ESTIMATION OF THE BEST CONSTANT INVOLVING THE L^2 NORM OF THE HIGHER-ORDER WENTE PROBLEM

SAMI BARAKET AND MAKKIA DAMMAK

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We study the best constant involving the L^2 norm of the p-derivative solution of Wente's problem in $\mathbb{R}^{2p}$. We prove that this best constant is achieved by the choice of some function u. We give also explicitly the expression of this constant in the special case $p = 2$.

1. Introduction and statement of the results

The Wente problem arises in the study of constant mean curvature immersions (see [6]). Let Ω be a smooth and bounded domain in $\mathbb{R}^2$. Given $u = (a,b)$ be function defined on Ω. Consider the following problem:

$$\begin{aligned} -\Delta\psi = \det\nabla u = a_{x_1}b_{x_2} - a_{x_2}b_{x_1} \quad &\text{in } \Omega, \\ \psi = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $x = (x_1, x_2)$ and a_{x_i} denote the partial derivative with respect to the variable x_i, for $i = 1,2$. If $\Omega = \mathbb{R}^2$, we consider the limit condition $\lim_{|x|\to+\infty}\psi(x) = 0$, where $|x| = r = (x_1^2 + x_2^2)^{1/2}$. When $u = (a,b) \in H^1(\Omega,\mathbb{R}^2)$, it is proven in [7] and [3] that ψ, the solution of (1.1) is in $L^\infty(\Omega)$. In particular, this provides control of $\nabla\psi$ in $L^2(\Omega)$ and continuity of ψ by simple arguments. We also have

$$\|\psi\|_\infty + \|\nabla\psi\|_2 \le C_0(\Omega)\|\nabla a\|_2\|\nabla b\|_2. \tag{1.2}$$

Denote

$$\begin{aligned} C_\infty(\Omega) &= \sup_{\nabla a,\nabla b\neq 0} \frac{\|\psi\|_\infty}{\|\nabla a\|_2\|\nabla b\|_2}, \\ C_1(\Omega) &= \sup_{\nabla a,\nabla b\neq 0} \frac{\|\nabla\psi\|_2}{\|\nabla a\|_2\|\nabla b\|_2}. \end{aligned} \tag{1.3}$$

It is proved in [1, 5, 7] that $C_\infty(\Omega) = 1/2\pi$ and in [4] that $C_1(\Omega) = \sqrt{(3/16\pi)}$.

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Here, we are interested to study a generalization of problem (1.1) in higher dimensions. More precisely, let $p \in \mathbb{N}^*$ and $u \in W^{1,2p}(\mathbb{R}^{2p}, \mathbb{R}^{2p})$. Consider the following problem:

$$\begin{gathered} (-\Delta)^p \varphi = \det \nabla u \quad \text{in } \mathbb{R}^{2p}, \\ \lim_{|x| \to +\infty} \varphi(x) = 0. \end{gathered} \tag{1.4}$$

It was proved in [2] that the solution φ of (1.4) is in $L^\infty(\mathbb{R}^{2p})$ and $\tilde{\Delta}^{k/2}\varphi$ is in $L^{2p/k}(\mathbb{R}^{2p})$ for $1 \le k \le p$, with the following estimates:

$$\|\varphi\|_\infty + \|\tilde{\Delta}^{k/2}\varphi\|_{2p/k} \le C \|\nabla u\|_{2p}^{2p}, \tag{1.5}$$

where

$$\|\tilde{\Delta}^{k/2}\varphi\|_{2p/k} = \begin{cases} \|\Delta^{k/2}\varphi\|_{2p/k} & \text{if } k \text{ is even}, \\ \|\nabla (\Delta^{(k-1)/2})\varphi\|_{2p/k} & \text{if } k \text{ is odd}. \end{cases} \tag{1.6}$$

Moreover, the best constant involving the L^∞ norm was determined. Here, we will focus our attention to the quantity $\|\tilde{\Delta}^{p/2}\varphi\|_2$. We will introduce some notations, denote by B^{2p} the unit ball in $\mathbb{R}^{2p}$, S^{2p} the unit sphere in $\mathbb{R}^{2p+1}$ and $\sigma_{2p+1} = \operatorname{vol}(S^{2p})$. Denote Ψ the function defined on $(0, +\infty)$ by

$$\Psi(s) = \frac{1}{s^p} \left(\int_{\mathbb{R}^{2p}} \left(s|\nabla\varphi|^2 + |\nabla u|^2 \right)^p \right)^{2p+1} = \frac{1}{s^p} \left(\sum_{k=0}^{p} C_p^k \| |\nabla\varphi|^k |\nabla u|^{p-k} \|_2^2 s^k \right)^{2p+1}. \tag{1.7}$$

Then, there exists a unique $\alpha = \alpha(\nabla\varphi, \nabla u) \in (0, +\infty)$ such that

$$\Psi(\alpha) = \inf_{s \in (0,+\infty)} \Psi(s) \tag{1.8}$$

satisfying

$$\sum_{k=0}^{p} \left[(2p+1)k - p \right] C_p^k \| |\nabla\varphi|^k |\nabla u|^{p-k} \|_2^2 \alpha^k = 0. \tag{1.9}$$

Finally, let

$$C_p = \sup_{\nabla u \neq 0} \frac{\|\tilde{\Delta}^{p/2}\varphi\|_2^2}{\Psi^{1/(2p)}(\alpha)}. \tag{1.10}$$

Our main result is the following theorem.

Theorem 1.1. *There exists*

$$C_p = \frac{1}{(2p+1)(2p)^{(2p+1)/2}\sigma_{2p+1}^{1/(2p)}}. \tag{1.11}$$

Moreover, the best constant C_p is achieved by a family of one parameter of functions $\bar{\varphi}$ and $\bar{u}$ given by

$$\bar{\varphi}(x) = \frac{2}{(2p)!(1+cr^2)}, \qquad \bar{u} = \frac{2\sqrt{c}x}{1+cr^2}, \tag{1.12}$$

where $c > 0$ is some arbitrary positive constant.

We can give for example more explicit expression of the best constant in the case where $p = 2$. Let $u \in W^{1,4}(\mathbb{R}^4, \mathbb{R}^4)$ and ξ is the solution of

$$\begin{gathered} \Delta^2\xi = \det \nabla u \quad \text{in } \mathbb{R}^4, \\ \lim_{|x| \to +\infty} \xi(x) = 0. \end{gathered} \tag{1.13}$$

We get that

$$\Psi(\alpha) = \frac{5^5\|\nabla u\|_4^{12}\Big(5\big\||\nabla\xi||\nabla u|\big\|_2^2 + \big(9\big\||\nabla\xi||\nabla u|\big\|_2^4 + 16\|\nabla\xi\|_4^4\|\nabla u\|_4^4\big)^{1/2}\Big)^5}{8^4\Big(3\big\||\nabla\xi||\nabla u|\big\|_2^2 + \big(9\big\||\nabla\xi||\nabla u|\big\|_2^4 + 16\|\nabla\xi\|_4^4\|\nabla u\|_4^4\big)^{1/2}\Big)^3}. \tag{1.14}$$

Corollary 1.2. *Let ξ be a solution of (1.13), then*

$$\begin{aligned} &\sup_{\nabla u \neq 0} \frac{\|\Delta\xi\|_2^2\Big(3\big\||\nabla\xi||\nabla u|\big\|_2^2 + \big(9\big\||\nabla\xi||\nabla u|\big\|_2^4 + 16\|\nabla\xi\|_4^4\|\nabla u\|_4^4\big)^{1/2}\Big)^{3/4}}{\|\nabla u\|_4^3\Big(5\big\||\nabla\xi||\nabla u|\big\|_2^2 + \big(9\big\||\nabla\xi||\nabla u|\big\|_2^4 + 16\|\nabla\xi\|_4^4\|\nabla u\|_4^4\big)^{1/2}\Big)^{5/4}} \\ &\quad = \frac{1}{2^8}\left(\frac{15}{8\pi^2}\right)^{1/4}, \end{aligned} \tag{1.15}$$

and the supremum is achieved by $\bar{\xi}$ and $\bar{u}$ given by

$$\bar{\xi}(x) = \frac{1}{12(1+cr^2)}, \qquad \bar{u}(x) = \frac{2\sqrt{c}x}{1+cr^2}, \tag{1.16}$$

where c is some arbitrary positive constant.

2. Proof of results

First, we introduce some notations which we will use later. Let Ω be a bounded subset of $\mathbb{R}^n$ and let $W:\Omega\to\mathbb{R}^{n+1}$ be a regular function. Denote $W=(w^1,w^2,\dots,w^n,w^{n+1})$ and $W_i=(w^1,\dots,w^{i-1},w^{i+1},\dots,w^n,w^{n+1})$, for $i=1,\dots,n+1$. Let V be the algebric volume of the image of W in $\mathbb{R}^{n+1}$ and denote by A the volume of the boundary of V. Then, we have

$$V=\frac{1}{n+1}\int_\Omega W\cdot W_{x_1}\times W_{x_2}\times\cdots\times W_{x_n}, \tag{2.1}$$

$$A=\int_\Omega |\,W_{x_1}\times W_{x_2}\times\cdots\times W_{x_n}\,|, \tag{2.2}$$

where $W_{x_1}\times W_{x_2}\times\cdots\times W_{x_n}$ is some vector of $\mathbb{R}^{n+1}$ given by

$$W_{x_1}\times W_{x_2}\times\cdots\times W_{x_n}=\begin{vmatrix} e_1 & w^1_{x_1} & \cdots & w^1_{x_n}\\ e_2 & w^2_{x_1} & \cdots & w^2_{x_n}\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ e_{n+1} & w^{n+1}_{x_1} & \cdots & w^{n+1}_{x_n}\end{vmatrix}=\sum_{i=1}^{n+1}(-1)^{i-1}\det(\nabla W_i)e_i. \tag{2.3}$$

Here $(e_i)_{1\le i\le n+1}$ is the canonic base of $\mathbb{R}^{n+1}$. We need the following Lemma.

Lemma 2.1. *Let $W:\Omega\to\mathbb{R}^{n+1}$ defined as above. Suppose that there exist $1\le i_0\le n$ such that $w^{i_0}=0$ on $\partial\Omega$, then*

$$\int_\Omega w^i\det(\nabla W_i)=(-1)^n\int_\Omega w^j\det(\nabla W_j), \tag{2.4}$$

for $1\le i<j\le n$.

2.1. Proof of Theorem 1.1. We will suppose that $u\in C^\infty(\mathbb{R}^{2p},\mathbb{R}^{2p})\cap W^{1,2p}(\mathbb{R}^{2p},\mathbb{R}^{2p})$. The general case can be obtained by approximating u by regular functions. Then we define W in $\mathbb{R}^{2p+1}$ as follows:

$$W(x)=\big(u(x),t\varphi(x)\big), \tag{2.5}$$

where t is a reel parameter which will be chosen later. Using (2.4) the algebric volume closed by the image of W in $\mathbb{R}^{2p+1}$ is

$$V=\int_{\mathbb{R}^{2p}}w^{2p+1}\det(\nabla W_{2p+1})dx=t\int_{\mathbb{R}^{2p}}\varphi\det\nabla u dx=t\int_{\mathbb{R}^{2p}}\varphi(-\Delta)^p\varphi dx. \tag{2.6}$$

Then we have

$$V=t||\tilde{\Delta}^{p/2}\varphi||_2^2. \tag{2.7}$$

Next, we will estimate A. We have by (2.2)

$$A \leq \int_{\mathbb{R}^{2p}} |W_{x_1}||W_{x_2}|\cdots|W_{x_{2p}}|\,dx = \int_{\mathbb{R}^{2p}} \prod_{i=1}^{2p}\left(|u_{x_i}|^2 + t^2\varphi_{x_i}^2\right)^{1/2}. \tag{2.8}$$

As $\left(\prod_{i=1}^{n}\alpha_i\right)^{1/n} \leq 1/n \sum_{i=1}^{n}\alpha_i$, we have

$$A \leq \frac{1}{(2p)^p}\int_{\mathbb{R}^{2p}}\left(\sum_{i=1}^{2p}\left(|u_{x_i}|^2 + t^2\varphi_{x_i}^2\right)\right)^p = \frac{1}{(2p)^p}\int_{\mathbb{R}^{2p}}(|\nabla u|^2 + t^2|\nabla\varphi|^2)^p. \tag{2.9}$$

Recall the isoperimetric inequality on a domains Ω of $\mathbb{R}^{2p+1}$. Denote by $V = \mathrm{Vol}(\Omega)$ and $A = \mathrm{Vol}(\partial\Omega)$, respectively, the volume of Ω and $\partial\Omega$, then

$$(2p+1)^{2p}\sigma_{2p+1}V^{2p} \leq A^{2p+1}. \tag{2.10}$$

By (2.7) and (2.9), we have

$$(2p+1)^{2p}\sigma_{2p+1}t^{2p}||\tilde{\Delta}^{p/2}\varphi||_2^{4p} \leq \frac{1}{(2p)^{p(2p+1)}}\left(\int_{\mathbb{R}^{2p}}\left(|\nabla u|^2 + t^2|\nabla\varphi|^2\right)^p\right)^{2p+1}. \tag{2.11}$$

We conclude that

$$||\tilde{\Delta}^{p/2}\varphi||_2^2 \leq \frac{1}{(2p+1)(2p)^{(2p+1)/2}\sigma_{2p+1}^{1/2p}}\Psi(t^2)^{1/2p}. \tag{2.12}$$

Then we obtain

$$C_p \leq \frac{1}{(2p+1)(2p)^{(2p+1)/2}\sigma_{2p+1}^{1/(2p)}}. \tag{2.13}$$

Next, we will show that C_p is achieved. We will consider a special case

$$u(x) = g(|x|)x, \tag{2.14}$$

where $g : \mathbb{R}_+ \to \mathbb{R}$ is a regular function which will be chosen later. Since

$$\det\nabla u = \frac{1}{2pr^{2p-1}}\frac{d}{dr}(r^{2p}g^{2p}(r)), \tag{2.15}$$

then, the solution φ of (1.4) is a radial function. Let χ a general radial function on $\mathbb{R}^{2p}$ and $W(x) = (g(|x|)x, t\chi(|x|))$. After a computation, we can show easily that in this case

$$|W_{x_1}\times W_{x_2}\times\cdots\times W_{x_{2p}}|^2 = g^{4p-2}(r)[g^2(r) + 2rg(r)g'(r) + r^2g'^2(r) + t^2\chi'^2(r)] \tag{2.16}$$

and for $1 \le i \le 2p$,

$$|W_{x_i}|^2 = g^2(r) + [2rg(r)g'(r) + r^2g'^2(r) + t^2\chi'^2(r)]\frac{x_i^2}{r^2}. \tag{2.17}$$

Next, we will suppose that χ and g satisfy

$$2rg(r)g'(r) + r^2g'^2(r) + t^2\chi'^2(r) = 0. \tag{2.18}$$

If we chose χ as the solution φ of (1.4) when $u = g(|x|)x$, then by (2.16), (2.17) and under the hypothesis (2.18), the inequality (2.9) becomes an equality. Let now

$$\bar{u}(x) = \bar{g}(|x|)x \quad \text{with } \bar{g}(r) = \frac{2\sqrt{c}}{1+cr^2}, \tag{2.19}$$

where $c > 0$ is some positive constant. Then the solution $\bar{\varphi}$ of (1.4) is given by

$$\bar{\varphi}(x) = \frac{1}{(2p)!}\frac{2}{1+cr^2}. \tag{2.20}$$

Indeed, the expression of $\Delta^k\varphi$, for $1 \le k \le p$ is

$$\Delta^k\bar{\varphi}(r) = \frac{2^{2k+1}(-1)^k k! c^k}{(2p)!(1+cr^2)^{2k+1}} \times \left(\prod_{l=0}^{k-1}(p+l) + \prod_{l=0}^{k-1}(p-2-l)c^k r^{2k} + \sum_{j=1}^{k-1} C_k^j \prod_{l=j}^{k-1}(p+l) \prod_{q=k-j}^{k-1}(p-2-q)c^j r^{2j}\right). \tag{2.21}$$

Remark that all the coefficients of r^{2j} for $2 \le j \le k$ in the expression of $\Delta^k\bar{\varphi}$ have the term $(p-k)$. Also, since

$$\det\nabla\bar{u} = \frac{1}{2pr^{2p-1}}\frac{d}{dr}(r^{2p}\bar{g}^{2p}(r)) = 2^{2p}c^p\frac{1-cr^2}{(1+cr^2)^{2p+1}}, \tag{2.22}$$

so, we have

$$(-\Delta)^p\bar{\varphi} = \det\nabla\bar{u} \quad \text{on } \mathbb{R}^{2p}. \tag{2.23}$$

If we choose $\bar{t} = (2p)!$ and $\bar{\chi}(r) = \bar{\varphi}(r) - 1/(2p)!$, we remark that $\bar{t}$, $\bar{\chi}$ and $\bar{g}$ satisfy (2.18). Since $\bar{W} = (\bar{u}, \bar{t}\bar{\chi}) : \mathbb{R}^{2p} \to S^{2p}$ and that the isoperimetric inequality (2.10) becomes equality, then we have

$$\frac{||\tilde{\Delta}^{p/2}\bar{\varphi}||_2^2}{\Psi(\bar{t}^2)^{1/(2p)}} = \frac{1}{(2p+1)(2p)^{(2p+1)/2}\sigma_{2p+1}^{1/(2p)}}. \tag{2.24}$$

We conclude that $\bar{\alpha} = \alpha(\nabla\bar{\varphi}, \nabla\bar{u})$ defined by (1.8) in this case is just $\bar{\alpha} = ((2p)!)^2$.

2.2. Proof of Corollary 1.2. Following step by step the proof of Theorem 1.1, we have

$$A=\int_{\mathbb{R}^4}|W_{x_1}\times W_{x_2}\cdots W_{x_4}|\le\frac{1}{16}\left(t^4\|\nabla\xi\|_4^4+2t^2\||\nabla\xi||\nabla u|\|_2^2+\|\nabla u\|_4^4\right). \tag{2.25}$$

Choosing

$$t^2=\alpha=\frac{2\|\nabla u\|_4^4}{3\||\nabla\xi||\nabla u|\|_2^2+\left(9\||\nabla\xi||\nabla u|\|_2^4+16\|\nabla\xi\|_4^4\|\nabla u\|_4^4\right)^{1/2}}, \tag{2.26}$$

and using the fact that

$$4\|\nabla\xi\|_4^4\alpha^2+3\||\nabla\xi||\nabla u|\|_2^2\alpha-\|\nabla u\|_4^4=0, \tag{2.27}$$

we have

$$\Psi(\alpha)=\frac{5^5\|\nabla u\|_4^{12}\left(5\||\nabla\xi||\nabla u|\|_2^2+\left(9\||\nabla\xi||\nabla u|\|_2^4+16\|\nabla\xi\|_4^4\|\nabla u\|_4^4\right)^{1/2}\right)^5}{8^4\left(3\||\nabla\xi||\nabla u|\|_2^2+\left(9\||\nabla\xi||\nabla u|\|_2^4+16\|\nabla\xi\|_4^4\|\nabla u\|_4^4\right)^{1/2}\right)^3}, \tag{2.28}$$

and then

$$\sup_{\nabla u\neq 0}\frac{\|\Delta\xi\|_2^2\left(3\||\nabla\xi||\nabla u|\|_2^2+\left(9\||\nabla\xi||\nabla u|\|_2^4+16\|\nabla\xi\|_4^4\|\nabla u\|_4^4\right)^{1/2}\right)^{3/4}}{\|\nabla u\|_4^3\left(5\||\nabla\xi||\nabla u|\|_2^2+\left(9\||\nabla\xi||\nabla u|\|_2^4+16\|\nabla\xi\|_4^4\|\nabla u\|_4^4\right)^{1/2}\right)^{5/4}}\le\frac{1}{2^8}\left(\frac{15}{8\pi^2}\right)^{1/4}. \tag{2.29}$$

By taking

$$\bar{\xi}(x)=\frac{1}{12(1+cr^2)},\qquad \bar{u}(x)=\frac{2\sqrt{c}x}{1+cr^2}, \tag{2.30}$$

we find

$$\|\nabla\bar{u}\|_4^4=\frac{2^6\times 3\times\pi^2}{7},$$
$$\|\Delta\bar{\xi}\|_2^2=\frac{\pi^2}{3^2\times 5},\qquad \|\nabla\bar{\xi}\|_4^4=\frac{\pi^2}{2^6\times 3^4\times 5\times 7},\qquad \||\nabla\bar{\xi}||\nabla\bar{u}|\|_2^2=\frac{11\pi^2}{3^3\times 5\times 7}. \tag{2.31}$$

Finally (1.15) follows.

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Sami Baraket: Départment de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisie

E-mail address: sami.baraket@fst.rnu.tn

Makkia Dammak: Départment de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisie

E-mail address: makkia.dammak@fst.rnu.tn