CAUCHY-DIRICHLET PROBLEM FOR THE NONLINEAR DEGENERATE PARABOLIC EQUATIONS

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We will investigate the nonexistence of positive solutions for the following nonlinear parabolic partial differential equation: $\partial u/\partial t = \mathcal{L}u + V(w)u^{p-1}$ in $\Omega \times (0,T)$, $1 < p < 2$, $u(w,0) = u_0(w) \geq 0$ in Ω, $u(w,t) = 0$ on $\partial\Omega \times (0,T)$ where $\mathcal{L}$ is the subelliptic p-Laplacian and $V \in L^1_{\text{loc}}(\Omega)$.

1. Introduction

This paper deals with the nonexistence of positive solutions to the following nonlinear parabolic equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \mathcal{L}u + V(w)u^{p-1} && \text{in } \Omega \times (0,T),\ 1 < p < 2, \\ &u(w,0) = u_0(w) \geq 0 && \text{in } \Omega, \\ &u(w,t) = 0 && \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{1.1}$$

where Ω is a Carnot-Carathéodory metric ball in $\mathbb{R}^{2n+1}$ and $V \in L^1_{\text{loc}}(\Omega)$. The nonlinear operator $\mathcal{L}$ is the subelliptic p-Laplacian:

$$\mathcal{L}u = \sum_{j=1}^{2n} X_j\left(|Xu|^{p-2}X_j u\right), \tag{1.2}$$

where

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial l}, \qquad X_{n+j} = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial l}, \quad j = 1,\dots,n. \tag{1.3}$$

are the smooth vector fields and satisfy Hörmander condition [25] for any $k \in \mathbb{N}$. Here $Xu = (X_1u,\dots,X_{2n}u)$ is the subelliptic gradient of a function u. Observe that

$$\mathcal{L}u = \sum_{j=1}^{2n} X_j\left(|Xu|^{p-2}X_j u\right) = 0 \tag{1.4}$$

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is the Euler-Lagrange equation of the variational integral

$$J_p(u) = \frac{1}{p}\int |Xu|^p, \tag{1.5}$$

where

$$|Xu| = \left[\sum_{j=1}^{2n} (X_j u)^2\right]^{p/2}, \quad p > 1. \tag{1.6}$$

The subelliptic p-Laplacian

$$Lu = \sum_{i=1}^{m} X_i\left(|Xu|^{p-2}X_i u\right) = 0 \tag{1.7}$$

was studied by Capogna et al. [7] for more general systems of C^∞ vector fields which satisfy Hörmander condition. They established sharp Sobolev embedding associated to the functional J_p and the Harnack inequality for positive solutions of (1.7). The fundamental solution of the subelliptic p-Laplacian (1.2) at the origin has been found by Zhang and Niu [36]. In that paper they established Hardy-type inequalities and Pohozaev-type identities associated with vector fields (1.3).

If $p = 2$ then subelliptic p-Laplacian (1.2) reduces to sub-Laplacian

$$\Delta_k = \sum_{j=1}^{2n} X_j^2 \tag{1.8}$$

which arise in a diverse area of mathematics including boundary value problems in several complex variables, harmonic analysis, quantum mechanics of anharmonic oscillators and electromagnetic fields. The hypoellipticity of Δ_k follows from the famous paper of Hörmander [25]. The fundamental solution of Δ_k has been studied extensively by Greiner [24] and Beals et al. [2, 3, 4]. In a recent paper, Kombe [32] studied nonlinear parabolic equations and found sharp Hardy-type inequality associated with Δ_k.

Note that if $k = 1$ then Δ_k becomes Kohn-Laplacian $\Delta_{\mathbb{H}^n}$ on the Heisenberg group $\mathbb{H}^n$ and today there is a large literature about partial differential equations on the Heisenberg group (see [9, 10, 12, 13, 16, 17, 21, 22, 23, 26, 27, 28] and references therein).

Problem (1.1) has been studied on the Euclidean space and Heisenberg group by Goldstein and Kombe [20, 21] and for the nonsmooth vector fields by Kombe [30, 31]. It turn outs that nonexistence of positive solutions that kind of problems largely depends on the size of bottom of the normalized p-energy forms

$$E(V) := \frac{\int_\Omega |X\phi|^p dw - \int_\Omega V|\phi|^p dw}{\int_\Omega |\phi|^p dw}, \tag{1.9}$$

where $\phi \in C_c^\infty(\Omega)$ and $V \in L^1_{\text{loc}}(\Omega)$. Clearly, in the case $k = 1$, our results recovers the results of Goldstein and Kombe [21]. The following is the main result of this paper.

THEOREM 1.1. *Let* $(4n+4k)/(2n+2k+1) \le p < 2$ *and* $V \in L^1_{\mathrm{loc}}(\Omega \setminus \mathcal{K})$*, where* $\mathcal{K}$ *is a closed Lebesgue null subset of* Ω*. If*

$$E_{\inf}((1-\epsilon)V) := \inf_{0 \not\equiv \phi \in C_c^\infty(\Omega \setminus \mathcal{K})} \frac{\int_\Omega |X\phi|^p dw - \int_\Omega (1-\epsilon)V|\phi|^p dw}{\int_\Omega |\phi|^p dw} = -\infty \tag{1.10}$$

for some $\epsilon > 0$*, then the problem (1.1) has no general positive local solution off of* $\mathcal{K}$*.*

The outline of this paper is the following. In Section 2, we introduce notations, spherical transformation, basic lemma and Hardy-type inequality. In Section 3, we prove our main Theorem and two corollary.

2. Preliminary and notations

The generic point is $w = (z,l) = (x,y,l) \in \mathbb{R}^{2n+1}$. For $w_0 \in \mathbb{R}^{2n+1}$ and $r > 0$, $\Omega = B_{d_c}(w_0, r) = \{w \in \mathbb{R}^{2n+1} \mid d_c(w,w_0) < r\}$ denotes the d_c-metric ball in $\mathbb{R}^{2n+1}$ with center w_0 and radius r. Here, d_c is the Carnot-Carathéodory distance (or control distance) generated by the vector fields (1.3) (see [11, 34]). We define the distance from the origin on $\mathbb{R}^{2n+1}$ by

$$\rho = \rho(w) = \left(|z|^{4k} + l^2\right)^{1/4k} \tag{2.1}$$

which is homogeneous of degree one with respect to the natural dilation

$$\delta_\tau(z,l) = \left(\tau z, \tau^{2k} l\right), \quad \tau > 0,\ (z,l) \in \mathbb{R}^{2n+1}. \tag{2.2}$$

The function ρ is related to the fundamental solution of subelliptic p-Laplacian and sub-Laplacian Δ_k at the origin (see, [2, 3, 4, 36]).

The sub-elliptic gradient is the $2n$ dimensional vector field given by

$$X = (X_1, \dots, X_{2n}), \tag{2.3}$$

where X_j and X_{j+n} are the smooth vector fields which is defined by (1.3). If ϕ is a smooth radial function then we have the following lemma.

LEMMA 2.1. *Let* $\phi = \phi(\rho)$ *be a smooth radial function (i.e.,* ϕ *only depends on the function* ρ *in (2.1)). Then*

$$|X\phi| = \frac{|z|^{2k-1}}{\rho^{2k-1}} \left|\phi'(\rho)\right|. \tag{2.4}$$

Proof. The proof is an easy computation (see [32]). □

Let

$$B_R(0) := \left\{(z,l) \in \mathbb{R}^{2n} \times \mathbb{R} : \rho < R\right\}. \tag{2.5}$$

be the ball with respect to ρ centered at the origin $(0,0) \in \mathbb{R}^{2n} \times \mathbb{R}$ with radius R. Let $D = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$ be an annulus with $0 \le R_1 < R_2 \le \infty$ and $\phi \in L^1(D)$. In order to compute $\int_D \phi(w)dw$, we use the following transformation which is a modification of spherical transformation in [5]

Let $w = (z,l) = (x,y,l)$ and

$$\begin{aligned}
x_1 &= \rho(\sin\varphi)^{1/2k}\cos\psi_1\cos\theta_1,\\
y_1 &= \rho(\sin\varphi)^{1/2k}\cos\psi_1\sin\theta_1,\\
&\ \ \vdots\\
x_{n-1} &= \rho(\sin\varphi)^{1/2k}\sin\psi_1\cdots\sin\psi_{n-2}\cos\psi_{n-1}\cos\theta_{n-1},\\
y_{n-1} &= \rho(\sin\varphi)^{1/2k}\sin\psi_1\cdots\sin\psi_{n-2}\cos\psi_{n-1}\sin\theta_{n-1},\\
x_n &= \rho(\sin\varphi)^{1/2k}\sin\psi_1\cdots\sin\psi_{n-2}\sin\psi_{n-1}\cos\theta_n,\\
y_n &= \rho(\sin\varphi)^{1/2k}\sin\psi_1\cdots\sin\psi_{n-2}\sin\psi_{n-1}\sin\theta_n,\\
l &= \rho^{2k}\cos\varphi,
\end{aligned}\tag{2.6}$$

for $R_1 < \rho < R_2$, $0 \le \varphi \le \pi$, $0 \le \psi_j \le \pi/2$, $j = 1,\dots,n-1$, and $0 \le \theta_j \le 2\pi$, $j = 1,\dots,n$. Then the volume element satisfies

$$dw = dx\,dy\,dl = \rho^{Q-1}d\rho(\sin\varphi)^{n-k}d\varphi\prod_{j=1}^{n-1}\Big[\cot\psi_j\big(\sin\psi_j\big)^{2(n-j)}d\psi_j\Big]\prod_{j=1}^{n}d\theta_j, \tag{2.7}$$

where

$$Q = 2k + 2n \tag{2.8}$$

is the homogeneous dimension.

Hardy-type inequalities. The prominent role of Hardy's inequality in partial differential equation has been known since the pioneering paper of Baras and Goldstein [1]. Their results stimulated several interesting results in the study of linear and nonlinear parabolic equations with singular potential (see [6, 14, 19, 20, 21, 18, 22, 23, 28, 29, 30, 31, 32]).

We should also mention that there has been great interest in the study of sharp Hardy-type inequalities on the sub-Riemanian space (see [8, 15, 16, 22, 32, 35, 36]). The following Hardy-type inequality associated with the vector fields (1.3) has been proved by Zhang and Niu [36]

Theorem 2.2. *Let $\phi \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{(0,0)\})$, $k > 1$ and $1 < p < 2k+2n$. Then*

$$c_p(k,n)\int\frac{|z|^{(2k-1)p}}{\big(|z|^{4k}+l^2\big)^{p/2}}|\phi|^p\,dz\,dl \le \int|X\phi|^p\,dz\,dl \tag{2.9}$$

where $c_p(k,n) = ((2k+2n-p)/p)^p$.

Singular potentials. In this paper, we will focus on some singular potentials (since they are critical). As a concrete example, we will treat positive singular potential

$$V(z,l) = \frac{\lambda|z|^{(2k-1)p}}{\big(|z|^{4k}+l^2\big)^{p/2}} \tag{2.10}$$

and highly singular, oscillatory potential

$$V(z,l) = \frac{\lambda|z|^{(2k-1)p}}{(|z|^{4k}+l^2)^{p/2}} + \frac{\beta|z|^{(2k-1)p}}{(|z|^{4k}+l^2)^{p/2}} \sin\left(\frac{1}{(|z|^{4k}+l^2)^{\alpha/4k}}\right), \tag{2.11}$$

where $k \in \mathbb{N}$, $\lambda > 0$, $\alpha > 0$ and $\beta \in \mathbb{R} \setminus \{0\}$. We prove that the nonexistence of positive solutions is intimately related with the Hardy's inequality.

We are concerned with the general positive local solution of problem (1.1) and we define the general positive local solution in the following sense.

Definition 2.3. By a *positive local solution continuous* off of $\mathcal{K}$, we mean

(i) $\mathcal{K}$ is a closed Lebesgue null subset of Ω,
(ii) $u : [0,T) \to L^1(\Omega)$ is continuous for some $T > 0$,
(iii) $(w,t) \to u(w,t) \in C((\Omega \setminus \mathcal{K}) \times (0,T))$,
(iv) $u(w,t) > 0$ on $(\Omega \setminus \mathcal{K}) \times (0,T)$,
(v) $\lim_{t\to 0} u(\cdot,t) = u_0$ in the sense of distributions,
(vi) $Xu \in L^p_{\mathrm{loc}}(\Omega)$ and u is a solution in the sense of distributions of the PDE.

Remark 2.4. If $0 < a < b < T$ and $\mathcal{K}_o$ is a compact subset of $\Omega \setminus \mathcal{K}$, then $u(w,t) \geq \epsilon_1 > 0$ for $(w,t) \in \mathcal{K}_o \times [a,b]$ for some $\epsilon_1 > 0$. We can weaken (iii), (iv) to be

(iii)′ $u(w,t)$ is positive and locally bounded on $(\Omega \setminus \mathcal{K}) \times (0,T)$,
(iv)′ $1/u(w,t)$ is locally bounded on $(\Omega \setminus \mathcal{K}) \times (0,T)$.

If a solution satisfies (i), (ii), (iii)′, (iv)′, (v), and (vi) then we call it a "*general positive local solution off of* $\mathcal{K}$." This is more general than a positive local solution continuous off of $\mathcal{K}$. If $\mathcal{K} = \emptyset$, we simply call u "*general positive local solution.*"

3. Proof of Theorem 1.1

We argue by contradiction. Given any $T > 0$, let $u : [0,T) \to L^1(\Omega)$ be a general positive local solution to (1.1) in $(\Omega \setminus \mathcal{K}) \times (0,T)$ with $u_0 \geq 0$ but not identically zero. Multiply both sides of (1.1) by the test function $|\phi|^p/u^{p-1}$ where $\phi \in C_c^\infty(\Omega \setminus \mathcal{K})$, and integrate over Ω, to get

$$\frac{1}{2-p}\frac{d}{dt}\int_\Omega u^{2-p}|\phi|^p dx - \int_\Omega \mathcal{L}u\left(\frac{|\phi|^p}{u^{p-1}}\right)dw = \int_\Omega V|\phi|^p dw. \tag{3.1}$$

It follows from the integration by parts that

$$\int_\Omega \mathcal{L}u\left(\frac{|\phi|^p}{u^{p-1}}\right)dw = -\int_\Omega |Xu|^{p-2}Xu \cdot X\left(\frac{|\phi|^p}{u^{p-1}}\right)dw. \tag{3.2}$$

Since

$$\begin{aligned} |Xu|^{p-2}Xu \cdot X\left(\frac{|\phi|^p}{u^{p-1}}\right) &= p|Xu|^{p-2}\frac{|\phi|^{p-1}}{u^{p-1}}Xu \cdot X|\phi| - (p-1)\frac{|\phi|^p}{u^p}|Xu|^p, \\ \int_\Omega \mathcal{L}u\left(\frac{|\phi|^p}{u^{p-1}}\right)dw &= (p-1)\int_\Omega |Xu|^p\frac{\phi^p}{u^p}dw - p\int_\Omega |Xu|^{p-2}\frac{\phi^{p-1}}{u^{p-1}}Xu \cdot X|\phi|dw, \end{aligned} \tag{3.3}$$

and then we have

$$\int_\Omega \mathcal{L}u\left(\frac{|\phi|^p}{u^{p-1}}\right)dw \geq (p-1)\int_\Omega |Xu|^p\frac{|\phi|^p}{u^p}\,dw - p\int_\Omega |Xu|^{p-1}|X\phi|\frac{\phi^{p-1}}{u^{p-1}}\,dw. \tag{3.4}$$

Here we can use the following elementary inequality: Let $p > 1$ and $s_1 \neq s_2$ be two positive real numbers. Then

$$s_1^p - s_2^p - ps_2^{p-1}(s_1 - s_2) > 0; \tag{3.5}$$

it follows that

$$(p-1)s_2^p - ps_2^{p-1}s_1 > -s_1^p. \tag{3.6}$$

We can take $s_2 = |(\phi/u)Xu|$, $s_1 = |X\phi|$; then we have

$$(p-1)\int_\Omega |Xu|^p\frac{|\phi|^p}{u^p}\,dw - p\int_\Omega |Xu|^{p-1}|X\phi|\frac{|\phi|^{p-1}}{u^{p-1}}\,dw \geq -\int_\Omega |X\phi|^p dw. \tag{3.7}$$

Therefore

$$\int_\Omega \mathcal{L}u\left(\frac{|\phi|^p}{u^{p-1}}\right)dw \geq -\int_\Omega |X\phi|^p dw. \tag{3.8}$$

Substituting (3.8) into (3.1) and integrating from t_1 to t_2, where $0 < t_1 < t_2 < T$, we obtain

$$\int_\Omega V(w)|\phi|^p dw - \int_\Omega |X\phi|^p dw \leq \frac{1}{(2-p)(t_2-t_1)}\int_\Omega \left(u^{2-p}(w,t_2) - u^{2-p}(w,t_1)\right)|\phi|^p dw. \tag{3.9}$$

Using Jensen's inequality for concave functions, we obtain

$$\int_\Omega \left(u^{(2-p)}(w,t_i)\right)^{(2k+2n)/p} dw \leq C(|\Omega|)\left(\int_\Omega u(w,t_i)\,dw\right)^{(2-p)(2k+2n)/p} < \infty. \tag{3.10}$$

Here we use the fact that Ω is bounded, whence $|\Omega|$ is finite. Therefore

$$u^{2-p}(w,t_i) \in L^{(2k+2n)/p}(\Omega). \tag{3.11}$$

We now use the following a priori inequality which is a consequence of the Sobolev-Poincaré inequality [7, 33]. For every $\epsilon > 0$ there exists $C(\epsilon)$ such that

$$\begin{aligned}\frac{1}{(2-p)(t_2-t_1)}&\int_\Omega \left(u^{2-p}(w,t_2) - u^{2-p}(w,t_1)\right)|\phi|^p dw\\ &\leq \frac{\epsilon}{1-\epsilon}\int_\Omega |X\phi|^p dw + C(\epsilon)\int_\Omega |\phi|^p dw.\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.9), we obtain

$$\int_\Omega V(w)|\phi|^p dw - \int_\Omega |X\phi|^p dw \le \frac{\epsilon}{1-\epsilon}\int_\Omega |X\phi|^p dw + C(\epsilon)\int_\Omega |\phi|^p dw. \tag{3.13}$$

Therefore

$$\inf_{0\not\equiv\phi\in C_c^\infty(\Omega\setminus\mathcal{H})} \frac{\int_\Omega |X\phi|^p dw - \int_\Omega (1-\epsilon)V(w)|\phi|^p dw}{\int_\Omega |\phi|^p dw} \ge -(1-\epsilon)C(\epsilon) > -\infty. \tag{3.14}$$

This contradicts our assumption. The proof of Theorem 1.1 is now complete.

Corollary 3.1. *Let $0\in\Omega$, $c_p(k,n) = ((2k+2n-p)/p)^p$ and $V(z,l) = \lambda|z|^{(2k-1)p}/(|z|^{4k}+l^2)^{p/2}$. Then problem (1.1) has no general positive local solution off of $\mathcal{H}$ if $\lambda > c_p(k,n)$ and $(4k+4n)/(2k+2n+1) \le p < 2$.*

Proof. Given $\epsilon > 0$, we define the radial function, $\phi \in C_c^1(\Omega)\bigcap W^{1,\infty}(\Omega)$, by

$$\phi(\rho) = \begin{cases} \epsilon^{-((2k+2n-p)/p)} & \text{if } 0 \le \rho \le \epsilon, \\ \rho^{-((2k+2n-p)/p)} & \text{if } \epsilon \le \rho \le 1, \\ 2-\rho & \text{if } 1 \le \rho \le 2, \\ 0 & \text{if } \rho \ge 2. \end{cases} \tag{3.15}$$

We are assuming that $0\in\Omega$. Without loss of generality we assume that $B_2(0) = \{(z,l)\in\mathbb{R}^{2n}\times\mathbb{R} : \rho < 2\} \subset \Omega$; if not, we simply redefine ϕ, replacing 2 by R where $B_R(0)\subset\Omega$. This only results in notational changes in the proof that follows. Then we assume (without loss of generality) that $\phi\in C_c^\infty(\Omega)$.

We want to show that

$$\inf_{0\not\equiv\phi\in C_c^\infty(\Omega\setminus\mathcal{H})} \frac{\int_\Omega |X\phi|^p dz dl - \int_\Omega \left(\lambda|z|^{(2k-1)p}/\left(|z|^{4k}+l^2\right)^{p/2}\right)|\phi|^2 dz dl}{\int_\Omega |\phi|^p dw} = -\infty. \tag{3.16}$$

Using Lemma 2.1, we get, for ϕ as in (3.15),

$$|X\phi(\rho)|^p = \begin{cases} 0 & \text{if } 0 \le \rho < \epsilon, \\ \left(\dfrac{2k+2n-p}{p}\right)^p \rho^{-2k-2n}\left(\dfrac{|z|}{\rho}\right)^{(2k-1)p} & \text{if } \epsilon < \rho < 1, \\ \left(\dfrac{|z|}{\rho}\right)^{(2k-1)p} & \text{if } 1 < \rho < 2, \\ 0 & \text{if } \rho > 2. \end{cases} \tag{3.17}$$

Using $|z| = \rho(\sin\varphi)^{1/2k}$ then we obtain

$$\begin{aligned}\int_\Omega |X\phi|^p dw &= \mu\left[\int_\epsilon^1 \left(\frac{2k+2n-p}{p}\right)^p \frac{d\rho}{\rho} + \int_1^2 \rho^{2k+2n-1} d\rho\right]\\ &= \mu\left[-\left(\frac{2k+2n-p}{p}\right)^p \log\epsilon + \frac{2^{2k+2n}-1}{2k+2n}\right],\end{aligned} \tag{3.18}$$

where

$$\mu = \int_0^\pi (\sin\varphi)^{n-k+(2k-1)p/2k} d\varphi \times \int_0^{\pi/2} \prod_{j=1}^{n-1}\left[\cot\psi_j\left(\sin\psi_j\right)^{2(n-j)} d\psi_j\right] \times \int_0^{2\pi} \prod_{j=1}^{n} d\theta_j. \tag{3.19}$$

Since

$$V(z,l) = \frac{\lambda|z|^{(2k-1)p}}{(|z|^{4k}+l^2)^{p/2}} = \frac{\lambda}{\rho^p}(\sin\varphi)^{(2k-1)p/2k}, \tag{3.20}$$

we write

$$\begin{aligned}&\int_\Omega \frac{\lambda|z|^{(2k-1)p}}{\left(|z|^{4k}+l^2\right)^{p/2}}|\phi|^p dw\\ &\quad= \lambda\mu\left[\epsilon^{p-2k-2n}\int_0^\epsilon \rho^{2k+2n-p-1} d\rho + \int_\epsilon^1 \frac{1}{\rho} d\rho + \int_1^2 (2-\rho)^p \rho^{2k+2n-p-1} d\rho\right]\\ &\quad= \lambda\mu\left(-\log\epsilon + \frac{1}{2k+2n-p} + \frac{\xi^{2k+2n-p}-1}{2k+2n-p}\right),\end{aligned} \tag{3.21}$$

where $\xi \in (1,2)$. Note that we applied the Generalized Second Mean Value Theorem for integrals to the last integral above.

Next,

$$\begin{aligned}\int_\Omega |\phi|^p(w) dw &= \gamma\left[\epsilon^{p-2k-2n}\int_0^\epsilon \rho^{2k+2n-1} d\rho + \int_\epsilon^1 \rho^{p-1} d\rho + \int_1^2 (2-\rho)^p \rho^{2k+2n-1} d\rho\right]\\ &= \gamma\left[\frac{\epsilon^p}{2k+2n} + \frac{1-\epsilon^p}{p} + \frac{\eta^{2k+2n}-1}{2k+2n}\right]\\ &= \gamma\left[\frac{1}{p} + \frac{\eta^{2k+2n}-1}{2k+2n}\right](1+o(1)) \quad \text{as } \epsilon \longrightarrow 0,\end{aligned} \tag{3.22}$$

where $\eta \in (1,2)$ and

$$\gamma = \int_0^\pi (\sin\varphi)^{n-k} d\varphi \times \int_0^{\pi/2} \prod_{j=1}^{n-1}\left[\cot\psi_j\left(\sin\psi_j\right)^{2(n-j)} d\psi_j\right] \times \int_0^{2\pi} \prod_{j=1}^{n} d\theta_j. \tag{3.23}$$

Substituting (3.18), (3.21) and (3.22) into the Rayleigh quotient,

$$\begin{aligned}\mathcal{R} &= \frac{\int_\Omega |X\phi|^p dzdl - \int_\Omega \frac{\lambda|z|^{(2k-1)p}}{(|z|^{4k}+l^2)^{p/2}} |\phi|^p dzdl}{\int_\Omega |\phi|^p dzdl} \\ &= \frac{\mu\Big[-\Big(\frac{2k+2n-p}{p}\Big)^p \log\epsilon + \frac{2^{2k+2n}-1}{2k+2n} - \lambda\Big(-\log\epsilon + \frac{1}{2k+2n-p} + \frac{\xi^{2k+2n-p}-1}{2k+2n-p}\Big)\Big]}{\gamma\Big[\frac{1}{p} + \frac{\eta^{2k+2n}-1}{2k+2n}\Big](1+o(1))} \\ &= \frac{\mu\Big[\Big(\lambda - \Big(\frac{2k+2n-p}{p}\Big)^p\Big) \log\epsilon + \frac{2^{2k+2n}-1}{2k+2n} - \lambda\Big(\frac{1}{2k+2n-p} + \frac{\xi^{2k+2n-p}-1}{2k+2n-p}\Big)\Big]}{\gamma\Big[\frac{1}{p} + \frac{\eta^{2k+2n}-1}{2k+2n}\Big](1+o(1))}.\end{aligned} \tag{3.24}$$

Therefore taking limits as $\epsilon \to 0^+$, we find that the right-hand side of (3.24) can become negative and arbitrarily large in magnitude, that is,

$$\inf_{0\not\equiv\phi\in(\Omega\setminus\mathcal{K})} \frac{\int_\Omega |X\phi|^p dzdl - \int_\Omega \Big(\lambda|z|^{(2k-1)p}/\big(|z|^{4k}+l^2\big)^{p/2}\Big)|\phi|^p dzdl}{\int_\Omega |\phi|^p dzdl} = -\infty \tag{3.25}$$

The proof of Corollary 3.1 is now complete. □

Corollary 3.2. *Let $0 \in \Omega$, $c_p(k,n) = ((2k+2n-p)/p)^p$ and $V(z,l)$ defined by (2.11). Then problem (1.1) has no general positive local solution off of $\mathcal{K}$ if $\lambda > c_p(k,n)$ and $(4k+4n)/(2k+2n+1) \leq p < 2$.*

Proof. The proof of Corollary 3.2 is similar and we omit the details. Although oscillating part of $V(z,l)$ is so singular at the origin, nonexistence of positive solutions only depends on the size of λ. The point is that oscillating part of the potential has very large positive and negative parts, in particular, it oscillates wildly, but important cancellations occur between the positive and the negative parts in the quadratic form. □

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