends to 0 as h tends to 0. Let U_b be a compact neighborhood of b; then for the K required by Definition 2.1, we can take the number $T^k r_{\tilde{t}+1} \cdot (\operatorname{diam}(U_b))^{k-t}$ so that, by finishing the proof of (2.9), we finish the proof of the "$C^{<k}$-extension on $\mathbb{R}$ property."

Lemma 2.11. *Let $n, p \in \mathbb{N}$, $p \leqslant n$, $k \in \mathbb{R}$, $k \geqslant 1$. Then there exists a continuous space-filling function*

$$\pi_{k,p}^n = (\pi_1, \pi_2) : [0,1] \xrightarrow{\text{onto}} [0,1]^n \tag{2.23}$$

such that $\pi_1 : [0,1] \xrightarrow{D^{(pk+n-p)/k}} [0,1]^p$ and $\pi_2 : [0,1] \xrightarrow{D^{pk+n-p}} [0,1]^{n-p}$ are component functions of $\pi_{k,p}^n$.

Proof. We consider the following:

(a) functions $\tilde{n}, \tilde{m} : \mathbb{N} \to \mathbb{N}$ such that for every $s \in \mathbb{N}$,

$$\begin{aligned} \tilde{n}(s) &= p \cdot ([ks] - [k(s-1)]), \\ \tilde{m}(s) &= n - p, \end{aligned} \tag{2.24}$$

where $[ks]$ is the integer part of ks;

(b) a function $H_{\tilde{n},\tilde{m}} : [0,1] \xrightarrow{\text{onto}} [0,1]^2$ defined in Lemma 2.7 and let $h_1, h_2 : [0,1] \to [0,1]$ be the component functions of $H_{\tilde{n},\tilde{m}}$ so that for all $t \in [0,1]$, $H_{\tilde{n},\tilde{m}}(t) = (h_1(t), h_2(t)) \in [0,1]^2$;

(c) A function $\pi_{p,k}^n = (\pi_1, \pi_2) : [0,1] \xrightarrow{\text{onto}} [0,1]^n$, where

$$\pi_1 = f_p \circ h_1, \qquad \pi_2 = f_{n-p} \circ h_2. \tag{2.25}$$

Additionally,

$$f_p : [0,1] \xrightarrow{\text{onto}} [0,1]^p, \qquad f_{n-p} : [0,1] \xrightarrow{\text{onto}} [0,1]^{n-p} \tag{2.26}$$

are some continuous space-filling cubes-preserving functions, the existence of which follows from Theorem 2.9.

We establish some properties of the functions π_1, π_2, which we will need to finish the proof of Lemma 2.11.

(I) If $\alpha \in K^1_{p[ks]+(n-p)s}$ for some $s \in \mathbb{N}$, then

$$\begin{aligned} f_p(h_1(\alpha)) &= (\pi_1(\alpha)) \in K^p_{[ks]}, \\ f_{n-p}(h_2(\alpha)) &= (\pi_2(\alpha)) \in K^{n-p}_s. \end{aligned} \tag{2.27}$$

We prove this property as follows. As $p[ks]+(n-p)s = \sum_{i=1}^{s} \tilde{n}(i)+\tilde{m}(i)$, by property (1) of Lemma 2.7, one has that $H_{\tilde{n},\tilde{m}}(\alpha) = \alpha' \times \alpha''$, where $h_1(\alpha) = \alpha' \in K^1_{\sum_{i=1}^{s}\tilde{n}(i)} = K^1_{p[ks]}$ and $h_2(\alpha) = \alpha'' \in K^1_{\sum_{i=1}^{s}\tilde{m}(i)} = K^1_{(n-p)s}$. Then according to Definition 2.8 of cubes-preserving functions f_p, f_{n-p}, we can see that

$$f_p(h_1(\alpha)) \in K^p_{[ks]}, \qquad f_{n-p}(h_2(\alpha)) \in K^{n-p}_s. \tag{2.28}$$

(II) If $\alpha \in K^1_{p[ks]+(n-p)s}$ for some $s \in \mathbb{N}$, then

$$|\alpha| = \big(S(\pi_1(\alpha))\big)^{(p[ks]+(n-p)s)/[ks]} = \big(S(\pi_2(\alpha))\big)^{(p[ks]+(n-p)s)/s}, \tag{2.29}$$

where $S(\pi_1(\alpha))$, $S(\pi_2(\alpha))$ are lengths of sides of cubes $\pi_1(\alpha)$, $\pi_2(\alpha)$, respectively.

We prove this property as follows. If $\alpha \in K^1_{p[ks]+(n-p)s}$ for some $s \in \mathbb{N}$, then by property (2.27), it means that

$$S(\pi_1(\alpha)) = \frac{1}{2^{[ks]}}, \qquad S(\pi_2(\alpha)) = \frac{1}{2^s}. \tag{2.30}$$

On the other hand, $\alpha \in K^1_{p[ks]+(n-p)s}$ so that

$$\begin{aligned} |\alpha| &= \frac{1}{2^{p[ks]+(n-p)s}}, \\ |\alpha| &= \big(S(\pi_1(\alpha))\big)^{(p[ks]+(n-p)s)/[ks]} \\ &= \big(S(\pi_2(\alpha))\big)^{(p[ks]+(n-p)s)/s}. \end{aligned} \tag{2.31}$$

(III) To prove that $\pi_1 \in D^{(pk+n-p)/k}$, $\pi_2 \in D^{pk+n-p}$, it suffices to show that there exists $K>0$ such that for all $a,b \in [0,1]$, $a<b$,

$$\big(\operatorname{diam}(\pi_1([a,b]))\big)^{p+(n-p)/k} < K(b-a) > \big(\operatorname{diam}(\pi_2([a,b]))\big)^{pk+n-p}. \tag{2.32}$$

We prove this property as follows. If $[a,b] \subseteq [0,1]$, then there exists $s_1 \in \mathbb{N}$, $s_1 \geqslant s_0$, such that

$$\frac{1}{2^{p[k(s_1+1)]+(n-p)(s_1+1)}} \leqslant b-a \leqslant \frac{1}{2^{p[ks_1]+(n-p)s_1}}; \tag{2.33}$$

then $[a,b] \subseteq \alpha' \cup \alpha''$ for some $\alpha', \alpha'' \in K^1_{p[ks_1]+(n-p)s_1}$, $\alpha' \cap \alpha'' \neq \varnothing$, and also

$$b-a \geqslant \frac{|\alpha'|}{2^{p([k(s_1+1)]-[ks_1])+(n-p)}} > \frac{|\alpha'|}{2^{2kp+n}}. \tag{2.34}$$

From property (2.29), we can see that

$$|\alpha'| = S(\pi_1(\alpha'))^{(p[ks_1]+(n-p)s_1)/[ks_1]} = S(\pi_2(\alpha'))^{(p[ks_1]+(n-p)s_1)/s_1} \tag{2.35}$$

and using the fact that

$$\begin{aligned} &\operatorname{diam}(\pi_1(\alpha' \cup \alpha'')) \leqslant 2\sqrt{p}S(\pi_1(\alpha')), \\ &\operatorname{diam}(\pi_2(\alpha' \cup \alpha'')) \leqslant 2\sqrt{n-p}S(\pi_2(\alpha')), \end{aligned} \tag{2.36}$$

we get

$$\operatorname{diam}(\pi_1(\alpha' \cup \alpha'')) \leqslant 2\sqrt{p}\big(2^{p([k(s_1+1)]-[ks_1])+n-p} \cdot (b-a)\big)^{[ks_1]/(p[ks_1]+(n-p)s_1)}, \tag{2.37}$$

$$\operatorname{diam}(\pi_2(\alpha' \cup \alpha'')) \leqslant 2\sqrt{n-p}\big(2^{p([k(s_1+1)]-[ks_1])+n-p} \cdot (b-a)\big)^{s_1/(p[ks_1]+(n-p)s_1)}. \tag{2.38}$$

Considering inequality (2.38), we may suppose that $\operatorname{diam}(\pi_2([a,b])) \leqslant 1$; also using

$$\begin{gathered} [a,b] \subseteq \alpha' \cup \alpha'', \\ [k(s_1+1)] - [ks_1] < 2k+1, \end{gathered} \tag{2.39}$$

and after the routine arithmetic transformation, we find that there exists $n_2 \in \mathbb{N}$, which does not depend on s_1, such that

$$\big(\operatorname{diam}(\pi_2([a,b]))\big)^{pk+n-p} \leqslant n_2(b-a). \tag{2.40}$$

Now we look at inequality (2.37). Knowing that $(b-a) \geqslant 1/2^{p[k(s_1+1)]+(n-p)(s_1+1)}$, inequality (2.37) can be transformed into

$$\begin{aligned} \operatorname{diam}(\pi_1([a,b])) \leqslant{}& 2\sqrt{p}\big(2^{2kp+n} \cdot (b-a)\big)^{1/(p+(n-p)/k)} \\ &\times \left(2^{2kp+n} \cdot \frac{1}{2^{p[k(s_1+1)]+(n-p)(s_1+1)}}\right)^{[ks_1]/(p[ks_1]+(n-p)s_1)-1/(p+(n-p)/k)}. \end{aligned} \tag{2.41}$$

Note that $[ks_1]/(p[ks_1]+(n-p)s_1) - 1/(p+(n-p)/k) \leqslant 0$ and there exists a number $n_1 \in \mathbb{N}$, which does not depend on s_1, such that

$$\big(\operatorname{diam}(\pi_1([a,b]))\big)^{p+(n-p)/k} \leqslant n_1(b-a). \tag{2.42}$$

The existence of such n_1 only depends on whether the expression

$$-(p[k(s_1+1)]+(n-p)(s_1+1))\left(\frac{[ks_1]}{p[ks_1]+(n-p)s_1} - \frac{1}{p+(n-p)/k}\right) \tag{2.43}$$

is bounded above. Expression (2.43) can be easily transformed into

$$\frac{p[k(s_1+1)]+(n-p)(s_1+1)}{p[ks_1]+(n-p)s_1}\times\frac{(n-p)(ks_1-[ks_1])}{pk+n-p}; \tag{2.44}$$

noticing that the first multiple is bounded and $ks_1-[ks_1]<1$, it follows that expression (2.43) is bounded above.

Now choosing $K=\max\{n_1,n_2\}$, we finish the proof of property (2.32).

(IV) $\pi^n_{k,p}$ is continuous space-filling function.

This is proved as follows. (1) Continuity of $\pi^n_{k,p}$ follows from the continuity of the component functions π_1,π_2 that are continuous as compositions of the continuous functions f_p with h_1 and f_{n-p} with h_2, respectively.

(2) Now let $y=(y_1,y_2)\in[0,1]^n$, where $y_1\in[0,1]^p$, $y_2\in[0,1]^{n-p}$.

Functions $f_p:[0,1]\to[0,1]^p$ and $f_{n-p}:[0,1]\to[0,1]^{n-p}$ are space filling so that there exist $z_1,z_2\in[0,1]$ such that $f_p(z_1)=y_1$, $f_{n-p}(z_2)=y_2$. On the other hand, the point $(z_1,z_2)\in[0,1]^2$ and $H_{\tilde{n},\tilde{m}}:[0,1]\xrightarrow{\text{onto}}[0,1]^2$ so that there exists $t\in[0,1]:H_{\tilde{n},\tilde{m}}(t)=(z_1,z_2)$ or $h_1(t)=z_1$, $h_2(t)=z_2$, and by the definition of $\pi^n_{k,p}:\pi^n_{k,p}(t)=y$.

From (1) and (2), it follows that $\pi^n_{k,p}$ is a continuous space-filling function. □

Lemma 2.12. *If $f\in C^{k\cdot\lambda}(\mathbb{R}^n,\mathbb{R}^n)$, $k\geqslant 1$, $\lambda\in[0,1)$, and Df_x is a linear isomorphism, then f is invertible in a neighborhood of x and f^{-1} is of class $C^{k\cdot\lambda}$.*

Proof. Similar to the proof of the $C^{k+\beta+}$ inverse function theorem in [7]. □

Lemma 2.13. *If $k\geqslant 1$, $f\in C^{k\cdot\lambda}(\mathbb{R}^n,R)$, $x\in\mathbb{R}^n$, $f(x)=0$, $Df(x)\neq 0$, then there is a neighborhood N of x in $\mathbb{R}^n$ and $C^{k\cdot\lambda}(n-1)$-submanifold $S\subseteq\mathbb{R}^n$ such that $f^{-1}(0)\cap N\subset S$.*

Proof. Similar to the proof of Zygmund preimage theorem in [8]. □

Lemmas 2.14 and 2.15 are generalized Morse vanishing lemma and Morse theorem; see Morse [5], and for more general version of the lemmas, see also Norton [7, 8] and Moreira [4].

Lemma 2.14. *Let k, n be nonnegative integers, $\lambda\in[0,1)$, and $A\subseteq\mathbb{R}^n=\mathbb{R}^{n-p}\times\mathbb{R}^p$ for some $p\leqslant n$. Then there are sets $A_1,A_2,\ldots\subseteq A$ such that $A=\bigcup_{i=1}^\infty A_i$, where for each $i=1,2,\ldots$, there is a function $\psi_i:V_i\times B_i\xrightarrow{C^1}\mathbb{R}^n$, V_i is a bounded ball in $\mathbb{R}^p$ and B_i is a bounded ball in some $\mathbb{R}^{r_i}$ $(0\leqslant r_i\leqslant n-p)$ such that*

$$\begin{gathered}\psi_i(x,y)=(x,\tilde{\psi}_i(x,y)),\\ |\psi_i(x_1,y_1)-\psi_i(x_2,y_2)|\geqslant|(x_1,y_1)-(x_2,y_2)|\quad\forall(x_1,y_1),(x_2,y_2)\in V_i\times B_i,\\ A_i\subset\psi_i(V_i\times B_i)\end{gathered} \tag{2.45}$$

with the following property: every $f\in C^{k\cdot\lambda}(\mathbb{R}^n,\mathbb{R})$ vanishing on A satisfies for each i and some $K_i\geqslant 0$,

$$\begin{aligned}&|f(\psi_i(x_0,y))-f(\psi_i(x_0,y_0))|\\ &\quad\leqslant K_i|y-y_0|^{k+\lambda}\quad\forall(x_0,y)\in V_i\times B_i,\ \psi_i(x_0,y_0)\in A_i.\end{aligned} \tag{2.46}$$

Proof. Fix λ. The proof is by double induction on n and k. Let $\langle n,k\rangle$ stand for the statement of the lemma for $\mathbb{R}^n$ and $C^{k\cdot\lambda}$. We will prove $\langle 0,k\rangle$ for all k, $\langle n,0\rangle$ for all n, and $\langle n-1,k\rangle$ and $\langle n,k-1\rangle$ imply $\langle n,k\rangle$.

(a) Proof of $\langle 0,k\rangle$ for all k is trivial.
(b) Proof of $\langle n,0\rangle$ for all n follows directly from the definition of $f\in C^{0\cdot\lambda}$.
(c) Induction step: we assume $\langle n-1,k\rangle$ and $\langle n,k-1\rangle$, and we prove $\langle n,k\rangle$.

Define

$$\begin{aligned} A^{**} &= \{(x,y)\in A : x\in\mathbb{R}^p,\ y\in\mathbb{R}^{n-p},\text{ and every } g\in C^{k\cdot\lambda}(\mathbb{R}^n,\mathbb{R}) \\ &\qquad \text{vanishing on } A \text{ satisfies } D_y g\equiv 0 \text{ on } A\}, \\ A^{*} &= A\setminus A^{**}. \end{aligned} \tag{2.47}$$

We prove the result separately for A^{**} and A^{*}.

On A^{**}. Since f *vanishes on* A, $D_y f=(D_{y_j}f)_{p<j\leqslant n}\equiv 0$ on A^{**}, where $y=(y_{p+1},\dots,y_j,\dots,y_n)$ so that for each j $(p<j\leqslant n)$, if any, $D_{y_j}f$ vanishes on A^{**}, and $D_{y_j}f\in C^{k-1\cdot\lambda}(\mathbb{R}^n,\mathbb{R})$. Hence by the $\langle n,k-1\rangle$ hypothesis, we have $A^{**}=\bigcup_{i=1}^{\infty}A_i^{**}$, $A_i^{**}\subset\psi_i(V_i\times B_i)$, ψ_i as in the statement, and

$$\begin{gathered} (x_0,y),(x_0,y_0)\in V_i\times B_i, \\ \psi_i(x_0,y_0)\in A_i^{**}\Longrightarrow \exists K_{ij}\geqslant 0 \text{ such that } \forall j\ (p<j\leqslant n), \\ \left|D_{y_j}f(\psi_i(x_0,y))-D_{y_j}f(\psi_i(x_0,y_0))\right|\leqslant K_{ij}\left|y-y_0\right|^{k-1+\lambda}, \end{gathered} \tag{2.48}$$

or let $K_i=\sqrt{n-p}\max_{p<j\leqslant n}K_{ij}$, then

$$\begin{gathered} (x_0,y),(x_0,y_0)\in V_i\times B_i, \\ \psi_i(x_0,y_0)\in A_i^{**}\Longrightarrow \left|D_y f(\psi_i(x_0,y))-D_y f(\psi_i(x_0,y_0))\right|\leqslant K_i\left|y-y_0\right|^{k-1+\lambda}. \end{gathered} \tag{2.49}$$

Now by the mean value theorem,

$$\begin{aligned} &(x_0,y),(x_0,y_0)\in V_i\times B_i, \\ &\psi_i(x_0,y_0)\in A_i^{**}\Longrightarrow f(\psi_i(x_0,y))-f(\psi_i(x_0,y_0)) \\ &\quad = D(f\circ\psi_i)(x_0,\theta)\cdot((x_0,y)-(x_0,y_0)) \\ &\qquad (\text{for some } \theta\in B_i \text{ lying on a line segment between } y \text{ and } y_0) \\ &\quad = (Df[\psi_i(x_0,\theta)]\cdot D\psi_i(x_0,\theta))\cdot(0,y-y_0) \\ &\quad = Df[\psi_i(x_0,\theta)]\cdot(D\psi_i(x_0,\theta)\cdot(0,y-y_0)). \end{aligned} \tag{2.50}$$

We recall that $\psi_i(x,y)=(x,\tilde{\psi}_i(x,y))$ so that $D\psi_i(x,y)$ is presented in the following matrix consisting of n rows, where the last $n-p$ rows constitute the Jacobian matrix for the function $\tilde{\psi}_i(x,y)$:

$$\left[\begin{array}{cccc} 1 & \text{Zeros} & & \\ 0 & 1 & \text{Zeros} & \\ & \cdots & \cdots & \\ \text{Zeros} & & 1 & \text{Zeros} \\ \hline & D\tilde{\psi}_i(x,y) & & \end{array}\right]. \tag{2.51}$$

Thus

$$D\psi_i(x_0,\theta)\cdot(0,y-y_0)=(0,D\tilde{\psi}_i(x_0,\theta)\cdot(0,y-y_0)). \tag{2.52}$$

Knowing that $(a,b)\cdot(0,c)=(0,b)\cdot(0,c)$, we get

$$\begin{aligned} &Df[\psi_i(x_0,\theta)]\cdot(0,D\tilde{\psi}_i(x_0,\theta)\cdot(0,y-y_0)) \\ &\quad=(0,D_yf[\psi_i(x_0,\theta)])\cdot(0,D\tilde{\psi}_i(x_0,\theta)\cdot(0,y-y_0)). \end{aligned} \tag{2.53}$$

Now using (2.52), (2.53), (2.49) in (2.50), we have

$$\begin{aligned} &|f(\psi_i(x_0,y))-f(\psi_i(x_0,y_0))| \\ &\quad\leqslant |D_yf[\psi_i(x_0,\theta)]|\cdot|0,D\tilde{\psi}_i(x_0,\theta)\cdot(0,y-y_0)| \\ &\quad\leqslant |D_yf[\psi_i(x_0,\theta)]-D_yf(\psi_i(x_0,y_0))|\,|\tilde{K}_i|\,|(0,y-y_0)| \\ &\qquad\text{(where } D_yf(\psi_i(x_0,y_0))=0 \text{ because } \psi_i(x_0,y_0)\in A^{**}, \\ &\qquad\text{and } \tilde{K}_i \text{ is a Lipschitz constant of the } C^1\text{-function } \tilde{\psi}_i \\ &\qquad\text{on the bounded cube } V_i\times B_i, \text{ that we may suppose to exist)} \\ &\quad\leqslant K_i|\theta-y_0|^{k-1+\lambda}\tilde{K}_i|y-y_0|\leqslant K_i\tilde{K}_i|y-y_0|^{k+\lambda}. \end{aligned} \tag{2.54}$$

On A^*. If $(x_0,y_0)\in A^*$, there is g as above, and by Lemma 2.13, there is $\varepsilon>0$ such that $g^{-1}(0)\cap B_\varepsilon(x_0,y_0)$ is contained in the image of $\psi: V\times B\xrightarrow{C^{k\cdot\lambda}}U$, where B is a ball in $\mathbb{R}^{n-p-1}$, V is a ball in $\mathbb{R}^p$ as in the statement, and $A\cap B_\varepsilon(x_0,y_0)\subseteq g^{-1}(0)$. Taking a countable subcovering of A^* by these balls, we reduce the proof in this case to a case with smaller n. □

LEMMA 2.15. *Let k, n be nonnegative integers, $\lambda\in[0,1)$, and $A\subseteq\mathbb{R}^n=\mathbb{R}^{n-p}\times\mathbb{R}^p$ for some $p\leqslant n$. Then there are sets $A_1,A_2,\ldots\subseteq A$ such that $A=\bigcup_{i=1}^{\infty}A_i$, where for each $i=1,2,\ldots$, there is a function $\psi_i: V_i\times B_i\xrightarrow{C^1}\mathbb{R}^n$, V_i is a bounded ball in $\mathbb{R}^p$ and B_i is a bounded ball in some $\mathbb{R}^{r_i}$ $(0\leqslant r_i\leqslant n-p)$ such that (2.45) holds with the following property: every $f\in C^{k\cdot\lambda}(\mathbb{R}^n,\mathbb{R})$, such that $D_yf\equiv 0$ in A, satisfies (2.46) for each i, and some $K_i\geqslant 0$.*

Proof. The same as in the case "On A^{**}" of Lemma 2.14 if we make there the following corrections:

(1) delete "f vanishes on A,"
(2) replace "$\langle n, k-1\rangle$ hypothesis" with "Lemma 2.14,"
(3) replace A^{**} with A,
(4) replace A_i^{**} with A_i. □

3. Proof of the main theorem

It follows from [2, Theorem 1] and Theorems 3.2, 3.3, and 3.4.

Definition 3.1. A set $A \subseteq \mathbb{R}^m$ is a Z_k*-set* for some positive $k \in \mathbb{R}$ if A is a subset of $\Phi([0,1])$ for some continuous function $\Phi : [0,1] \to \mathbb{R}^m$ such that there exists $P > 0$ such that for all $a \in \Phi^{-1}(A)$, $b \in [0,1]$, the following is true:

$$|\Phi(a) - \Phi(b)| \leqslant P|a-b|^k. \tag{3.1}$$

If a set $A = \bigcup_{i\in\mathbb{N}} A_i$ and every A_i is a Z_k-set for some fixed k, then the set A is called a $\sigma - Z_k$-set.

Theorem 3.2. *Let* $F : \mathbb{R}^n \xrightarrow{C^{k\cdot\lambda}} \mathbb{R}^m$, $k \in \mathbb{N}$, $\lambda \in [0,1)$. *Then* $F(C_p(F))$ *is* $\sigma - Z_{1/(p+(n-p)/(k+\lambda))}$*-set in* $\mathbb{R}^m$ *for every* $p < \min\{m,n\}$.

Proof. Since $C_p(F) = \bigcup_{r=0}^{p}\{x \in \mathbb{R}^n : \operatorname{rank}(DF(x)) = r\}$ and $r + (n-r)/(k+\lambda) \leqslant p + (n-p)/(k+\lambda)$ for $0 \leqslant r \leqslant p$, we may restrict our attention to $\tilde{C}_p(F) = \{x \in \mathbb{R} : \operatorname{rank}(DF(x)) = p\}$.

If $x_0 \in \tilde{C}_p(F)$, we can consider with accuracy to within a change of coordinates of class $C^{k\cdot\lambda}$ that

$$\begin{aligned} &F(z,y) = (z, G(z,y)), \quad (z,y) \in \mathbb{R}^p \times \mathbb{R}^{n-p}, \\ &G(z,y) \in \mathbb{R}^{m-p} \quad \text{in a neighborhood } U \text{ of } x_0 = (z_0, y_0). \end{aligned} \tag{3.2}$$

We have $x = (z,y) \in \tilde{C}_p(F)$ if and only if $D_yG(z,y) = 0$. By applying the results of Lemma 2.15 to a set $A = \{(z,y) \in U : D_yG(z,y) = 0\}$, we obtain the decomposition $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \subset \psi_i(V_i \times B_i)$. It is not difficult to see that the proof of the theorem is reducible to a proof of the following statement: for each i, and some $K_i \geqslant 0$,

$$F(A_i \cap \psi_i(V_i \times B_i)) \text{ is a } Z_{p+(n-p)(k+\lambda)}\text{-}set. \tag{3.3}$$

Since every component function $F_j : \mathbb{R}^n \to \mathbb{R}$ $(1 \leqslant j \leqslant m)$ of the function $F = (F_1, \dots, F_j, \dots, F_m)$ satisfies $F_j \in C^{k\cdot\lambda}$, $D_yF_j \equiv 0$ on A, then by Lemma 2.15, F_j satisfies (2.46), and then it is not difficult to see that F itself satisfies (2.46), it means that for each i; and for some $K_i \geqslant 0$,

$$\begin{aligned} &|F(\psi_i(x_0,y)) - F(\psi(x_0,y_0))| \\ &\quad \leqslant K_i|y-y_0|^{k+\lambda} \quad \forall (x_0,y) \in V_i \times B_i,\ (x_0,y_0) \in A_i. \end{aligned} \tag{3.4}$$

Now we fix such A_i. We may suppose without loss of generality that $V_i \in K^p_{[(k+\lambda)s_0]}$, $B_i \in K^{n-p}_{s_0}$ for some $s_0 \in \mathbb{N}$, where $[(k+\lambda)s_0]$ is an integer part of $(k+\lambda)s_0$.

We consider the following:

(a) a set $\alpha_0 \in K^1_{p[(k+\lambda)s_0]+(n-p)s_0}$;
(b) a set $D = \{\alpha \in K^1_{p[(k+\lambda)s]+(n-p)s},\ \alpha \subseteq \alpha_0\}$;
(c) a function $\pi = (\pi^*, \pi^{**})$, where $\pi^* = I_1 \circ \pi_1 \restriction \alpha_0$, $\pi^{**} = I_2 \circ \pi_2 \restriction \alpha_0$; π_1, π_2 are the component functions of a function $\pi^n_{k+\lambda,p}$, defined in Lemma 2.11; $I_1 : \pi^*(\alpha_0) \xrightarrow{\text{onto}} V_i$, $I_2 : \pi^{**}(\alpha_0) \xrightarrow{\text{onto}} B_i$ are the identity maps;
(d) and finally a function $\Phi : [0,1] \to F(\psi_i(V_i \times B_i)) \subseteq \mathbb{R}^m$ such that

$$\begin{aligned}
&\Phi(a) = F\big(\psi_i(\pi(a))\big) \quad \forall \alpha \in \alpha_0,\\
&\Phi(a) = \Phi(\underline{a}) \quad \forall a \leqslant \underline{a} = \min_{x \in \alpha_0} x,\\
&\Phi(a) = \Phi(\overline{a}) \quad \forall a \geqslant \overline{a} = \max_{x \in \alpha_0} x.
\end{aligned} \tag{3.5}$$

It follows from property (3.2) of F and the property of ψ_i (see Lemma 2.15) that $\Phi(a) = (\pi^*(a), G(\psi_i(\pi(a))))$ for every $a \in [0,1]$.

Now, we are ready to prove (3.3).

From (c) and (d), we see that $F(A_i \cap \psi_i(V_i \times B_i)) \subseteq \Phi([0,1])$.

The function $\Phi : [0,1] \to \mathbb{R}^m$ is continuous as a composition of continuous functions (see (d)). To finish the proof of property (3.3), we need to evaluate $\Phi([a,b])$ for any $[a,b] \subseteq \alpha_0$ such that $\psi_i(\pi(a)) \in A_i$ or $\psi_i(\pi(b)) \in A_i$.

We suppose that $\psi_i(\pi(a)) \in A_i$ (the other case is similar to this one). Then

$$\begin{aligned}
&\big|\Phi(b) - \Phi(a)\big|\\
&\quad = \big|F\big(\psi_i(\pi(b))\big) - F\big(\psi_i(\pi(a))\big)\big|\\
&\quad \leqslant \big|F\big(\psi_i(\pi^*(a), \pi^{**}(b))\big) - F\big(\psi_i(\pi^*(a), \pi^{**}(a))\big)\big|\\
&\qquad + \big|F\big(\psi_i(\pi^*(b), \pi^{**}(b))\big) - F\big(\psi_i(\pi^*(a), \pi^{**}(b))\big)\big|\\
&\quad \leqslant K_i \big|\pi^{**}(b) - \pi^{**}(a)\big|^{(k+\lambda)} + L\big|\pi^*(b) - \pi^*(a)\big|\\
&\qquad \text{(by (3.4) and that } L \text{ is a Lipschitz constant of the } C^1\\
&\qquad \text{function } F \circ \psi_i \restriction V_i \text{ which we may suppose to exist)}\\
&\quad \leqslant K_i\big(K^*|a-b|\big)^{(k+\lambda)/(p(k+\lambda)+n-p)} + L\big(K^{**}|a-b|\big)^{(k+\lambda)/(p(k+\lambda)+n-p)}\\
&\qquad \text{(for some positive numbers } K^*, K^{**} \text{ by (c) and Lemma 2.11)}\\
&\quad \leqslant P \cdot |a-b|^{(k+\lambda)/(p(k+\lambda)+n-p)},
\end{aligned} \tag{3.6}$$

where $P = (K_i + L)(\max\{K^*, K^{**}\})^{(k+\lambda)/(p(k+\lambda)+n-p)}$.

We finish the proof of property (3.3) and thereby the proof of Theorem 3.2. □

THEOREM 3.3. *If $A \subseteq \mathbb{R}^m$ is a Z_k-set in $\mathbb{R}^m$, then $A \subseteq f(\Sigma_k f)$ for some $f : [a,b] \subseteq \mathbb{R}^1 \xrightarrow{C^{<k}} \mathbb{R}^m$.*

Proof. If $A \subseteq \mathbb{R}^m$ is a Z_k-set in $\mathbb{R}^m$, then by Definition 3.1, A is a subset of $\Phi([0,1])$ for some continuous function $\Phi : [0,1] \to \mathbb{R}^m$ such that there exists $P > 0$ such that for all $a \in \Phi^{-1}(A)$, $b \in [0,1]$, the following is true:

$$\left|\Phi(a) - \Phi(b)\right| \leqslant P|a-b|^k \tag{3.7}$$

so that a function $F = \Phi \upharpoonright \Phi^{-1}(A)$ is a $D^{1/k}$-function such that $A \subseteq \operatorname{range}(F)$.

Now, the conclusion of this theorem follows from the "$C^{<k}$-extension on $\mathbb{R}$ property." □

THEOREM 3.4. *If $A = \bigcup_{i\in\mathbb{N}} A_i \subseteq \mathbb{R}^m$, and for all $i \in \mathbb{N}$, there exist $f_i : [a_i,b_i] \subseteq \mathbb{R}^1 \xrightarrow{C^{k(<k)}} \mathbb{R}^m$ such that $A_i \subseteq f_i(\Sigma_k f_i)$. Then there exists $f : \mathbb{R}^1 \xrightarrow{C^{k(<k)}} \mathbb{R}^m$ such that $A \subseteq f(\Sigma_k f)$.*

Proof. We may suppose without loss of generality that for all $i \in \mathbb{N}$,

(i) A_i is closed,
(ii) $a_i, b_i \in \Sigma_k f_i$,
(iii) $\{[a_i,b_i], i \in \mathbb{N}\}$ is disjoint,
(iv) $|[b_i,a_{i+1}]| > \max\{|f_{i+1}(a) - f_j(b)|\}$; $j \leqslant i$, $a \in [a_{i+1},b_{i+1}]$, $b \in \bigcup_{j=1}^{i}[a_j,b_j]$.

Using functions similar to (2.7), we can construct C^∞-function $f_0 : \mathbb{R}^1 \setminus \bigcup_{i\in\mathbb{N}}(a_i,b_i) \to \mathbb{R}^m$ such that $\{a_i,b_i;\ i \in \mathbb{N}\} \subseteq \Sigma_\infty f_0$. Then define the required function $f : \mathbb{R}^1 \to \mathbb{R}^m$ as follows:

$$\begin{gathered} f \upharpoonright \left(\mathbb{R}^1 \setminus \bigcup_{i\in\mathbb{N}} (a_i,b_i)\right) = f_0, \\ f \upharpoonright [a_i,b_i] = f_i, \quad i \in \mathbb{N}. \end{gathered} \tag{3.8}$$

□

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