PRECISE SPECTRAL ASYMPTOTICS FOR NONAUTONOMOUS LOGISTIC EQUATIONS OF POPULATION DYNAMICS IN A BALL

TETSUTARO SHIBATA

Received 17 August 2004

We consider the semilinear elliptic eigenvalue problem $-\Delta u + k(|x|)u^p = \lambda u$, $u > 0$ in B_R, $u = 0$ on ∂B_R, where $p > 1$ is a constant, $B_R := \{x \in \mathbf{R}^N : |x| < R\}(N \geq 1)$, and $\lambda > 0$ is a parameter. We investigate the global structure of the branch of (λ, u_λ) of bifurcation diagram from a point of view of L^2-theory. To do this, we establish a precise asymptotic formula for $\lambda = \lambda(\alpha)$ as $\alpha \to \infty$, where $\alpha := \|u_\lambda\|_2$.

1. Introduction

We consider the semilinear nonautonomous logistic equation of population dynamics

$$-\Delta u + k(|x|)u^p = \lambda u \quad \text{in } B_R, \tag{1.1}$$

$$u > 0 \quad \text{in } B_R, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial B_R, \tag{1.3}$$

where $p > 1$ is a constant, $B_R := \{x \in \mathbf{R}^N : |x| < R\}(N \geq 1)$ and $\lambda > 0$ is a parameter.

We assume the following conditions.

(A.1) $k(r) \in C^2([0,R])$ and $k(r) > 0$ for $r \in [0,R]$.
(A.2) $dk(r)/dr > 0$ for $r \in [0,R]$.
(A.3) Let $K(r) := k(r)^{-1/(p-1)}$. Then $-\Delta K(|x|) \geq 0$ for $x \in B_R$.

The typical example is $k(r) = (1+r)^2$, $p = 3$ and $N \geq 3$.

The equation (1.1)–(1.3) is a model of population density for some species when $p = 2$. Here, $\lambda > 0$ is regarded as the reciprocal number of its diffusion rate, and k the effect of crowding for the species. Since (1.1)–(1.3) is motivated by this biological background, it has been investigated by many authors. In particular, it is shown by the arguments in [1, 6] (we also refer to [7, 10, 11, 12]) that:

(a) for each $\lambda > \lambda_1$, there exists a unique solution $(\lambda, u_\lambda) \in \mathbf{R}_+ \times C^2(\bar{B}_R)$ of (1.1)–(1.3), where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet zero boundary condition,
(b) the set $\{(\lambda, u_\lambda), \lambda > \lambda_1\}$ gives all the solutions of (1.1)–(1.3) and is an unbounded curve of class C^1 in $\mathbf{R}_+ \times C^2(\bar{B}_R)$ emanating from $(\lambda_1, 0)$.

Abstract and Applied Analysis 2005:6 (2005) 563–573
DOI: 10.1155/AAA.2005.563

Therefore, one of the main interests to study (1.1)–(1.3) is to understand well this bifurcation diagram.

The purpose here is to study precisely the global structure of this bifurcation diagram from a view point of L^2-theory to know the relationship between the reciprocal parameter λ of the diffusion rate and the number of the population of the species. To do this, we parametrize the solution set by its L^2-norm as follows. For a given $\alpha > 0$, we denote by $(\lambda(\alpha), u_\alpha) \in \{\lambda > \lambda_1\} \times C^2(\bar{B}_R)$ the solution pair of (1.1)–(1.3) with $\|u_\alpha\|_{L^2(B_R)} = \alpha$, which uniquely exists (i.e. $u_\alpha = u_{\lambda(\alpha)}$). We call the graph $\lambda = \lambda(\alpha)(\alpha > 0)$ the L^2-bifurcation diagram of (1.1)–(1.3).

To gain a clearer picture of the L^2-bifurcation diagram locally and globally, several attempts have been made to study the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \to 0$ and $\alpha \to \infty$. For the works in this direction, we refer to [2, 3, 4, 5, 9, 14, 15, 16, 17] and the references therein. In particular, [3, 4] obtained the asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to 0$ by using a variational method. On the other hand, if we focus our attention to the behavior of $\lambda(\alpha)$ as $\alpha \to \infty$, then since

$$u_\lambda(x)/\lambda^{1/(p-1)} \longrightarrow K(|x|) \tag{1.4}$$

uniformly on any compact subset in B_R as $\lambda \to \infty$ (cf. Lemma 2.1 in Section 2), we easily see from (1.4) that

$$\alpha^2 = ||u_\alpha||_2^2 = (1 + o(1))\lambda(\alpha)^{2/(p-1)} \int_{B_R} K(|x|)^2\, dx. \tag{1.5}$$

This implies

$$\lambda(\alpha) = A(K)\alpha^{p-1} + o(\alpha^{p-1}), \tag{1.6}$$

where $A(K) = \|K\|_{L^2(B_R)}^{1-p}$. It is therefore natural to ask: *what is the exact second term of $\lambda(\alpha)$ as $\alpha \to \infty$?*

Before answering this question, we recall some known results for the autonomous case. Let

$$\begin{aligned} C_0 &:= \int_0^1 \sqrt{\frac{p-1}{p+1} - s^2 + \frac{2}{p+1} s^{p+1}}\, ds, \\ B(K) &:= \frac{p+3}{2} C_0 A(K)^{(p+3)/(2(p-1))} R^{N-1} |S^{N-1}|, \end{aligned} \tag{1.7}$$

where $|S^{N-1}|$ is the measure of the unit sphere in $\mathbf{R}^N$.

Theorem 1.1 [15, 16]. (i) *Let $N = 1$ and $k \equiv 1$. Furthermore, replace B_R by $(0,1)$ in (1.1)–(1.3). Then as $\alpha \to \infty$*

$$\lambda(\alpha) = \alpha^{p-1} + (p+3)C_0\alpha^{(p-1)/2} + O(1). \tag{1.8}$$

(ii) *Let $k \equiv 1$ in (1.1)–(1.3). Then as $\alpha \to \infty$*

$$\lambda(\alpha) = A(1)\alpha^{p-1} + B(1)\alpha^{(p-1)/2} + o(\alpha^{(p-1)/2}). \tag{1.9}$$

We remark that m-th ($m \in \mathbf{N}$) term of $\lambda(\alpha)$ has been obtained in [15] for the case $N = 1$. Furthermore, if $k \equiv 1$, then (1.9) has been extended to the case of general bounded domain (cf. [17]).

On the other hand, for the nonautonomous case, only the following result seems to have been given.

Theorem 1.2 [14, Theorem 1]. *Let $N = 1$ and $R = 1$ in (1.1)–(1.3). Then the following asymptotic formula holds as $\alpha \to \infty$:*

$$\lambda(\alpha) = A(K)\alpha^{p-1} + O\left(\alpha^{(p-1)/2}\right). \tag{1.10}$$

We see that for the nonautonomous case, not the exact second term has been obtained even if $N = 1$.

The main aim here, therefore, is to establish the exact second term for the nonautonomous case when $N \geq 1$.

Now we state our result.

Theorem 1.3. *Let $C(K) := K(R)^2B(K)$. Then as $\alpha \to \infty$*

$$\lambda(\alpha) = A(K)\alpha^{p-1} + C(K)\alpha^{(p-1)/2} + o\left(\alpha^{(p-1)/2}\right). \tag{1.11}$$

Comparing (1.11) with (1.9), we find that the leading term of $\lambda(\alpha)$ is affected by the value of K in B_R, while the second term is affected also by $K(R)$, the value of K at the boundary.

We explain briefly the reason why the nonautonomous equation is difficult to treat. When we consider the autonomous case, to obtain the generalization of (1.9) in [17] for general bounded domain Ω, the Pohozaev identity and the precise study of the asymptotic behavior of u_λ near $\partial\Omega$ have played crucial roles. Unfortunately, the Pohozaev identity is not useful any more for nonautonomous case. Furthermore, the precise analysis of the asymptotic behavior of u_λ near $\partial\Omega$ is quite difficult to study when the equation is nonautonomous and Ω is a general bounded domain. Therefore, we restrict our attention to the case where $\Omega = B_R$ here as a first step to investigate the nonautonomous cases. The foreseeable extension of this research would certainly be to investigate the case where Ω is a general bounded domain.

The approach of the proof of Theorem 1.3 is based on the arguments developed in [16], since in this case, by (A.2), we see from [8] that (1.1)–(1.3) is equivalent to the ODE

$$u''(r) + \frac{N-1}{r}u'(r) + \lambda u(r) - k(r)u(r)^p = 0, \quad 0 < r < R, \tag{1.12}$$

$$u(r) > 0, \quad 0 \leq r < R, \tag{1.13}$$

$$u'(0) = u(R) = 0. \tag{1.14}$$

In [16], the key ingredient for obtaining the second term of $\lambda(\alpha)$ in (1.9) is to find the asymptotic formulas for $\|\nabla u_\alpha\|_2$, $\|u_\alpha\|_{p+1}^{p+1}$ and $\int_0^R (N-1)u_\alpha'(s)^2/s\,ds$ as $\alpha \to \infty$. However, if $k \not\equiv 1$, then clearly these terms are affected by k and we encounter several difficulties caused by k. Therefore, the calculation of these terms are more complicated than that to obtain (1.9) and in particular, the effect from the boundary on these terms must be taken into careful consideration.

2. Proof of Theorem 1.3

For convenience, we identify (1.1)–(1.3) with (1.12)–(1.14). Furthermore, for the notation of the solution pair of (1.1)–(1.3) and (1.12)–(1.14), we use both (λ, u_λ) and $(\lambda(\alpha), u_\alpha)$. Furthermore, we write $A = A(K)$ for simplicity.

Lemma 2.1. *Let an arbitrary* $0 < R_0 < R$ *be fixed. Then as* $\lambda \to \infty$

$$\frac{u_\lambda(r)}{\lambda^{1/(p-1)}} = K(r)(1+o(1)) \tag{2.1}$$

uniformly on $[0, R_0]$.

Proof. Let an arbitrary $x_0 \in \bar{B}_{R_0}$ be fixed. Furthermore, let $r_0 = |x_0|$ and $B_{x_0,\delta} := \{|x - x_0| < \delta\} \subset B_R$. By (A.2), we see that for $x \in B_{x_0,\delta}$

$$k(r_0 - \delta) < k(|x|) < k(r_0 + \delta). \tag{2.2}$$

Let w_λ be a unique solution of

$$\begin{aligned} -\Delta w + k(|x|)w^p &= \lambda w \quad \text{in } B_{x_0,\delta}, \\ w &> 0 \quad \text{in } B_{x_0,\delta}, \\ w &= 0 \quad \text{on } \partial B_{x_0,\delta}. \end{aligned} \tag{2.3}$$

Then a comparison and maximum principle show that $u_\lambda \geq w_\lambda$ in $B_{x_0,\delta}$. Furthermore, let v_λ be a unique solution of

$$\begin{aligned} -\Delta v + k(r_0+\delta)v^p &= \lambda v \quad \text{in } B_{x_0,\delta}, \\ v &> 0 \quad \text{in } B_{x_0,\delta}, \\ v &= 0 \quad \text{on } \partial B_{x_0,\delta}. \end{aligned} \tag{2.4}$$

Then by (2.2), for $x \in B_{x_0,\delta}$, we obtain

$$-\Delta v_\lambda + k(|x|)v_\lambda^p - \lambda v_\lambda = \big(k(|x|) - k(r_0+\delta)\big)v_\lambda \leq 0. \tag{2.5}$$

This implies that v_λ is a subsolution of (2.3). Furthermore, since a constant function $(\lambda/k(r_0+\delta))^{1/(p-1)}$ is a supersolution of (2.4), by (2.2), we see that for $x \in B_{x_0,\delta}$

$$v_\lambda(x) \leq \left(\frac{\lambda}{k(r_0+\delta)}\right)^{1/(p-1)} \leq W_\lambda(x) := \left(\frac{\lambda}{k(|x|)}\right)^{1/(p-1)}. \tag{2.6}$$

By (A.3), we see that W_λ is a super-solution of both (1.1)–(1.3) and (2.3). Therefore, by (2.6),

$$\begin{gathered} v_\lambda(x) \le w_\lambda(x) \le W_\lambda(x), \quad x \in B_{r_0,\delta}, \\ u_\lambda(x) \le W_\lambda(x), \quad x \in B_R. \end{gathered} \tag{2.7}$$

Since $u_\lambda \ge w_\lambda$, consequently,

$$v_\lambda(x) \le w_\lambda(x) \le u_\lambda(x) \le W_\lambda(x), \quad x \in B_{x_0,\delta}. \tag{2.8}$$

On the other hand, it is known (cf. [6]) that for $x \in \bar{B}_{x_0,\delta/2} = \{|x - x_0| \le \delta/2\}$, as $\lambda \to \infty$

$$\frac{v_\lambda(x)}{\lambda^{1/(p-1)}} \longrightarrow k(r_0+\delta)^{-1/(p-1)} \tag{2.9}$$

uniformly. Since $\bar{B}_{R_0}$ is covered by finite number of balls with radius $\delta/2$ such as $B_{x_0,\delta/2}$, and δ is arbitrary, by (2.8) and (2.9), we obtain our conclusion. □

Corollary 2.2. *As $\lambda \to \infty$*

$$||u_\lambda||_\infty = \lambda^{1/(p-1)}(K(0)+o(1)). \tag{2.10}$$

Lemma 2.3. *As $\lambda \to \infty$*

$$\int_0^R \frac{N-1}{r} u_\lambda'(r)^2\,dr = o\big(\lambda^{(p+1)/(p-1)}\big). \tag{2.11}$$

Proof. Multiply (1.12) by $u_\lambda'(r)$. Then

$$\left\{u_\lambda''(r) + \frac{N-1}{r}u_\lambda'(r) + \lambda u_\lambda(r) - k(r)u_\lambda(r)^p\right\}u_\lambda'(r) = 0. \tag{2.12}$$

This implies

$$\begin{aligned} \frac{1}{2}u_\lambda'(r)^2 &+ \int_0^r \frac{N-1}{s}u_\lambda'(s)^2\,ds + \frac{1}{2}\lambda u_\lambda(r)^2 \\ &\quad - \frac{1}{p+1}k(r)u_\lambda(r)^{p+1} + \int_0^r \frac{1}{p+1}k'(s)u_\lambda(s)^{p+1}\,ds \\ &= \frac{1}{2}u_\lambda'(R)^2 + \int_0^R \frac{N-1}{r}u_\lambda'(r)^2\,ds + \int_0^R \frac{1}{p+1}k'(r)u_\lambda(r)^{p+1}\,dr \\ &= \frac{1}{2}\lambda||u_\lambda||_\infty^2 - \frac{1}{p+1}k(0)||u_\lambda||_\infty^{p+1}. \end{aligned} \tag{2.13}$$

We fix $0 < R_0 < R$. Then by Lemma 2.1, for $0 \le r \le R_0$

$$\begin{aligned} \int_0^r k'(s)u_\lambda(s)^{p+1}\,ds &= (1+o(1))\lambda^{(p+1)/(p-1)}\int_0^r k'(s)K(s)^{p+1}\,ds \\ &= (1+o(1))\lambda^{(p+1)/(p-1)}\int_{k(0)}^{k(r)} \theta^{-(p+1)/(p-1)}\,d\theta \\ &= -\frac{p-1}{2}(1+o(1))\lambda^{(p+1)/(p-1)}\big(K(r)^2 - K(0)^2\big). \end{aligned} \tag{2.14}$$

By this, Lemma 2.1 and Corollary 2.2, for $0 \le r \le R_0$, we obtain

$$\begin{aligned}
L_\lambda(r) :=&\ \frac{1}{2}\lambda\|u_\lambda\|_\infty^2 - \frac{1}{p+1}k(0)\|u_\lambda\|_\infty^{p+1} - \frac{1}{2}\lambda u_\lambda(r)^2 \\
&+ \frac{1}{p+1}k(r)u_\lambda(r)^{p+1} - \frac{1}{p+1}\int_0^r k'(s)u_\lambda(s)^{p+1}\,ds \\
=&\ \frac{1}{2}(1+o(1))\lambda^{(p+1)/(p-1)}K(0)^2 - \frac{1}{p+1}(1+o(1))K(0)^2\lambda^{(p+1)/(p-1)} \\
&- \frac{1}{2}(1+o(1))\lambda^{(p+1)/(p-1)}K(r)^2 + \frac{1}{p+1}(1+o(1))\lambda^{(p+1)/(p-1)}K(r)^2 \\
&+ \frac{p-1}{2(p+1)}(1+o(1))\lambda^{(p+1)/(p-1)}\left(K(r)^2 - K(0)^2\right) \\
=&\ o\left(\lambda^{(p+1)/(p-1)}\right).
\end{aligned} \tag{2.15}$$

By this and (2.13), we obtain

$$\int_0^{R_0} \frac{N-1}{s}u_\lambda'(s)^2\,ds \le L_\lambda(r) = o\left(\lambda^{(p+1)/(p-1)}\right). \tag{2.16}$$

Multiply (1.1) by u_λ. Then integration by parts along with Lemma 2.1 implies that as $\lambda \to \infty$

$$\|\nabla u_\lambda\|_2^2 = \lambda\|u_\lambda\|_2^2 - \int_{B_R} k(|x|)u_\lambda(x)^{p+1}\,dx = o\left(\lambda^{(p+1)/(p-1)}\right). \tag{2.17}$$

By this, for $\lambda \gg 1$, we obtain

$$\begin{aligned}
\int_{R_0}^{R} \frac{N-1}{s}u_\lambda'(s)^2\,ds &\le \frac{N-1}{R_0^N}\int_{R_0}^{R} s^{N-1}u_\lambda'(s)^2\,ds \\
&\le C\|u_\lambda\|_2^2 = o\left(\lambda^{(p+1)/(p-1)}\right).
\end{aligned} \tag{2.18}$$

By this and (2.16), we obtain our conclusion. Thus the proof is complete. $\square$

Lemma 2.4. *As $\lambda \to \infty$*

$$\|\nabla u_\lambda\|_2^2 = K(R)^2R^{N-1}\,|\,S^{N-1}\,|\,C_0\lambda^{(p+3)/(2(p-1))}(1+o(1)). \tag{2.19}$$

Proof. Let an arbitrary $0 < \delta \ll 1$ be fixed. We first show that for $\lambda \gg 1$

$$\int_0^{R-\delta} r^{N-1} u_\lambda'(r)^2\, dr = o\big(\lambda^{(p+3)/(2(p-1))}\big). \tag{2.20}$$

By (2.13) and (2.15), for $r \in [0, R-\delta]$,

$$\frac{1}{2} u_\lambda'(r)^2 \le L_\lambda(r). \tag{2.21}$$

By this, (2.15), Lemma 2.1 and Corollary 2.2, we obtain

$$\begin{aligned}
\int_0^{R-\delta} r^{N-1} u_\lambda'(r)^2\, dr &\le R^{N-1} \int_0^{R-\delta} u_\lambda'(r)^2\, dr \\
&\le R^{N-1} \int_0^{R-\delta} -\sqrt{L_\lambda(r)}\, u_\lambda'(r)\, dr \\
&= o\big(\lambda^{(p+1)/(2(p-1))}\big) \int_0^{R-\delta} -u_\lambda'(r)\, dr \\
&= o\big(\lambda^{(p+1)/(2(p-1))}\big) \big(\|u_\lambda\|_\infty - u_\lambda(R-\delta)\big) \\
&= o\big(\lambda^{(p+3)/(2(p-1))}\big).
\end{aligned} \tag{2.22}$$

Next, by Lemma 2.3, (2.13) and (2.14), for a fixed $0 \le r < R$, as $\lambda \to \infty$

$$\begin{aligned}
\frac{1}{2} u_\lambda'(r)^2 = {}& \frac{1}{2}\lambda \|u_\lambda\|_\infty^2 - \frac{1}{p+1} k(0) \|u_\lambda\|_\infty^{p+1} - \frac{1}{2}\lambda u_\lambda(r)^2 \\
&+ \frac{1}{p+1} k(r) u_\lambda(r)^{p+1} + \frac{p-1}{2(p+1)} \lambda^{(p+1)/(p-1)} \big(K(r)^2 - K(0)^2\big) \\
&+ o\big(\lambda^{(p+1)/(p-1)}\big).
\end{aligned} \tag{2.23}$$

By this, (A.2) and Lemma 2.1, for a fixed $r \in [R-\delta, R)$

$$\begin{aligned}
\frac{1}{2} u_\lambda'(r)^2 &\le \frac{1}{2} M_\lambda(r) \\
&:= \frac{p-1}{2(p+1)} K(R-\delta)^2 \lambda^{(p+1)/(p-1)} - \frac{1}{2}\lambda u_\lambda(r)^2 \\
&\quad + \frac{1}{p+1} k(R) u_\lambda(r)^{p+1} + o\big(\lambda^{(p+1)/(p-1)}\big).
\end{aligned} \tag{2.24}$$

By this, we obtain

$$\begin{aligned}
&\limsup_{\lambda\to\infty} \lambda^{-(p+3)/(2(p-1))} \int_{R-\delta}^{R} r^{N-1} u_\lambda'(r)^2\,dr \\
&\quad \le \limsup_{\lambda\to\infty} \lambda^{-(p+3)/(2(p-1))} R^{N-1} \int_{R-\delta}^{R} -\sqrt{M_\lambda(r)}\,u_\lambda'(r)\,dr \\
&\quad = \limsup_{\lambda\to\infty} \lambda^{-(p+3)/(2(p-1))} R^{N-1} \\
&\qquad \times \int_0^{u_\lambda(R-\delta)} \sqrt{\frac{p-1}{p+1}K(R-\delta)^2\lambda^{(p+1)/(p-1)}(1+o(1)) - \lambda\theta^2 + \frac{2}{p+1}k(R)\theta^{p+1}}\,d\theta \\
&\quad = \limsup_{\lambda\to\infty} R^{N-1}\bigl(1+\epsilon_1(\delta)\bigr) \\
&\qquad \times \int_0^{u_\lambda(R-\delta)/\lambda^{1/(p-1)}} \sqrt{\frac{p-1}{p+1}K(R)^2 - s^2 + \frac{2}{p+1}k(R)s^{p+1}}\,ds \\
&\quad \le R^{N-1}\bigl(1+\epsilon_2(\delta)\bigr) \\
&\qquad \times \int_0^{K(R-\delta)} \sqrt{\frac{p-1}{p+1}K(R)^2 - s^2 + \frac{2}{p+1}k(R)s^{p+1}}\,ds \quad (\text{put } s = K(R)t) \\
&\quad = R^{N-1}\bigl(1+\epsilon_3(\delta)\bigr)K(R)^2 \int_0^1 \sqrt{\frac{p-1}{p+1} - t^2 + \frac{2}{p+1}t^{p+1}}\,dt,
\end{aligned} \tag{2.25}$$

where $\epsilon_1(\delta), \epsilon_2(\delta), \epsilon_3(\delta) \to 0$ as $\delta \to 0$. By the same arguments as those above, we obtain

$$\begin{aligned}
&\liminf_{\lambda\to\infty} \lambda^{-(p+3)/(2(p-1))} \int_{R-\delta}^{R} r^{N-1} u_\lambda'(r)^2\,dr \\
&\qquad \ge \bigl(1-\epsilon_4(\delta)\bigr)K(R)^2(R-\delta)^{N-1} \int_0^1 \sqrt{\frac{p-1}{p+1} - s^2 + \frac{2}{p+1}s^{p+1}}\,ds,
\end{aligned} \tag{2.26}$$

where $\epsilon_4(\delta) \to 0$ as $\delta \to 0$. Since δ is arbitrary, by this and (2.25), we obtain

$$\lim_{\lambda\to\infty} \lambda^{-(p+3)/(2(p-1))} \int_{R-\delta}^{R} r^{N-1} u_\lambda'(r)^2\,dr = K(R)^2 C_0 R^{N-1}. \tag{2.27}$$

By this and (2.22), we obtain (2.19). Thus the proof is complete. $\square$

The following Corollary follows directly from (1.6) and Lemma 2.4.

Corollary 2.5. *Let* $C_3 = K(R)^2 R^{N-1}|S^{N-1}|A^{(p+3)/(2(p-1))}C_0$. *Then as* $\alpha \to \infty$

$$||\nabla u_\alpha||_2^2 = C_3\alpha^{(p+3)/2}(1+o(1)). \tag{2.28}$$

We define the critical level $\eta(\alpha)$ by

$$\eta(\alpha) = ||\nabla u_\alpha||_2^2 + \frac{2}{p+1}\int_{B_R} k(|x|)u_\alpha(x)^{p+1}\,dx. \tag{2.29}$$

The following lemma is a variant of [16, Lemma 2.1].

Lemma 2.6. $d\eta(\alpha)/d\alpha = 2\lambda(\alpha)\alpha$.

Proof. We have

$$\begin{aligned}
\frac{d\eta(\alpha)}{d\alpha} &= 2\int_{B_R} \nabla u_\alpha \cdot \nabla\left(\frac{du_\alpha}{d\alpha}\right)dx + 2\int_{B_R} k(|x|)u_\alpha^p \frac{du_\alpha}{d\alpha}\,dx \\
&= 2\int_{B_R}\left(-\Delta u_\alpha + k(|x|)u_\alpha^p\right)\frac{du_\alpha}{d\alpha}\,dx \\
&= 2\lambda(\alpha)\int_{B_R} u_\alpha \frac{du_\alpha}{d\alpha}\,dx = \lambda(\alpha)\frac{d||u_\alpha||_2^2}{d\alpha} \\
&= 2\lambda(\alpha)\alpha.
\end{aligned} \tag{2.30}$$

Thus the proof is complete. □

Proof of Theorem 1.3. By (2.17) and (2.29)

$$\eta(\alpha) - \frac{2}{p+1}\lambda(\alpha)\alpha^2 = \frac{p-1}{p+1}||\nabla u_\alpha||_2^2. \tag{2.31}$$

We put

$$\lambda_1(\alpha) := \lambda(\alpha) - A\alpha^{p-1}, \tag{2.32}$$

$$\eta_1(\alpha) := \eta(\alpha) - \frac{2}{p+1}A\alpha^{p+1}. \tag{2.33}$$

Then it is easy to see from (1.6), Lemma 2.1 and Corollary 2.5 that as $\alpha \to \infty$

$$\lambda_1(\alpha) = o(\alpha^{p-1}), \qquad \eta_1(\alpha) = o(\alpha^{p+1}). \tag{2.34}$$

Furthermore, by (2.31), (2.32), (2.33), Corollary 2.5 and Lemma 2.6

$$\eta_1(\alpha) - \frac{2}{p+1}\lambda_1(\alpha)\alpha^2 = \frac{p-1}{p+1}||\nabla u_\alpha||_2^2 = \frac{p-1}{p+1}C_3\alpha^{(p+3)/2}(1+o(1)), \tag{2.35}$$

$$\frac{d\eta_1(\alpha)}{d\alpha} = 2\lambda_1(\alpha)\alpha. \tag{2.36}$$

By (2.35) and (2.36), we obtain

$$\eta_1(\alpha) - \frac{\alpha}{p+1}\eta_1'(\alpha) = \frac{p-1}{p+1}C_3\alpha^{(p+3)/2}(1+o(1)). \tag{2.37}$$

We put $g(\alpha) := \eta_1(\alpha)/\alpha^{p+1}$. Then by (2.37),

$$g'(\alpha) = -(p-1)C_3(1+o(1))\alpha^{-(p+1)/2}. \tag{2.38}$$

Since $g(\alpha) \to 0$ as $\alpha \to \infty$, we obtain

$$g(\alpha) = -\int_\alpha^\infty g'(s)\,ds = 2C_3(1+o(1))\alpha^{(1-p)/2}. \tag{2.39}$$

This implies

$$\eta_1(\alpha) = 2C_3(1+o(1))\alpha^{(p+3)/2}. \tag{2.40}$$

By this and (2.35), we obtain

$$\frac{2}{p+1}\lambda_1(\alpha)\alpha^2 = \frac{p+3}{p+1}C_3\alpha^{(p+3)/2}(1+o(1)). \tag{2.41}$$

Since $(p+3)C_3/2 = C(K)$, this along with (2.32) implies (1.11). Thus the proof is complete. □

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Tetsutaro Shibata: Applied Mathematics Research Group, Graduate School of Engineering, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8527, Japan

E-mail address: shibata@amath.hiroshima-u.ac.jp