INFINITE PRODUCTS OF HOLOMORPHIC MAPPINGS

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Let X be a complex Banach space, $\mathcal{N}$ a norming set for X, and $D \subset X$ a bounded, closed, and convex domain such that its norm closure $\overline{D}$ is compact in $\sigma(X, \mathcal{N})$. Let $\varnothing \neq C \subset D$ lie strictly inside D. We study convergence properties of infinite products of those self-mappings of C which can be extended to holomorphic self-mappings of D. Endowing the space of sequences of such mappings with an appropriate metric, we show that the subset consisting of all the sequences with divergent infinite products is σ-porous.

1. Introduction

Let K be a nonempty, bounded, closed, and convex subset of a Banach space. The convergence of infinite products of self-mappings of such sets is of interest in many areas of mathematics and its applications. See, for instance, [19] and references therein. In a recent paper [20], it is proved that the subsets consisting of all those sequences of non-expansive self-mappings of K with divergent infinite products are not only of the first Baire category, but also σ-porous in several spaces, endowed with appropriate metrics, of sequences of such mappings. In the present paper, we establish analogous results for holomorphic mappings. After discussing some basic facts regarding holomorphic mappings and the Kobayashi distance in the next section, we study weak ergodicity in Section 3. The convergence of infinite products to a (unique) common fixed point is considered in Section 4. In the last section of our paper, we study the convergence of infinite products to a retraction.

2. Preliminaries

In this section, we recall several basic facts concerning the Kobayashi distance and holomorphic mappings. These facts will be used throughout our paper.

In the sequel, all Banach spaces X will be complex and D will always denote a bounded, convex domain in X. Let k_D be the Kobayashi distance in D [15] (see also [10, 11, 12, 13, 17]).

We first quote the following very useful lemma regarding convex combinations of points.

Lemma 2.1 [18] (see also [17]). *Let D be a bounded, convex domain in a Banach space $(X, \| \cdot \|)$.*

(i) *If $x, y, w, z \in D$ and $s \in [0,1]$, then*

$$k_D(sx + (1-s)y, sw + (1-s)z) \leq \max [k_D(x,w), k_D(y,z)]. \tag{2.1}$$

(ii) *If $x, y \in D$ and $s, t \in [0,1]$, then*

$$k_D(sx + (1-s)y, tx + (1-t)y) \leq k_D(x,y). \tag{2.2}$$

There are also simple connections between k_D and the norm $\| \cdot \|$ of X.

Theorem 2.2 [12, 17]. *If D is a bounded, convex domain in a Banach space $(X, \| \cdot \|)$, then*

$$\arg\tanh\left(\frac{\|x - y\|}{\operatorname{diam}_{\|\cdot\|} D}\right) \leq k_D(x,y) \tag{2.3}$$

for all $x, y \in D$ and

$$k_D(x,y) \leq \arg\tanh\left(\frac{\|x - y\|}{\operatorname{dist}_{\|\cdot\|}(x, \partial D)}\right) \tag{2.4}$$

whenever $\|x - y\| < \operatorname{dist}_{\|\cdot\|}(x, \partial D)$.

This theorem shows that the Kobayashi distance k_D is locally equivalent to the norm $\| \cdot \|$.

Next, we observe that in analogy with the norm, the Kobayashi distance is lower semi-continuous with respect to a suitably chosen topology. Let $\mathcal{N}$ be a nonempty subset of the dual X^* of X. If there exist positive constants r and R such that

$$\sup\{|l(x)| : l \in \mathcal{N}, \|l\| \leq R\} \geq r\|x\| \tag{2.5}$$

for each $x \in X$, then we say that $\mathcal{N}$ is a norming set for X [8]. It is obvious that a norming set generates a Hausdorff linear topology $\sigma(X, \mathcal{N})$ on X which is weaker than the weak topology $\sigma(X, X^*)$.

Theorem 2.3 [14] (see also [2, 16, 17]). *Let X be a Banach space, $\mathcal{N}$ a norming set for X, and $D \subset X$ a bounded, convex domain such that its norm closure $\overline{D}$ is compact in the $\sigma(X, \mathcal{N})$ topology. If $\{x_\beta\}_{\beta \in J}$ and $\{y_\beta\}_{\beta \in J}$ are nets in D which are convergent in $\sigma(X, \mathcal{N})$ to x and y, respectively, and $x, y \in D$, then*

$$k_D(x,y) \leq \liminf_\beta k_D(x_\beta, y_\beta). \tag{2.6}$$

If the nets $\{x_\beta\}_{\beta \in J}$ and $\{y_\beta\}_{\beta \in J}$ are replaced by the sequences $\{x_k\}_{k \in \mathbb{N}}$ and $\{y_k\}_{k \in \mathbb{N}}$, then the compactness of $\overline{D}$ in $\sigma(X, \mathcal{N})$ can be replaced by its sequential compactness in $\sigma(X, \mathcal{N})$.

Recall that a subset C of D is said to lie strictly inside D if $\operatorname{dist}_{\|\cdot\|}(C, \partial D) > 0$. Thus any closed subset C lying strictly inside a bounded, convex domain D is complete with respect to k_D.

THEOREM 2.4 [12]. *Let D be a bounded, convex domain in a Banach space $(X, \| \cdot \|)$. A subset C of D is k_D-bounded if and only if C lies strictly inside D.*

Now, we present a few results concerning holomorphic mappings. We begin with the definition of a holomorphic mapping. Let X_1 and X_2 be two complex normed linear spaces and let D_1 be a domain in X_1. A mapping $f : D_1 \to X_2$ is said to be holomorphic in D_1 if it is Fréchet differentiable at each point of D_1. An equivalent definition is given in the following theorem.

THEOREM 2.5 [8] (see also [3]). *Let $(X_1, \| \cdot \|_1)$ and $(X_2, \| \cdot \|_2)$ be Banach spaces, D a domain in X_1, and let $\mathcal{N}$ be a norming set for $(X_2, \| \cdot \|_2)$. For $a \in D$ and $x \in X_1 \setminus \{0\}$, let $D(a,x)$ denote the set*

$$D(a,x) = \{z \in \mathbb{C} : a + zx \in D\}. \tag{2.7}$$

Then the mapping $f : D \to X_2$ is holomorphic in D if and only if f is locally bounded on D and for each $a \in D$, $x \in X_1 \setminus \{0\}$, and $l \in \mathcal{N}$, the function

$$l \circ f_{|D(a,x)} : D(a,x) \longrightarrow \mathbb{C} \tag{2.8}$$

is holomorphic in $D(a,x)$ in the classical one-variable sense.

Directly from Theorem 2.5, we get the following simple lemma regarding nets of holomorphic mappings. We note here that $\overline{C}$ always stands for the norm closure of each subset C of a Banach space.

LEMMA 2.6 [14] (see also [16]). *Let D_1 and D_2 be bounded, convex domains in the Banach spaces $(X_1, \| \cdot \|_1)$ and $(X_2, \| \cdot \|_2)$, respectively, and $\mathcal{N}$ a norming set for $(X_2, \| \cdot \|_2)$. If $\{f_\lambda\}_{\lambda \in J}$ is a net of holomorphic mappings $f_\lambda : D_1 \to D_2$ which is pointwise convergent in the topology $\sigma(X_2, \mathcal{N})$ to a mapping $f : D_1 \to \overline{D_2}$ and there exists a point $z_0 \in D_1$ such that $w_0 = f(z_0) \in D_2$, then f maps D_1 holomorphically into D_2.*

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. We say that a mapping $f : M_1 \to M_2$ is nonexpansive if

$$d_2(f(x), f(y)) \le d_1(x,y) \tag{2.9}$$

for all $x, y \in M_1$.

An immediate consequence of the definition of the Kobayashi distance is the following property of holomorphic mappings. If D_1 and D_2 are bounded domains in the Banach spaces $(X_1, \| \cdot \|_1)$ and $(X_2, \| \cdot \|_2)$, respectively, and k_{D_1} and k_{D_2} are the Kobayashi distances on D_1 and D_2, respectively, then each holomorphic $f : D_1 \to D_2$ is nonexpansive, that is,

$$k_{D_1}(f(x), f(y)) \le k_{D_2}(x,y) \tag{2.10}$$

for all $x, y \in D_1$ [15].

We also recall the Earle-Hamilton theorem.

THEOREM 2.7 [9]. *Let D be a bounded, convex domain in a Banach space $(X, \| \cdot \|)$ and let $\widetilde{R} = \mathrm{diam}_{\|\cdot\|} D$. Let a holomorphic $f : D \to D$ map D strictly inside itself. If $\epsilon > 0$ is such that $\mathrm{dist}_{\|\cdot\|}(f(D), \partial D) \geq \epsilon$ and $t = \epsilon/2\widetilde{R}$, then for $0 < s = 1/(1+t) < 1$,*

$$k_D(f(x), f(y)) \leq s k_D(x, y) \tag{2.11}$$

for all $x, y \in D$, and therefore f has a unique fixed point. Moreover, for any x in D, the sequence of iterates $\{f^k(x)\}$ converges to this fixed point.

Now we introduce basic notions and notations concerning those spaces which we will investigate in subsequent sections.

Throughout this paper, we let X be a complex Banach space, $\mathcal{N}$ a norming set for X, and $D \subset X$ a bounded, closed, and convex domain such that its norm closure $\overline{D}$ is compact in $\sigma(X, \mathcal{N})$. Let C be a k_D-bounded subset of D and let the set $\mathcal{U}_H$ consist of all those self-mappings of C which can be extended to holomorphic self-mappings of D. We endow $\mathcal{U}_H$ with the metric $\rho_{\mathcal{U}_H}$ defined by

$$\rho_{\mathcal{U}_H}(f, g) = \sup_{x \in C} k_D(f(x), g(x)) \tag{2.12}$$

for $f, g \in \mathcal{U}_H$. It is not difficult to see that, by Lemma 2.6, the metric space $(\mathcal{U}_H, \rho_{\mathcal{U}_H})$ is complete.

Denote by $\mathcal{A}_H$ the set of all sequences $\{f_t\}_{t=1}^{\infty}$, where each $f_t \in \mathcal{U}_H$. We equip the space $\mathcal{A}_H$ with the metric $d_{\mathcal{A}_H}$ defined by

$$d_{\mathcal{A}_H}\left(\{f_t\}_{t=1}^{\infty}, \{g_t\}_{t=1}^{\infty}\right) = \sup_{t \geq 1} \rho_{\mathcal{U}_H}(f_t, g_t), \tag{2.13}$$

where $\{f_t\}_{t=1}^{\infty}, \{g_t\}_{t=1}^{\infty} \in \mathcal{A}_H$. Clearly, the metric space $(\mathcal{A}_H, d_{\mathcal{A}_H})$ is also complete.

Finally, we recall the concept of porosity. We will use the rather strong notion which appears in [4, 5, 6, 7]. In the literature, one can also find other notions of porosity [1, 21, 22, 23].

Definition 2.8. Let (Y, d) be a complete metric space. Denote by $B(x, R)$ the closed ball centered at $x \in Y$ and of radius $R > 0$. A subset $E \subset Y$ is called porous in (Y, d) if there exist $\alpha \in (0, 1)$ and $R_0 > 0$ such that for each $R \in (0, R_0)$ and each $y \in Y$, there is a point $z \in Y$ for which

$$B(z, \alpha R) \subset B(y, R) \setminus E. \tag{2.14}$$

A subset of the space Y is called σ-porous in (Y, d) if it is a countable union of porous subsets in (Y, d).

To end this section, we introduce the following notations which will be used throughout this paper.

Let $D \subset X$ be a bounded, closed, and convex domain and let C be a k_D-bounded subset of D. The positive numbers R_1, R_2 are such that for each $\widetilde{x}_0 \in C$, the closed ball

$B_{\|\cdot\|}(\widetilde{x}_0, R_1)$ in $(X, \|\cdot\|)$ lies in D and $\mathrm{diam}_{\|\cdot\|}D < R_2$. Next, $0 < L_1 < L_2$ satisfy

$$L_1 k_D(x, y) \leq \|x - y\| \leq L_2 k_D(x, y) \tag{2.15}$$

for all $x, y \in C$. It is obvious that $0 < R_1/R_2 < 1$ and $K = L_2/L_1 > 1$.

If $\varnothing \neq C \subset D$, then $d(C)$ denotes the diameter of C in (D, k_D), that is,

$$d(C) = \sup_{x,y \in C} k_D(x, y). \tag{2.16}$$

3. Weak ergodicity

This section is devoted to weak ergodicity in the sense of population biology (see [19] and the references therein). Our result is analogous to [20, Theorem 1.1].

A sequence $\{f_t\}_{t=1}^{\infty} \in \mathcal{A}_H$ is called regular if for any $\epsilon > 0$, there exists a number $N \in \mathbb{N}$ such that for each $x, y \in C$, each integer $T \geq N$, and each mapping $\pi : \{1, \dots, T\} \to \{1, 2, \dots\}$, we have

$$k_D\big((f_{\pi(T)} \circ \cdots \circ f_{\pi(1)})(x), (f_{\pi(T)} \circ \cdots \circ f_{\pi(1)})(y)\big) \leq \epsilon. \tag{3.1}$$

A mapping $f \in \mathcal{U}_H$ is called regular if the sequence $\hat{f} = \{f_t\}_{t=1}^{\infty}$, where $f_t = f$ $(t \geq 1)$, is regular. It is easy to verify that if $f \in \mathcal{U}_H$ is regular, then there exists a unique $x_f \in C$ such that $f(x_f) = x_f$ and $f^n(x) \to x_f$ as $n \to \infty$, uniformly on C.

Denote by $\mathcal{F}$ the set of all regular elements of $\mathcal{A}_H$ and by $\mathcal{F}^{(0)}$ the set of all regular elements of $\mathcal{U}_H$.

For each $n \in \mathbb{N}$, we denote by $\mathcal{F}_n$ the set of all sequences $\{f_t\}_{t=1}^{\infty} \in \mathcal{A}_H$ which have the following property.

There exists an integer $N \in \mathbb{N}$ such that for each $x, y \in C$, each integer $T \geq N$, and each mapping $\pi : \{1, \dots, T\} \to \{1, 2, \dots\}$,

$$k_D\big((f_{\pi(T)} \circ \cdots \circ f_{\pi(1)})(x), (f_{\pi(T)} \circ \cdots \circ f_{\pi(1)})(y)\big) \leq \frac{1}{n}. \tag{3.2}$$

It is not difficult to see that $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n$.

Similarly, $\mathcal{F}_n^{(0)}$ is the set of all $f \in \mathcal{U}_H$ such that $\hat{f} \in \mathcal{F}_n$. Clearly, $\mathcal{F}^{(0)} = \bigcap_{n=1}^{\infty} \mathcal{F}_n^{(0)}$.

THEOREM 3.1. *Under the assumptions given in the definitions of $\mathcal{A}_H$, $\mathcal{U}_H$, $\mathcal{F}$, and $\mathcal{F}^{(0)}$,*
   (i) *the set $\mathcal{A}_H \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}_H$,*
   (ii) *the set $\mathcal{U}_H \setminus \mathcal{F}^{(0)}$ is σ-porous in $\mathcal{U}_H$.*

Proof. We will show that $\mathcal{A}_H \setminus \mathcal{F}_n$ is porous in $\mathcal{A}_H$ and that $\mathcal{U}_H \setminus \mathcal{F}_n^{(0)}$ is porous in $\mathcal{U}_H$ for each $n \in \mathbb{N}$. To this end, fix $n \in \mathbb{N}$ and choose $\alpha \in (0, 1)$ such that

$$\alpha < \frac{1-\alpha}{8Kn(d(C)+1)} \cdot \frac{R_1}{3R_2}, \tag{3.3}$$

where R_1, R_2, K, and $d(C)$ are defined at the end of the previous section. Assume that $\{f_t\}_{t=1}^{\infty} \in \mathcal{A}_H$ and $R \in (0,1]$. Set

$$\gamma = \frac{(1-\alpha)R}{2K(d(C)+1)} \cdot \frac{R_1}{3R_2} = s\frac{R_1}{3R_2}, \tag{3.4}$$

where $0 < s < 1$, and choose $0 < N \in \mathbb{N}$ such that

$$d(C) + 1 < 2N\alpha R. \tag{3.5}$$

Observe that

$$4\alpha nR < \gamma. \tag{3.6}$$

Fix $x' \in C$. For each $t \in \mathbb{N}$, define

$$f_{\gamma t}(x) = (1-s)f_t(x) + sx', \tag{3.7}$$

where $x \in D$. Note that if $f_t = f$ $(t \geq 1)$ with $f \in \mathcal{U}_H$, then $f_{\gamma t} = f_\gamma$ $(t \geq 1)$, where

$$f_\gamma(x) = (1-s)f(x) + sx' \tag{3.8}$$

for $x \in D$. Hence, we get

$$k_D(f_{\gamma t}(x), f_t(x)) \leq \frac{1}{L_1}s\|f_t(x) - x'\| \leq \frac{L_2}{L_1}sk_D(f_t(x), x') \leq K\gamma\frac{3R_2}{R_1}d(C)$$
$$= K \cdot \frac{(1-\alpha)R}{2K(d(C)+1)} \cdot \frac{R_1}{3R_2} \cdot \frac{3R_2}{R_1} \cdot d(C) < \frac{(1-\alpha)R}{2} \tag{3.9}$$

for each $x \in C$, and

$$k_D(f_{\gamma t}(x), f_{\gamma t}(y)) \leq \frac{1}{1+sR_1/2R_2}k_D(x,y) = \left(1 - \frac{sR_1}{2R_2+sR_1}\right)k_D(x,y)$$
$$\leq \left(1 - s\frac{R_1}{3R_2}\right)k_D(x,y) = (1-\gamma)k_D(x,y) \tag{3.10}$$

for every $x, y \in D$. Now, assume that $\{g_t\}_{t=1}^{\infty} \in \mathcal{A}_H$ and that

$$d_{\mathcal{A}_H}\left(\{f_{\gamma t}\}_{t=1}^{\infty}, \{g_t\}_{t=1}^{\infty}\right) \leq \alpha R. \tag{3.11}$$

Then we see that

$$d_{\mathcal{A}_H}\left(\{f_t\}_{t=1}^{\infty}, \{g_t\}_{t=1}^{\infty}\right) \leq d_{\mathcal{A}_H}\left(\{f_{\gamma t}\}_{t=1}^{\infty}, \{g_t\}_{t=1}^{\infty}\right) + d_{\mathcal{A}_H}\left(\{f_{\gamma t}\}_{t=1}^{\infty}, \{f_t\}_{t=1}^{\infty}\right)$$
$$< \alpha R + \frac{(1-\alpha)R}{2} = \frac{(1+\alpha)R}{2} < R. \tag{3.12}$$

To prove that for each $x, y \in C$, each integer $T \geq N$, and each mapping $\pi : \{1,\ldots,T\} \to \{1,2,\ldots\}$, we have

$$k_D\left((g_{\pi(T)} \circ \cdots \circ g_{\pi(1)})(x), (g_{\pi(T)} \circ \cdots \circ g_{\pi(1)})(y)\right) \leq \frac{1}{n}, \tag{3.13}$$

it is sufficient to verify that for each $x, y \in C$, each integer $T \geq N$, and each mapping $\pi : \{1,\ldots,N\} \to \{1,2,\ldots\}$, there is an integer $m \in \{1,\ldots,N\}$ such that

$$k_D\big((g_{\pi(m)} \circ \cdots \circ g_{\pi(1)})(x), (g_{\pi(m)} \circ \cdots \circ g_{\pi(1)})(y)\big) \leq \frac{1}{n}. \tag{3.14}$$

To this end, assume that $x, y \in C$ and $\pi : \{1,\ldots,N\} \to \{1,2,\ldots\}$. Assume, contrary to our claim, that for each integer $j \in \{1,\ldots,N\}$,

$$k_D\big((g_{\pi(j)} \circ \cdots \circ g_{\pi(1)})(x), (g_{\pi(j)} \circ \cdots \circ g_{\pi(1)})(y)\big) > \frac{1}{n},$$

$$k_D(x, y) > \frac{1}{n}. \tag{3.15}$$

Set

$$x_0 = x, \qquad x_{j+1} = g_{\pi(j+1)} x_j,$$

$$y_0 = y, \qquad y_{j+1} = g_{\pi(j+1)} y_j \tag{3.16}$$

for each $j \in \{0,\ldots,N-1\}$.

Now fix $j \in \{0,\ldots,N-1\}$. Then

$$k_D(x_j, y_j) > \frac{1}{n}. \tag{3.17}$$

Moreover,

$$k_D\big(f_{\gamma\pi(j+1)}(x_j), f_{\gamma\pi(j+1)}(y_j)\big) \leq (1-\gamma)k_D(x_j, y_j). \tag{3.18}$$

It follows from the above inequalities that

$$
\begin{aligned}
k_D(x_{j+1}, y_{j+1}) &= k_D\big(g_{\pi(j+1)}(x_j), g_{\pi(j+1)}(y_j)\big) \\
&\leq k_D\big(g_{\pi(j+1)}(x_j), f_{\gamma\pi(j+1)}(x_j)\big) + k_D\big(f_{\gamma\pi(j+1)}(x_j), f_{\gamma\pi(j+1)}(y_j)\big) \\
&\quad + k_D\big(f_{\gamma\pi(j+1)}(y_j), g_{\pi(j+1)}(y_j)\big) \\
&\leq (1-\gamma)k_D(x_j, y_j) + 2\alpha R = k_D(x_j, y_j) - \gamma k_D(x_j, y_j) + 2\alpha R \\
&< k_D(x_j, y_j) - \frac{\gamma}{n} + 2\alpha R < k_D(x_j, y_j) - \frac{4\alpha n R}{n} + 2\alpha R \\
&= k_D(x_j, y_j) - 2\alpha R.
\end{aligned}
\tag{3.19}
$$

Therefore, we obtain

$$k_D(x_N, y_N) \leq k_D(x_0, y_0) - 2N\alpha R \leq d(C) - 2N\alpha R < 0. \tag{3.20}$$

This contradiction yields the existence of an integer $m \in \{1,\ldots,N\}$ for which the inequality

$$k_D\big((g_{\pi(m)} \circ \cdots \circ g_{\pi(1)})(x), (g_{\pi(m)} \circ \cdots \circ g_{\pi(1)})(y)\big) \leq \frac{1}{n} \tag{3.21}$$

is valid. Hence $\{g_t\}_{t=1}^{\infty} \in \mathscr{F}_n$. Thus we have shown that

$$
\begin{aligned}
&\left\{\{g_t\}_{t=1}^{\infty} \in \mathscr{A}_H : d_{\mathscr{A}_H}\left(\{f_{\gamma t}\}_{t=1}^{\infty}, \{g_t\}_{t=1}^{\infty}\right) \le \alpha R\right\} \\
&\quad \subset \left\{\{\tilde{f}_t\}_{t=1}^{\infty} \in \mathscr{A}_H : d_{\mathscr{A}_H}\left(\{f_t\}_{t=1}^{\infty}, \{\tilde{f}_t\}_{t=1}^{\infty}\right) < R\right\} \cap \mathscr{F}_n.
\end{aligned}
\tag{3.22}
$$

If $f \in \mathscr{U}_H$ and $f_t = f$ for all $t \ge 1$, then $f_{\gamma t} = f_\gamma$ $(t \ge 1)$ and

$$
\{g \in \mathscr{U}_H : \rho_{\mathscr{U}_H}(f_\gamma, g) \le \alpha r\} \subset \{\tilde{f} \in \mathscr{U}_H : \rho_{\mathscr{U}_H}(f, \tilde{f}) < r\} \cap \mathscr{F}_n^{(0)}.
\tag{3.23}
$$

Consequently, the set $\mathscr{A}_H \setminus \mathscr{F}_n$ is porous in $\mathscr{A}_H$ and the set $\mathscr{U}_H \setminus \mathscr{F}_n^{(0)}$ is porous in $\mathscr{U}_H$ for each $n \in \mathbb{N}$. This completes the proof. $\square$

4. Convergence to common fixed points

In this section, we will study the convergence of unrestricted infinite products to a common fixed point.

First we introduce the following notations. Let $\mathscr{A}_H^*$ denote the set of all sequences $\mathbf{f} = \{f_t\}_{i=1}^{\infty} \in \mathscr{A}_H$ for which there exists $x_{\mathbf{f}} \in C$ such that

$$
f_t(x_{\mathbf{f}}) = x_{\mathbf{f}}
\tag{4.1}
$$

for all $t \ge 1$. The closure of $\mathscr{A}_H^*$ in the metric space $(\mathscr{A}_H, d_{\mathscr{A}_H})$ will be denoted by $\overline{\mathscr{A}_H^*}$.

THEOREM 4.1. *Let $\mathscr{F}$ be the set of all $\mathbf{f} = \{f_t\}_{i=1}^{\infty} \in \overline{\mathscr{A}_H^*}$ which satisfy the following conditions:*
  (i) *there exists $x_* \in C$ such that $f_t(x_*) = x_*$ for all $t \ge 1$;*
  (ii) *for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that*

$$
k_D\left((f_{\pi(n)} \circ \cdots \circ f_{\pi(1)})(x), x_*\right) \le \epsilon
\tag{4.2}
$$

  for each integer $n \ge N$, each mapping $\pi : \{1,\ldots,n\} \to \{1,2,\ldots\}$, and each $x \in C$.
Then the set $\overline{\mathscr{A}_H^*} \setminus \mathscr{F}$ is σ-porous in $\overline{\mathscr{A}_H^*}$.

Proof. For each $n \in \mathbb{N}$, let $\mathscr{F}_n$ be the set of all sequences $\{f_t\}_{i=1}^{\infty} \in \overline{\mathscr{A}_H^*}$ for which there exist $x^{(n)} \in C$ and an $N \in \mathbb{N}$ such that

$$
k_D\left((f_{\pi(T)} \circ \cdots \circ f_{\pi(1)})(x), x^{(n)}\right) \le \frac{1}{n}
\tag{4.3}
$$

for each integer $T \ge N$, each mapping $\pi : \{1,\ldots,T\} \to \{1,2,\ldots\}$, and each point $x \in C$. It is obvious that $\mathscr{F} = \bigcap_{n=1}^{\infty} \mathscr{F}_n$. Now, fix $n \in \mathbb{N}$. We will show that the set $\overline{\mathscr{A}_H^*} \setminus \mathscr{F}_n$ is porous in $\overline{\mathscr{A}_H^*}$. To see this, let $\alpha \in (0,1)$ be such that

$$
\alpha < \frac{1}{8Kn(d(C)+1)} \cdot \frac{R_1}{3R_2},
\tag{4.4}
$$

where R_1, R_2, K, and $d(C)$ are defined in Section 2. Clearly, $0 < \alpha < 1/2$. Assume that $\tilde{\mathbf{f}} = \{\tilde{f}_t\}_{i=1}^{\infty} \in \overline{\mathscr{A}_H^*}$ and $R \in (0, 1]$. Then there exists $\mathbf{f} = \{f_t\}_{i=1}^{\infty} \in \mathscr{A}_H^*$ such that

$$d_{\mathscr{A}_H}\left(\{\tilde{f}_t\}_{i=1}^{\infty}, \{f_t\}_{i=1}^{\infty}\right) \leq \frac{R}{4}. \tag{4.5}$$

Let $x_{\mathbf{f}} \in C$ satisfy

$$f_t(x_{\mathbf{f}}) = x_{\mathbf{f}} \tag{4.6}$$

for all $t \geq 1$. Set

$$\gamma = \frac{R}{4K(d(C)+1)} \cdot \frac{R_1}{3R_2} = s\frac{R_1}{3R_2}. \tag{4.7}$$

It is obvious that $0 < \gamma < 1$ and $0 < s < 1$. Next, choose $2 < N \in \mathbb{N}$ such that

$$(1 - \gamma)^N (d(C) + 1) < \frac{1}{2n}. \tag{4.8}$$

Finally, for each $t \in \mathbb{N}$, define

$$f_{\gamma t}(x) = (1 - s) f_t(x) + s x_{\mathbf{f}} \tag{4.9}$$

for $x \in C$. It is obvious that $\{f_{\gamma t}\}_{t=1}^{\infty} \in \mathscr{A}_H^*$. Moreover, we have

$$
\begin{aligned}
k_D\left(f_{\gamma t}(x), f_{\gamma t}(y)\right) &\leq \frac{1}{1 + sR_1/2R_2} k_D(x, y) = \left(1 - \frac{sR_1}{sR_1 + 2R_2}\right) k_D(x, y) \\
&\leq \left(1 - s\frac{R_1}{3R_2}\right) k_D(x, y) = (1 - \gamma) k_D(x, y)
\end{aligned}
\tag{4.10}
$$

for every $x, y \in D$. Next, we obtain

$$
\begin{aligned}
k_D\left(f_{\gamma t}(x), f_t(x)\right) &\leq \frac{1}{L_1} s\|f_t(x) - x_{\mathbf{f}}\| \leq \frac{L_2}{L_1} s k_D\left(f_t(x), x_{\mathbf{f}}\right) \\
&\leq K \frac{3R_2}{R_1} \gamma d(C) = K \cdot \frac{3R_2}{R_1} \cdot \frac{R}{4K(d(C)+1)} \cdot \frac{R_1}{3R_2} \cdot d(C) < \frac{R}{4}
\end{aligned}
\tag{4.11}
$$

for each $x \in C$. Assume now that $\{g_t\}_{i=1}^{\infty} \in \mathscr{A}_H$ and

$$d_{\mathscr{A}_H}\left(\{f_{\gamma t}\}_{i=1}^{\infty}, \{g_t\}_{i=1}^{\infty}\right) \leq \alpha R. \tag{4.12}$$

Then we get

$$
\begin{aligned}
d_{\mathscr{A}_H}\left(\{\tilde{f}_t\}_{i=1}^{\infty}, \{g_t\}_{i=1}^{\infty}\right) &\leq d_{\mathscr{A}_H}\left(\{\tilde{f}_t\}_{i=1}^{\infty}, \{f_t\}_{i=1}^{\infty}\right) + d_{\mathscr{A}_H}\left(\{f_t\}_{i=1}^{\infty}, \{f_{\gamma t}\}_{i=1}^{\infty}\right) \\
&\quad + d_{\mathscr{A}_H}\left(\{f_{\gamma t}\}_{i=1}^{\infty}, \{g_t\}_{i=1}^{\infty}\right) \\
&< \frac{R}{4} + \frac{R}{4} + \alpha R < R.
\end{aligned}
\tag{4.13}
$$

We will show that the following property holds.

(P1) For each $x \in C$, each integer $T \geq N$, and each mapping $\pi : \{1,\dots,T\} \to \{1,2,\dots\}$,

$$k_D\big((g_{\pi(T)} \circ \cdots \circ g_{\pi(1)})(x),x_{\mathrm{f}}\big) \leq \frac{1}{n}. \tag{4.14}$$

So, let $y \in C$ and $t \in \mathbb{N}$. We have

$$k_D(f_{\gamma t}(y),x_{\mathrm{f}}) = k_D(f_{\gamma t}(y),f_{\gamma t}(x_{\mathrm{f}})) \leq (1-\gamma)k_D(y,x_{\mathrm{f}}), \tag{4.15}$$

which implies that

$$k_D(g_t(y),x_{\mathrm{f}}) \leq k_D(g_t(y),f_{\gamma t}(y)) + k_D(f_{\gamma t}(y),x_{\mathrm{f}}) \leq \alpha R + (1-\gamma)k_D(y,x_{\mathrm{f}}) \tag{4.16}$$

for each $t \in \mathbb{N}$ and each $y \in C$. Assume that $x \in C$, $T \geq N$, and $\pi : \{1,\dots,T\} \to \{1,2,\dots\}$. Set

$$x_0 = x,$$
$$x_{i+1} = g_{\pi(i+1)}(x_i) \quad (i \geq 0). \tag{4.17}$$

Then, for any integer $i \in \mathbb{N}_0$, we get

$$k_D(x_{i+1},x_{\mathrm{f}}) = k_D(g_{\pi(i+1)}(x_i),x_{\mathrm{f}}) \leq \alpha R + (1-\gamma)k_D(x_i,x_{\mathrm{f}}). \tag{4.18}$$

Using induction, we obtain

$$k_D(x_i,x_{\mathrm{f}}) \leq (1-\gamma)^i k_D(x_0,x_{\mathrm{f}}) + \alpha R \cdot \left[\sum_{j=0}^{i-1}(1-\gamma)^j\right] \tag{4.19}$$

for $i = 1,\dots,T$, and therefore we see that

$$\begin{aligned}
k_D\big(&(g_{\pi(T)} \circ \cdots \circ g_{\pi(1)})(x),x_{\mathrm{f}}\big) \\
&= k_D(x_T,x_{\mathrm{f}}) < (1-\gamma)^T k_D(x_0,x_{\mathrm{f}}) + \alpha R \frac{1}{\gamma} \\
&\leq (1-\gamma)^N d(C) + \alpha R \frac{1}{\gamma} \\
&< (1-\gamma)^N d(C) + \frac{1}{8Kn(d(C)+1)} \cdot \frac{R_1}{3R_2} \cdot R \cdot \frac{4K(d(C)+1)}{R} \cdot \frac{3R_2}{R_1} \\
&= (1-\gamma)^N d(C) + \frac{1}{2n} < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.
\end{aligned} \tag{4.20}$$

Hence property (P1) holds. Therefore, for $\{g_t\}_{t=1}^{\infty} \in \overline{\mathscr{A}_H^*}$ with

$$d_{\mathscr{A}_H}\big(\{f_{\gamma t}\}_{t=1}^{\infty},\{g_t\}_{t=1}^{\infty}\big) \leq \alpha R, \tag{4.21}$$

we have

$$d_{\mathscr{A}_H}\big(\{\tilde{f}_t\}_{t=1}^{\infty},\{g_t\}_{t=1}^{\infty}\big) < R \tag{4.22}$$

and $\{g_t\}_{t=1}^{\infty}$ satisfies (P1). This means that $\{g_t\}_{t=1}^{\infty} \in \mathcal{F}_n$ which implies, in turn, that the set $\overline{\mathcal{A}_H^*} \setminus \mathcal{F}_n$ is porous in $\overline{\mathcal{A}_H^*}$. The proof is complete. $\qquad\square$

5. Convergence to a retraction

In this section, we continue to use the notations and definitions introduced in the previous sections.

Recall that C is a nonempty, closed, and convex subset of a bounded, convex domain D in a Banach space X, and that C lies strictly inside D. Let F be a nonempty, closed, and convex subset of the set C. By $\mathcal{U}_H^{(F)}$, we denote the set of all $f \in \mathcal{U}_H$ such that $f(x) = x$ for each $x \in F$. It is obvious that $\mathcal{U}_H^{(F)}$ is a closed subset of $\mathcal{U}_H$. Next, we let $\mathcal{A}_H^{(F)}$ consist of all $\{f_t\}_{t=1}^{\infty} \in \mathcal{A}_H$ such that $f_t \in \mathcal{U}_H^{(F)}$ for all $t \geq 1$. Clearly, $\mathcal{A}_H^{(F)}$ is a closed subset of $\mathcal{A}_H$.

Assume additionally that there exists a mapping $r \in \mathcal{U}_H^{(F)}$ such that $r(C) = F$.

A sequence $\{f_t\}_{t=1}^{\infty} \in \mathcal{A}_H^{(F)}$ is called normal if the following two properties hold.

(i) For each mapping $\widetilde{\pi} : \mathbb{N} \to \mathbb{N}$, there exists a mapping $p_{\widetilde{\pi}} : C \to F$ such that

$$\lim_{t \to \infty} \left(f_{\widetilde{\pi}(t)} \circ \cdots \circ f_{\widetilde{\pi}(1)} \right)(x) = p_{\widetilde{\pi}}(x) \tag{5.1}$$

for all $x \in C$.

(ii) For each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for each integer $T \geq N$, each mapping $\widetilde{\pi} : \mathbb{N} \to \mathbb{N}$, and each $x \in C$,

$$k_D \left((f_{\widetilde{\pi}(T)} \circ \cdots \circ f_{\widetilde{\pi}(1)})(x), p_{\widetilde{\pi}}(x) \right) \leq \epsilon. \tag{5.2}$$

We observe that $p_{\widetilde{\pi}} \in \mathcal{U}_H^{(F)}$ as the pointwise limit of a sequence in $\mathcal{U}_H^{(F)}$ (by the compactness of $\overline{D}$ in $\sigma(X, \mathcal{N})$ and Lemma 2.6). Denote by $\mathcal{F}$ the set of all normal sequences in $\mathcal{A}_H^{(F)}$. For $n \in \mathbb{N}$, denote by $\mathcal{F}_n$ the set of all $\{f_t\}_{t=1}^{\infty} \in \mathcal{A}_H^{(F)}$ for which there exists an $N \in \mathbb{N}$ such that for each $x \in C$, each integer $T \geq N$, and each mapping $\pi : \{1, \ldots, T\} \to \{1, 2, \ldots\}$,

$$\mathrm{dist}_{k_D} \left((f_{\pi(T)} \circ \cdots \circ f_{\pi(1)})(x), F \right) = \inf_{y \in F} k_D \left((f_{\pi(T)} \circ \cdots \circ f_{\pi(1)})(x), y \right) < \frac{1}{n}. \tag{5.3}$$

It is easy to see that $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n$.

A mapping $f \in \mathcal{U}_H^{(F)}$ is called normal if the constant sequence $\{f_t\}_{t=1}^{\infty}$, with $f_t = f$ for all $t \geq 1$, is normal. Denote by $\mathcal{F}^{(0)}$ the set of all normal mappings $f \in \mathcal{U}_H^{(F)}$.

THEOREM 5.1. (i) *The set $\mathcal{A}_H^{(F)} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{A}_H^{(F)}, d_{\mathcal{A}_H})$.*

(ii) *The set $\mathcal{U}_H^{(F)} \setminus \mathcal{F}^{(0)}$ is σ-porous in $(\mathcal{U}_H^{(F)}, \rho_{\mathcal{U}_H})$.*

Proof. Let $n \in \mathbb{N}$ and choose $\alpha \in (0, 1)$ such that

$$\alpha < \frac{1}{4Kn(d(C)+1)} \cdot \frac{R_1}{3R_2} \tag{5.4}$$

(here we use notations from the previous sections). Obviously, $\alpha < 1/4$. Choose any $\{f_t\}_{t=1}^{\infty}$ $\in \mathcal{A}_H^{(F)}$ and $R \in (0,1]$. Put

$$\gamma = \frac{R}{2K(d(C)+1)} \cdot \frac{R_1}{3R_2} = s\frac{R_1}{3R_2}. \tag{5.5}$$

Clearly, $0 < \gamma < 1$ and $0 < s < 1$. Assume that a natural number $N > 2$ is such that

$$(1-\gamma)^N (d(C)+1) < \frac{1}{2n}. \tag{5.6}$$

Now, for each $t \in \mathbb{N}$, define

$$f_{\gamma t}(x) = (1-s)f_t(x) + sr(x) \tag{5.7}$$

for $x \in C$. Hence we have

$$f_{\gamma t}(x) = x \tag{5.8}$$

for every $x \in F$ and $t \geq 1$. We also have

$$
\begin{aligned}
k_D(f_{\gamma t}(y),x) &= k_D((1-s)f_t(y) + sr(y), (1-s)f_t(x) + sr(x)) \\
&\leq \max\left[k_D(f_t(y),f_t(x)), k_D(r(y),r(x))\right] \leq k_D(y,x)
\end{aligned} \tag{5.9}
$$

for each $x \in F$, $y \in C$, and $t \geq 1$. Therefore $\{f_{\gamma t}\}_{t=1}^{\infty} \in \mathcal{A}_H^{(F)}$ if $\{f_t\}_{t=1}^{\infty} \in \mathcal{A}_H^{(F)}$, and $f_\gamma \in \mathcal{U}_H^{(F)}$ if $f \in \mathcal{U}_H^{(F)}$. Next, we get

$$
\begin{aligned}
k_D(f_{\gamma t}(x), f_t(x)) &= k_D((1-s)f_t(x) + sr(x), f_t(x)) \leq \frac{1}{L_1}s\|f_t(x) - r(x)\| \\
&\leq \frac{L_2}{L_1}sk_D(f_t(x),r(x)) \leq K\frac{3R_2}{R_1}\gamma d(C) \\
&= K\frac{3R_2}{R_1}\frac{R}{2K(d(C)+1)} \cdot \frac{R_1}{3R_2}d(C) < \frac{1}{2}R
\end{aligned} \tag{5.10}
$$

for $x \in C$ and $t \geq 1$, and hence for $\{g_t\}_{t=1}^{\infty} \in \mathcal{A}_H^{(F)}$ with

$$d_{\mathcal{A}_H}\left(\{g_t\}_{t=1}^{\infty}, \{f_{\gamma t}\}_{t=1}^{\infty}\right) \leq \alpha R, \tag{5.11}$$

we obtain

$$
\begin{aligned}
d_{\mathcal{A}_H}\left(\{g_t\}_{t=1}^{\infty}, \{f_t\}_{t=1}^{\infty}\right) &\leq d_{\mathcal{A}_H}\left(\{g_t\}_{t=1}^{\infty}, \{f_{\gamma t}\}_{t=1}^{\infty}\right) + d_{\mathcal{A}_H}\left(\{f_{\gamma t}\}_{t=1}^{\infty}, \{g_t\}_{t=1}^{\infty}\right) \\
&< \alpha R + \frac{1}{2}R < \frac{1}{4}R + \frac{1}{2}R < R.
\end{aligned} \tag{5.12}
$$

Let $T \geq N$ be an integer, $x \in C$, and $\pi : \{1,\ldots,T\} \to \{1,2,\ldots\}$. We will show that

$$\operatorname{dist}_{k_D}\left((g_{\pi(T)} \circ \cdots \circ g_{\pi(1)})(x), F\right) = \inf_{y \in F} k_D\left((g_{\pi(T)} \circ \cdots \circ g_{\pi(1)})(x), y\right) < \frac{1}{n}. \tag{5.13}$$

It is sufficient to show that

$$\text{dist}_{k_D}\left(\left(g_{\pi(N)}\circ\cdots\circ g_{\pi(1)}\right)(x),F\right)<\frac{1}{n}. \tag{5.14}$$

Take $w\in C$ and $t\in\mathbb{N}$. Then, for each $z\in F$, we have $(1-s)z+sr(w)\in F$, and therefore

$$\begin{aligned}
\text{dist}_{k_D}\left(f_{\gamma\pi(t)}(w),F\right) &= \inf_{y\in F}k_D\left(f_{\gamma\pi(t)}(w),y\right)\le k_D\left(f_{\gamma\pi(t)}(w),(1-s)z+sr(w)\right)\\
&= k_D\left((1-s)f_{\pi(t)}(w)+sr(w),(1-s)f_{\pi(t)}(z)+sr(w)\right)\\
&\le \frac{1}{1+sR_1/2R_2}k_D(w,z)=\left(1-\frac{sR_1}{sR_1+2R_2}\right)k_D(w,z)\\
&\le \left(1-s\frac{R_1}{3R_2}\right)k_D(w,z)=(1-\gamma)k_D(w,z).
\end{aligned} \tag{5.15}$$

This leads to

$$\text{dist}_{k_D}\left(f_{\gamma\pi(t)}(w),F\right)\le(1-\gamma)\,\text{dist}_{k_D}(w,F), \tag{5.16}$$

which implies, in turn, that

$$\begin{aligned}
\text{dist}_{k_D}\left(g_{\pi(t)}(w),F\right) &\le \text{dist}_{k_D}\left(f_{\gamma\pi(t)}(w),F\right)+k_D\left(f_{\gamma\pi(t)}(w),g_{\pi(t)}(w)\right)\\
&\le (1-\gamma)\,\text{dist}_{k_D}(w,F)+\alpha R.
\end{aligned} \tag{5.17}$$

By induction,

$$\text{dist}_{k_D}\left(\left(g_{\pi(i)}\circ\cdots\circ g_{\pi(1)}\right)(x),F\right)\le(1-\gamma)^i\text{dist}_{k_D}(w,F)+\alpha R\cdot\left[\sum_{j=0}^{i-1}(1-\gamma)^j\right]. \tag{5.18}$$

Hence

$$\begin{aligned}
&\text{dist}_{k_D}\left(\left(g_{\pi(N)}\circ\cdots\circ g_{\pi(1)}\right)(x),F\right)\\
&\quad<(1-\gamma)^N d(C)+\frac{1}{\gamma}\cdot\alpha R\\
&\quad\le (1-\gamma)^N d(C)+\frac{2K(d(C)+1)}{R}\cdot\frac{3R_2}{R_1}\cdot\frac{1}{4Kn(d(C)+1)}\cdot\frac{R_1}{3R_2}\cdot R\\
&\quad=(1-\gamma)^N d(C)+\frac{1}{2n}<\frac{1}{2n}+\frac{1}{2n}=\frac{1}{n}.
\end{aligned} \tag{5.19}$$

Thus we have shown that

$$\text{dist}_{k_D}\left(\left(g_{\pi(T)}\circ\cdots\circ g_{\pi(1)}\right)(x),F\right)<\frac{1}{n} \tag{5.20}$$

for each point $x\in C$, each integer $T\ge N$, and each mapping $\pi:\{1,\dots,T\}\to\{1,2,\dots\}$. This means that $\{g_t\}_{t=1}^\infty\in\mathscr{F}_n$ and yields the porosity of $\mathscr{A}_H^{(F)}\setminus\mathscr{F}_n$ in $(\mathscr{A}_H^{(F)},d_{\mathscr{A}_H})$. Since $\mathscr{F}=\bigcap_{n=1}^\infty\mathscr{F}_n$, we conclude that $\mathscr{A}_H^{(F)}\setminus\mathscr{F}$ is σ-porous in $(\mathscr{A}_H^{(F)},d_{\mathscr{A}_H})$. It is not difficult to see that the second statement of the theorem has also been proved. $\square$

Remark 5.2. Finally, we observe that all the results proved in this paper have a metric character and therefore analogous results are valid in appropriately defined spaces of sequences of k_D-nonexpansive (not necessarily holomorphic) mappings.

Acknowledgments

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