FLOW OF ELECTRORHEOLOGICAL FLUID UNDER CONDITIONS OF SLIP ON THE BOUNDARY

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We study a mathematical model describing flows of electrorheological fluids. A theorem of existence of a weak solution is proved. For this purpose the approximating-topological method is used.

1. Introduction

Electrorheological fluids are smart materials that are concentrated suspensions of polarizable particles in a nonconducting dielectric liquid. In moderately large electric fields the particles form chains along the field lines and these chains then aggregate into the form of columns. These chainlike and columnar structures yield dramatic changes in the rheological properties of the suspensions. The fluid becomes anisotropic, the apparent viscosity (the resistance to flow) in the direction, orthogonal to that of the electric field, abruptly increases, while the apparent viscosity in the direction of the electric field changes not so drastically.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, in which a fluid flows, $n \in \{2,3\}$. Let the boundary S of Ω be Lipschitz continuous. As it is well known, the stationary movement of any fluid is described by the equation in Cauchy form:

$$\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \sum_{j=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_j} = F_i, \tag{1.1}$$

where $x \in \Omega$, $u = (u_1,\dots,u_n)$ is the velocity field of the fluid, $\{\sigma_{ij}\}_{i,j=1}^{n}$ is the stress tensor, $F = (F_1,\dots,F_n)$ is the volume force. We also add the condition of incompressibility to (1.1):

$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0. \tag{1.2}$$

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We will consider the following constitutive equation (see [2]):

$$\sigma_{ij}(p,u) = -p\delta_{ij} + 2\varphi(I(u),|E|,\mu(u,E))\varepsilon_{ij}(u), \tag{1.3}$$

where δ_{ij} are the components of the unit tensor, $\varepsilon_{ij}(u)$ are the components of the rate of strain tensor, $\varepsilon_{ij}(u) = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, p is the spherical part of the stress tensor, φ is the viscosity function, $I(u) = \sum_{i,j=1}^{n}(\varepsilon_{ij}(u))^2$,

$$\mu(u,E)(x) = \left(\frac{\alpha\theta + u(x)}{\alpha\sqrt{n} + |u(x)|}, \frac{E(x)}{|E(x)|}\right)^2_{\mathbb{R}^n}, \tag{1.4}$$

α is a small positive constant, and $\mathbb{R}^n \ni \theta = (1,\dots,1)$, $E = (E_1,\dots,E_n)$ is the electric field strength.

We consider the condition of slip on S (see [4, 6]). Let $f = (f_1,\dots,f_n)$ be an external surface force acting on the fluid,

$$f_i = \sum_{j=1}^{n}\left[-p\delta_{ij} + 2\varphi(I(u),|E|,\mu(u,E))\varepsilon_{ij}(u)\right]\eta_j|_S, \tag{1.5}$$

where $\eta = (\eta_1,\dots,\eta_n)$ is the unit outward normal to S. We represent f in the form

$$f(s) = f^{\eta}(s) + f^{\tau}(s) \quad \forall s \in S, \tag{1.6}$$

where f^{η} and f^{τ} are the normal and the tangent vectors:

$$\begin{gathered} f^{\eta}(s) = f_{\eta}(s)\eta(s), \qquad f_{\eta}(s) = \sum_{i=1}^{n} f_i(s)\eta_i(s), \\ f^{\tau}(s) = f(s) - f^{\eta}(s) = \sum_{i=1}^{n} f_{\tau i}(s)e_i, \qquad f_{\tau i}(s) = f_i(s) - f_{\eta}(s)\eta_i(s), \end{gathered} \tag{1.7}$$

$\{e_1,\dots,e_n\}$ is an orthonormal basis in $\mathbb{R}^n$.

For the field u, a similar decomposition is valid.

The slip conditions on the boundary are the following [4]:

$$u_{\eta}(s) = 0 \quad \forall s \in S, \tag{1.8}$$

$$f^{\tau}(s) = -\chi\left(f_{\eta}(s), |u^{\tau}(s)|^2\right)u^{\tau}(s) \quad \forall s \in S \tag{1.9}$$

(by $|\cdot|$ we denote the norm in Euclidian space $\mathbb{R}^n$). Instead of (1.9) we will consider the regularized condition

$$f^\tau(s) = -\chi\Big(f_{r\eta}(s), |u^\tau(s)|^2\Big)u^\tau(s) \quad \forall s \in S, \tag{1.10}$$

$$\begin{gathered} f_{r\eta}(p,u) = \left[-Pp + \sum_{i,j=1}^{n} 2\varphi(I(Pu), |E|, \mu(Pu,E))\varepsilon_{ij}(Pu)\eta_i\eta_j \right]\Bigg|_S, \\ Pv(x) = \int_{\mathbb{R}^n} \omega(|x-\acute{x}|)v(\acute{x})d\acute{x}, \quad x \in \overline{\Omega}, \end{gathered} \tag{1.11}$$

where $\omega \in C^\infty(\mathbb{R}_+)$, $\operatorname{supp}\omega \in [0,a]$, $a \in \mathbb{R}_+$, $\omega(z) \geqslant 0$ at $z \in \mathbb{R}_+$, $\int_{\mathbb{R}^n} \omega(|x|)dx = 1$.

Here $\mathbb{R}_+$ is the set of nonnegative numbers.

Equation (1.10) means that the model of slip is not local, this is natural from the physical view point (see [1]).

We assume that

$$\int_\Omega p(x)dx = 0. \tag{1.12}$$

Let us describe the concept of a weak solution of (1.1)–(1.3), (1.8), (1.10), (1.12). We introduce some Hilbert spaces (see [4]):

$$\begin{gathered} Z = \left\{ v : v \in H^1(\Omega)^n,\ v_\eta|_S = 0, \int_\Omega [\operatorname{div} v](x)dx = 0 \right\}, \\ W = \{v : v \in Z,\ \operatorname{div} v = 0\}. \end{gathered} \tag{1.13}$$

The expression

$$(u,v)_Z = \sum_{i,j=1}^{n} \int_\omega \varepsilon_{ij}[u](x)\varepsilon_{ij}[v](x)dx + \sum_{i=1}^{n} \int_S u_{\tau i}(s)v_{\tau i}(s)ds \tag{1.14}$$

defines a scalar product on Z (and in W).

Multiplying (1.1) by a function h in $L_2(\Omega)^n$, and using Green's formula and (1.3), (1.8), (1.10), we see that

$$\begin{aligned} &\sum_{i,j=1}^{n} \int_\omega 2\varphi(I(u)(x), |E(x)|, \mu(u,E)[x])\varepsilon_{ij}[u](x)\varepsilon_{ij}[h](x)dx \\ &\quad + \sum_{i=1}^{n} \int_S \chi[f_{rn}(p(s),u(s)), |u_\tau(s)|^2]u_{\tau i}(s)h_{\tau i}(s)ds \\ &\quad + \sum_{i,j=1}^{n} \int_\omega u_j(x)\frac{\partial u_i}{\partial x_j}(x)h_i(x)dx - \int_\omega p(x)[\operatorname{div} h](x)dx = \sum_{i=1}^{n} \int_\omega F_i(x)h_i(x)dx \end{aligned} \tag{1.15}$$

(here we suppose that $F \in L_2(\Omega)^n$).

Definition 1.1. A couple of functions $(u,p)\in W\times L_2(\Omega)$ is a weak solution of problem (1.1)–(1.3), (1.8), (1.10), (1.12) if it satisfies equality (1.15) for all $h\in Z$.

The following conditions are imposed on the functions φ and χ.

(C1) There are positive constants a_1 and a_2 such that

$$a_1\leqslant\varphi(y_1,y_2,y_3)\leqslant a_2\quad\forall(y_1,y_2,y_3)\in\mathbb{R}_+^2\times[0,1].\tag{1.16}$$

(C2) The function $\varphi(y_1,\cdot,y_3):\mathbb{R}_+\to\mathbb{R}$ is measurable in y_2 for all specified $(y_1,y_3)\in\mathbb{R}_+\times[0,1]$.

(C3) The function $\varphi(\cdot,y_2,\cdot):\mathbb{R}_+\times[0,1]\to\mathbb{R}$ is jointly continuous in (y_1,y_3) for all $y_2\in\mathbb{R}_+$.

(C4) The function $y_1\mapsto\varphi(y_1^2,y_2,y_3)y_1$ is not decreasing at nonnegative values of y_1.

(C5) There are positive constants b_1 and b_2 such that

$$b_1\leqslant\chi(z_1,z_2)\leqslant b_2\quad\forall(z_1,z_2)\in\mathbb{R}\times\mathbb{R}_+.\tag{1.17}$$

(C6) The function $\chi:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ is continuous.

Note that conditions (C1)–(C6) have a physical meaning (see [2, 4]).

The main result of this paper is the following theorem.

Theorem 1.2. *Suppose that conditions (C1)–(C6) are satisfied. Then there exists a weak solution of problems (1.1)–(1.3), (1.8), (1.10), (1.12).*

For the proof of Theorem 1.2 we use the approximating-topological method [8]. For this purpose, in the beginning, we determine an equivalent operational treatment of the problem under consideration. After that for the obtained operational equation, we introduce an approximating family of equations depending on a parameter δ and by use of Skrypnik's version of the topological degree [7], on the basis of a priori estimates, we prove existence of solutions of the approximating equations. As a result, making limiting transition for $\delta\to 0$, we obtain the solvability of problem (1.1)–(1.3), (1.8), (1.10), (1.12).

2. Operational treatment

Let us introduce some notations. By X^* we denote the space, conjugate to some Banach space X, $\langle g,y\rangle$ denotes the action of the functional $g\in X^*$ on the element $y\in X$, X^m is the topological product of m copies of the space X.

Determine several mappings as follows:

$$\begin{gathered}A:Z\longrightarrow Z^*,\quad\langle A(u),h\rangle=\sum_{i,j=1}^n\int_\omega 2\varphi(I(u),|E|,\mu(u,E))\varepsilon_{ij}(u)\varepsilon_{ij}(h)dx,\\ K:Z\times L_2(\Omega)\longrightarrow Z^*,\\ \langle K(u,p),h\rangle=\sum_{i=1}^n\int_S\chi\left[f_{rn}(p(s),u(s)),|u_\tau(s)|^2\right]u_{\tau i}(s)h_{\tau i}(s)ds,\\ M:Z\longrightarrow Z^*,\quad\langle M(u),h\rangle=\sum_{i,j=1}^n\int_\omega u_j(x)\frac{\partial u_i}{\partial x_j}(x)h_i(x)dx.\end{gathered}\tag{2.1}$$

Take

$$D : Z \longrightarrow L_2(\Omega), \qquad D(u) = \operatorname{div} u. \tag{2.2}$$

Identifying $L_2(\Omega)^n$ and $(L_2(\Omega)^n)^*$ we obtain

$$D^* : L_2(\Omega)^n \equiv \left(L_2(\Omega)^n\right)^* \longrightarrow Z^*, \qquad \langle D^*(p), h\rangle = \int_\Omega p(x)[\operatorname{div} h](x)dx. \tag{2.3}$$

It is obvious that the set of weak solutions of problem (1.1)–(1.3), (1.8), (1.10), (1.12) coincides with the set of couples $(u,p) \in W \times L_2(\Omega)$ that satisfy the following operational equation:

$$A(u) + K(u,p) + M(u) - D^*(p) = F. \tag{2.4}$$

3. Properties of operators

Everywhere below the expressions $\mathbf{v}_k \xrightarrow[k\to\infty]{} \mathbf{v}_0$ and $\mathbf{v}_k \rightharpoonup_{k\to\infty} \mathbf{v}_0$ will denote strong and weak convergences, respectively, of the sequence $\{\mathbf{v}_k\}_{k=1}^\infty$ to an element $\mathbf{v}_0$. The case when the sequence $\{\mathbf{v}_k\}_{k=1}^\infty$ does not converge to $\mathbf{v}_0$ in strong sense is denoted as $\mathbf{v}_k \not\xrightarrow[k\to\infty]{} \mathbf{v}_0$.

Lemma 3.1. *The following statements hold.*

(1) *The operator A is bounded and demicontinuous (the latter means that if $u_k \xrightarrow[k\to\infty]{} u_0$ in Z, then $\langle A(u_k), h\rangle \xrightarrow[k\to\infty]{} \langle A(u_0), h\rangle$ $\forall h \in Z$).*

(2) *For the operator*

$$\langle \mathcal{A}(u,v), h\rangle = \sum_{i,j=1}^{n} \int_\omega 2\varphi\big(I(u), |E|, \mu(v,E)\big)\varepsilon_{ij}(u)\varepsilon_{ij}(h)dx, \tag{3.1}$$

the following inequality holds

$$\langle \mathcal{A}(u_1,v) - \mathcal{A}(u_2,v), u_1 - u_2\rangle \geqslant 0 \quad \forall u_1, u_2, v \in Z. \tag{3.2}$$

(3) *Specify an element $u \in Z$. Then from any bounded sequence $\{v_k\}_{k=1}^\infty$ in Z it is possible to choose a subsequence $\{v_{k_l}\}_{l=1}^\infty$ such that $\mathcal{A}(u, v_{k_l}) \xrightarrow[l\to\infty]{} \mathcal{A}(u, v_0)$ in Z^*.*

Proof. (a) Boundedness of the operator A is obvious. Let us show that it is demicontinuous. So, let $u_k \xrightarrow[k\to\infty]{} u_0$ in Z. Clearly, there is a subsequence $\{u_{k_l}\}_{l=1}^\infty$, for which

$$I(u_{k_l})[x] \xrightarrow[l\to\infty]{} I(u_0)[x], \quad u_{k_l}[x] \xrightarrow[l\to\infty]{} u_0[x] \quad \text{for a.e. } x \in \Omega. \tag{3.3}$$

Assume that $\langle A(u_k), \check{h}\rangle \not\xrightarrow[k\to\infty]{} \langle A(u_0), \check{h}\rangle$ for some $\check{h} \in Z$. Without loss of generality for a sequence $\{u_{k_l}\}_{l=1}^\infty$ the following inequality holds with some $\zeta > 0$:

$$\left| \langle A(u_{k_l}), \check{h}\rangle - \langle A(u_0), \check{h}\rangle \right| > \zeta. \tag{3.4}$$

We estimate

$$
\begin{aligned}
&|\langle A(u_{k_l}) - A(u_0), \check{h}\rangle| \\
&\quad \leqslant \left| \sum_{i,j=1}^{n} 2\int_\Omega \varphi(I(u_{k_l}), |E|, \mu(u_{k_l}, E))\varepsilon_{ij}(u_{k_l} - u_0)\varepsilon_{ij}(\check{h})dx \right| \\
&\qquad + \left| \sum_{i,j=1}^{n} 2\int_\Omega [\varphi(I(u_{k_l}), |E|, \mu(u_{k_l}, E)) - \varphi(I(u_0), |E|, \mu(u_0, E))]\varepsilon_{ij}(u_0)\varepsilon_{ij}(\check{h})dx \right| \\
&\quad \leqslant 2a_2\|u_{k_l} - u_0\|_Z\|\check{h}\|_Z \\
&\qquad + 2\left\{ \int_\Omega [\varphi(I(u_{k_l}), |E|, \mu(u_{k_l}, E)) - \varphi(I(u_0), |E|, \mu(u_0, E))]^2 I(u_0)dx \right\}^{1/2} \|\check{h}\|_Z.
\end{aligned}
\tag{3.5}
$$

The first component in the last part of inequality (3.5) tends to zero because of the convergence of $u_k \xrightarrow[k\to\infty]{} u_0$ in Z, the second component tends to zero by the Lebesgue theorem. Hence, $\langle A(u_{k_l}) - A(u_0), \check{h}\rangle \xrightarrow[l\to\infty]{} 0$, that contradicts inequality (3.4).

(b) Introduce the notation $\phi(u,v) = \varphi(I(u), |E|, \mu(v,E))$. With the help of the Cauchy inequality and condition (C4) we obtain that

$$
\begin{aligned}
&\langle \mathcal{A}(u_1, v) - \mathcal{A}(u_2, v), u_1 - u_2\rangle \\
&\quad = \sum_{i,j=1}^{n} 2\int_\Omega [\phi(u_1, v)\varepsilon_{ij}(u_1) - \phi(u_2, v)\varepsilon_{ij}(u_2)][\varepsilon_{ij}(u_1 - u_2)]dx \\
&\quad = 2\int_\Omega [\phi(u_1, v)I(u_1) + \phi(u_2, v)I(u_2)]dx \\
&\qquad - \sum_{i,j=1}^{n} 2\int_\Omega [\phi(u_1, v)\varepsilon_{ij}(u_1)\varepsilon_{ij}(u_2) - \phi(u_2, v)\varepsilon_{ij}(u_1)\varepsilon_{ij}(u_2)]dx \\
&\quad \geqslant 2\int_\Omega [\phi(v, u_1)I^{1/2}(u_1) - \phi(u_2, v)I^{1/2}(u_2)](I^{1/2}(u_1) - I^{1/2}(u_2))dx \geqslant 0.
\end{aligned}
\tag{3.6}
$$

(c) Let $\{v_k\}_{k=1}^{\infty}$ be a bounded sequence in $Z, u \in Z$. There is a subsequence $\{v_{k_l}\}_{l=1}^{\infty}$ and an element $v_0 \in Z$ such that

$$
v_{k_l}(x) \xrightarrow[l\to\infty]{} v_0(x) \quad \text{for a.e. } x \in \Omega. \tag{3.7}
$$

We have

$$
\begin{aligned}
&||\mathcal{A}(u, v_{k_l}) - \mathcal{A}(u, v_0)||_{Z^*} \\
&\quad = \sup_{\|h\|_Z = 1} \sum_{i,j=1}^{n} 2\int_\omega [\varphi(I(u), E, \mu(v_{k_l}, E))\varepsilon_{ij}(u) - \varphi(I(u), E, \mu(v_0, E))\varepsilon_{ij}(u)]\varepsilon_{ij}(h)dx \\
&\quad \leqslant 2n\left\{ \int_\omega [\varphi(I(u), E, \mu(v_{k_l}, E)) - \varphi(I(u), E, \mu(v_0, E))]^2 I(u)dx \right\}^{1/2}.
\end{aligned}
\tag{3.8}
$$

The last expression in inequality (3.8) tends to zero by conditions (C1)–(C3) and by the Lebesgue theorem. □

Lemma 3.2. *The operator K possesses the following properties.*

(1) *For any sequence $\{u^k,h^k,p^k\}_{k=1}^{\infty}$ from $Z\times Z\times L_2(\Omega)^n$, for which $(u^k,h^k,p^k)\xrightarrow[k\to\infty]{}(u^0,h^0,p^0)$, there is a subsequence $\{u^{k_l},h^{k_l},p^{k_l}\}_{l=1}^{\infty}$ such that $\langle K(u^{k_l},p^{k_l}),h^{k_l}\rangle\xrightarrow[l\to\infty]{}\langle K(u^0,p^0),h^0\rangle$ (from this property, in particular, it follows that the operator K is bounded and demicontinuous).*

(2) *The operator $K(\cdot,T(\cdot))$ is compact for any operator $T:Z\to L_2(\Omega)$.*

Proof. (a) So, let the limits from the condition take place. Using the fact that the embedding $Z\hookrightarrow L_2(S)^n$ is compact, we take a subsequence $\{u^{k_l},h^{k_l}\}_{l=1}^{\infty}$ such that

$$(u^{k_l},h^{k_l})\xrightarrow[l\to\infty]{}(u^0,h^0)\quad \text{in } L_2(S)^n\times L_2(S)^n, \tag{3.9}$$

$$(u^{k_l}[s],h^{k_l}[s])\xrightarrow[l\to\infty]{}(u^0[s],h^0[s])\quad \text{in } \mathbb{R}^{2n}\quad \text{for a.e. } s\in S. \tag{3.10}$$

Extend the functions of the sequence $\{p^k\}_{k=1}^{\infty}$ (and the function p^0) onto the entire space $\mathbb{R}^n$. For this purpose we take $\tilde{p}^k(x)=p^k(x)$ if $x\in\Omega$, otherwise $\tilde{p}^k(x)=0$. It is obvious that $\tilde{p}^{k_l}\xrightarrow[l\to\infty]{}\tilde{p}^0$ in $L_2(\mathbb{R}^n)$. Thus we have

$$\int_{\mathbb{R}^n}\omega(|\xi-\acute{x}|)\tilde{p}^{k_l}(\acute{x})d\acute{x}=P(p^{k_l})[\xi]\xrightarrow[l\to\infty]{}P(p^0)[\xi]\quad \forall\xi\in S. \tag{3.11}$$

Similarly

$$\begin{aligned}&\Big[-Pp^{k_l}+\sum_{i,j=1}^{n}2\varphi(I(Pu^{k_l}),|E|,\mu(Pu^{k_l},E))\varepsilon_{ij}(Pu^{k_l})\eta_i\eta_j\Big](s)\\&\quad=f_{r\eta}(p^{k_l},u^{k_l})[s]\xrightarrow[l\to\infty]{}f_{r\eta}(p^0,u^0)[s]\end{aligned} \tag{3.12}$$

for all $s\in S$.

We estimate

$$\begin{aligned}&|\langle K(u^{k_l},p^{k_l}),h^{k_l}\rangle-\langle K(u^0,p^0),h^0\rangle|\\&\quad\leqslant\sum_{i=1}^{n}\Big|\int_S\Big\{\chi\Big(f_{r\eta}(p^{k_l},u^{k_l}),|u_\tau^{k_l}|^2\Big)-\chi\Big(f_{r\eta}(p^0,u^0),|u_\tau^0|^2\Big)\Big\}u_{\tau i}^0h_{\tau i}^0ds\Big|\\&\qquad+\sum_{i=1}^{n}\Big|\int_S\Big\{\chi\Big(f_{r\eta}(p^{k_l},u^{k_l}),|u_\tau^{k_l}|^2\Big)\Big\}(u_{\tau i}^0-u_{\tau i}^{k_l})h_{\tau i}^0ds\Big|\\&\qquad+\sum_{i=1}^{n}\Big|\int_S\Big\{\chi\Big(f_{r\eta}(p^{k_l},u^{k_l}),|u_\tau^{k_l}|^2\Big)\Big\}u_{\tau i}^{k_l}(h_{\tau i}^0-h_{\tau i}^{k_l})ds\Big|.\end{aligned} \tag{3.13}$$

Using condition (C5) and (3.11), (1.2) we conclude that the first summand in the right-hand side of inequality (3.13) tends to zero by the Lebesgue theorem, the other summands tend to zero by (3.9).

(b) We will use Gelfand's criterion, having reformulated it according to the following lemma.

LEMMA 3.3. *The subset $\mathfrak{M}$ of a separable Banach space $\mathcal{X}$ is relatively compact, if from any sequence of functionals $\{\mathbf{f}_k\}_{k=1}^{\infty}$ belonging to $\mathcal{X}^*$ and such that*

$$\mathbf{f}_k(y) \xrightarrow[k\to\infty]{} 0 \quad \forall y \in \mathcal{X}, \tag{3.14}$$

it is possible to take a subsequence $\{\mathbf{f}_{k_l}\}_{l=1}^{\infty}$ such that (3.14) is valid for it uniformly for all y from $\mathfrak{M}$.

The proof of Lemma 3.3 is similar to that of [3, Theorem 3(1.IX), page 274], minor changes are connected with transition to a subsequence in the formulation of the statement.

Now, let $T : Z \to L_2(\Omega)$ be an operator, let $\mathfrak{M}$ be some bounded set from Z, and let $h_k \xrightharpoonup[k\to\infty]{} 0$ in the space $(Z^*)^* \equiv Z$. For all u from $\mathfrak{M}$ we have

$$\langle h_k, K(u, Tu)\rangle \leqslant b_2 \|u\|_{L_2(S)^n} \|h_k\|_{L_2(S)^n}. \tag{3.15}$$

However, the embedding $Z \hookrightarrow L_2(S)^n$ is compact, therefore for some subsequence $\{h_{k_l}\}_{l=1}^{\infty}$ we obtain that $\langle h_{k_l}, K(u, Tu)\rangle \xrightarrow[l\to\infty]{} 0$ uniformly for all u from $\mathfrak{M}$. Hence, the set $K(\mathfrak{M}, T(\mathfrak{M}))$ is relatively compact as it was required to show. □

LEMMA 3.4. *Let M_δ, $\delta > 0$, be an approximation of the operator M, that is*

$$M_\delta : Z \longrightarrow Z^*, \quad \langle M_\delta(u), h\rangle = \frac{1}{1+\delta^{1/4}\|u\|_{L_4(\Omega)^n}} \sum_{i,j=1}^{n} \int_\Omega u_j \frac{\partial u_i}{\partial x_j} h_i dx. \tag{3.16}$$

(1) *The operator M_δ is compact.*
(2) *Take any sequences $\{u^k\}_{k=1}^{\infty}$ and $\{h^k\}_{k=1}^{\infty}$ in the space Z such that $u^k \xrightharpoonup[k\to\infty]{} u^0$ in Z, $h^k \xrightharpoonup[k\to\infty]{} h^0$ in Z. Then there are subsequences $\{u^{k_l}\}_{l=1}^{\infty}$ and $\{h^{k_l}\}_{l=1}^{\infty}$ such that $\langle M_\delta(u^{k_l}), h^{k_l}\rangle \xrightarrow[l\to\infty]{} \langle M_\delta(u^0), h^0\rangle$.*

Proof. (1) The proof of boundedness and continuity of the operator M_δ is standard. The property of compactness is shown similarly to item (2) of Lemma 3.2. Here the following inequality is in use:

$$\langle M_\delta(u), h\rangle \leqslant \frac{\|u\|_{L_4(\Omega)^n}\|u\|_{H^1(\Omega)^n}}{1+\delta^{1/4}\|u\|_{L_4(\Omega)^n}} \|h\|_{L_4(\Omega)^n}. \tag{3.17}$$

(2) Let $u^k \xrightharpoonup[k\to\infty]{} u^0$ and $h^k \xrightharpoonup[k\to\infty]{} h^0$ in Z. It follows from here that there are the subsequences $\{u^{k_l}\}_{l=1}^{\infty}$ and $\{h^{k_l}\}_{l=1}^{\infty}$ such that

$$(u^{k_l}, h^{k_l}) \xrightarrow[l\to\infty]{} (u^0, h^0) \quad \text{in } (L_4(\Omega)^n)^2. \tag{3.18}$$

Thus

$$
\begin{aligned}
&\left|\langle M_\delta(u^0),h^0\rangle-\langle M_\delta(u^{k_l}),h^{k_l}\rangle\right| \\
&\quad\leqslant\left|\frac{1}{1+\delta^{1/4}\|u^0\|_{L_4(\Omega)^n}}\sum_{i,j=1}^n\int_\omega u_j^0\left(\frac{\partial u_i^0}{\partial x_j}-\frac{\partial u_i^{k_l}}{\partial x_j}\right)h_i^0\,dx\right| \\
&\qquad+\left|\frac{1}{1+\delta^{1/4}\|u^0\|_{L_4(\Omega)^n}}\sum_{i,j=1}^n\int_\omega (u_j^0-u_j^{k_l})\frac{\partial u_i^{k_l}}{\partial x_j}h_i^0\,dx\right| \\
&\qquad+\left|\frac{1}{1+\delta^{1/4}\|u^0\|_{L_4(\Omega)^n}}\sum_{i,j=1}^n\int_\omega u_j^{k_l}\frac{\partial u_i^{k_l}}{\partial x_j}(h_i^0-h_i^{k_l})\,dx\right| \\
&\qquad+\left|\left(\frac{1}{1+\delta^{1/4}\|u^0\|_{L_4(\Omega)^n}}-\frac{1}{1+\delta^{1/4}\|u^{k_l}\|_{L_4(\Omega)^n}}\right)\sum_{i,j=1}^n\int_\omega u_j^{k_l}\frac{\partial u_i^{k_l}}{\partial x_j}h_i^{k_l}\,dx\right|.
\end{aligned}
\tag{3.19}
$$

The first summand on the right-hand side of (3.19) tends to zero since $u^{k_l}\xrightarrow[l\to\infty]{}u^0$ in Z, the other summands tend to zero by (3.18). □

4. Approximation equation and an a priori estimate

For any $\delta>0$ we introduce an auxiliary equation in the unknown function u^δ:

$$
A^+(u^\delta)+A(u^\delta)+K_\delta(u^\delta)+M_\delta(u^\delta)+\delta^{-1}D^*D(u^\delta)=F,
\tag{4.1}
$$

where

$$
\begin{aligned}
\langle K_\delta(u),h\rangle&=\sum_{i=1}^n\int_S\chi\Big(f_{rn}(-\delta^{-1}Du,u),|u_\tau|^2\Big)u_{\tau i}h_{\tau i}\,ds,\\
\langle A^+(u),h\rangle&=\delta\sum_{i,j=1}^n\int_\omega\varepsilon_{ij}(u)\varepsilon_{ij}(h)\,dx.
\end{aligned}
\tag{4.2}
$$

Let $\langle K^+(u),h\rangle=(b_1/2)\sum_{i=1}^n\int_S u_{\tau i}h_{\tau i}\,ds$.

Lemma 4.1. *For the following family of the operational equations, depending on the parameter $t\in[0,1]$:*

$$
\begin{aligned}
\Lambda_t^\delta(u^\delta)&=A^+(u^\delta)+K^+(u^\delta)\\
&\quad+t\big(A(u^\delta)+K_\delta(u^\delta)-K^+(u^\delta)+M_\delta(u^\delta)+\delta^{-1}D^*D(u^\delta)-F\big)=0,
\end{aligned}
\tag{4.3}
$$

the estimate

$$
\|u^\delta\|_Z\leqslant C(\|F\|_{Z^*},a_1,b_1,n,\Omega)
\tag{4.4}
$$

holds for $0<\delta\leqslant c(a_1,b_1,n,\Omega)$. Here c and C are variables that depend only on the specified parameters.

Proof. For $t = 0$ there is only a zero solution, in this case estimation (4.4) is obvious. Let u^δ be a solution of (4.3) for some $t \in (0,1]$. Apply both sides of (4.3) to u^δ. We obtain

$$\langle A^+(u^\delta) + K^+(u^\delta), u^\delta \rangle \geqslant 0, \tag{4.5}$$

thus

$$\langle A(u^\delta), u^\delta \rangle + \langle K_\delta(u^\delta), u^\delta \rangle - \langle K^+(u^\delta), u^\delta \rangle + \langle M_\delta(u^\delta), u^\delta \rangle + \delta^{-1} \langle D^* D(u^\delta), u^\delta \rangle \leqslant \langle \mathbf{f}, u^\delta \rangle. \tag{4.6}$$

Notice that

$$\langle A(u^\delta), u^\delta \rangle + \langle K_\delta(u^\delta), u^\delta \rangle - \langle K^+(u^\delta), u^\delta \rangle \geqslant \min(2a_1, b_1/2) \|u^\delta\|_Z^2. \tag{4.7}$$

At the same time

$$\begin{aligned}
|\langle M_\delta(u^\delta), u^\delta \rangle| &= \frac{1}{1 + \delta^{1/4} \|u^\delta\|_{L_4(\Omega)^n}} \left| \sum_{i,j=1}^n \int_\omega u_j^\delta \frac{\partial u_i^\delta}{\partial x_j} u_i^\delta \, dx \right| \\
&= \frac{1}{1 + \delta^{1/4} \|u^\delta\|_{L_4(\Omega)^n}} \left| \left(-\frac{1}{2} \int_\Omega D(u^\delta) |u^\delta|^2 dx + \frac{1}{2} \sum_{i=1}^n \int_S u_{\tau i}^\delta \eta_i |u^\delta|^2 ds \right) \right| \\
&\leqslant \frac{n}{2} \|D(u^\delta)\|_{L_2(\Omega)} \frac{\|u^\delta\|_{L_4(\Omega)^n}^2}{1 + \delta^{1/4} \|u^\delta\|_{L_4(\Omega)^n}} \leqslant \frac{n}{2} \|D(u^\delta)\|_{L_2(\Omega)} \frac{\|u^\delta\|_{L_4(\Omega)^n}^2}{\delta^{1/4} \|u^\delta\|_{L_4(\Omega)^n}} \\
&\leqslant \frac{\|D(u^\delta)\|_{L_2(\Omega)}}{\delta^{1/2}} \delta^{1/4} \frac{n}{2} \vartheta \|u^\delta\|_Z \leqslant \frac{\|D(u^\delta)\|_{L_2(\Omega)}^2}{2\delta} + \delta^{1/2} \frac{n^2}{8} \vartheta^2 \|u^\delta\|_Z^2
\end{aligned} \tag{4.8}$$

(ϑ is the norm of the operator of embedding of the space Z into the space $L_4(\Omega)^n$).

It is easy to see that

$$\begin{gathered}
\delta^{-1} \langle D^* D(u^\delta), u^\delta \rangle = \delta^{-1} \langle D(u^\delta), D(u^\delta) \rangle = \delta^{-1} \|D(u^\delta)\|_{L_2(\Omega)}^2, \\
\|F\|_{Z^*} \|u^\delta\|_Z \leqslant \frac{l^2}{2} \|F\|_{Z^*}^2 + \frac{1}{2l^2} \|u^\delta\|_Z^2.
\end{gathered} \tag{4.9}$$

Now for sufficiently large l and small δ we obtain the required estimate (4.4). □

5. Existence of a solution of the approximation equation

For the proof of existence of a solution of the approximation equation we apply the method of topological degree for generalized monotonous maps (see [7]). We show that the family Λ_t^δ carries out homotopy of the maps Λ_0^δ and Λ_1^δ. For this purpose, we will notice first that from the a priori estimate (4.4) it follows that there is a sphere of nonzero radius R, with the center at zero, such that on its boundary there are no solutions of the equation $\Lambda_t^\delta(u^\delta) = 0 (t \in [0,1])$.

For our purpose it is necessary to prove first the following statements.

(a) For any sequence $\{u_k\}_{k=1}^{\infty}$ on the border of the sphere $\partial\Theta_R$, and for any sequence $\{t_k\}_{k=1}^{\infty}$ of points from the interval $[0,1]$ such that $u_k \xrightharpoonup[k\to\infty]{} u_0$ in Z, $\Lambda_{t_k}^{\delta}(u_k) \xrightharpoonup[k\to\infty]{} 0$ in Z^*, and $\langle \Lambda_{t_k}^{\delta}(u_k), u_k - u_0\rangle \xrightarrow[k\to\infty]{} 0$, the convergence $u_k \xrightarrow[k\to\infty]{} u_0$ in Z also takes place.

(b) For any sequence $\{u_k\}_{k=1}^{\infty}$ from the closure of the sphere $\overline{\Theta}_R$ such that $u_k \xrightarrow[k\to\infty]{} u_0$ in Z, for any sequence of points $\{t_k\}_{k=1}^{\infty}$ ($t_k \in [0,1]$) such that $t_k \xrightarrow[k\to\infty]{} t_0$, there is a limit $\Lambda_{t_k}^{\delta}(u_k) \xrightharpoonup[k\to\infty]{} \Lambda_{t_0}^{\delta}(u_0)$ in Z^*.

Proof. (a) Assume the contrary, that is, let $u_k \not\xrightarrow[k\to\infty]{} u_0$ in Z. Then for some subsequence $\{u_{k_l}\}_{l=1}^{\infty}$ and some fixed number $\epsilon > 0$ the following inequality:

$$||u_{k_l} - u_0||_Z > \epsilon \tag{5.1}$$

holds. We will show that this inequality is not valid.

From the hypothesis of statement (a) we have

$$\lim_{l\to\infty}\langle \Lambda_{t_{k_l}}^{\delta}(u_{k_l}), u_{k_l} - u_0\rangle = \lim_{l\to\infty}\langle \Lambda_{t_{k_l}}^{\delta}(u_{k_l}) - \Lambda_{t_0}^{\delta}(u_{k_l}) + \Lambda_{t_0}^{\delta}(u_{k_l}) - \Lambda_{t_0}^{\delta}(u_0), u_{k_l} - u_0\rangle = 0. \tag{5.2}$$

Without loss of generality we may suppose that the subsequence $\{t_{k_l}\}_{l=1}^{\infty}$ is such that $t_{k_l} \xrightarrow[l\to\infty]{} t_0 \in [0,1]$. All operators, that we have determined, are bounded. Hence

$$\begin{aligned}
&\lim_{l\to\infty}\langle \Lambda_{t_{k_l}}^{\delta}(u_{k_l}) - \Lambda_{t_0}^{\delta}(u_{k_l}), u_{k_l} - u_0\rangle \\
&\quad = \lim_{l\to\infty}\langle (t_{k_l} - t_0)[A(u_{k_l}) + K_\delta(u_{k_l}) - K^+(u_{k_l}) \\
&\qquad\qquad + M_\delta(u_{k_l}) + \delta^{-1}D^*D(u_{k_l}) - F], u_{k_l} - u_0\rangle = 0.
\end{aligned} \tag{5.3}$$

At the same time, from the properties of the operator, that we have proved, it follows that from the sequence $\{u_{k_l}\}_{l=1}^{\infty}$ it is possible to take a subsequence $\{u_{k_{l_m}}\}_{m=1}^{\infty}$ such that

$$\begin{gathered}
\lim_{m\to\infty}\langle A(u_{k_{l_m}}) - \mathcal{A}(u_{k_{l_m}}, u_0), u_{k_{l_m}} - u_0\rangle = 0, \\
\lim_{m\to\infty}\langle \mathcal{A}(u_{k_{l_m}}, u_0) - A(u_0), u_{k_{l_m}} - u_0\rangle \geqslant 0, \\
\lim_{m\to\infty}\langle K_\delta(u_{k_{l_m}}) - K_\delta(u_0) + K^+(u_{k_{l_m}}) - K^+(u_0) + M_\delta(u_{k_{l_m}}) - M_\delta(u_0), u_{k_{l_m}} - u_0\rangle = 0.
\end{gathered} \tag{5.4}$$

Clearly,

$$\lim_{m\to\infty}\langle \delta^{-1}D^*D(u_{k_{l_m}} - u_0), u_{k_{l_m}} - u_0\rangle \geqslant 0. \tag{5.5}$$

From (5.3)–(5.5) it follows that

$$\lim_{m\to\infty}\langle [A^+ + K^+](u_{k_{l_m}} - u_0), u_{k_{l_m}} - u_0\rangle \leqslant 0. \tag{5.6}$$

The limiting relation (5.6) contradicts (5.1), since the operator $A^{+}+K^{+}$ is strictly monotonous:

$$\langle [A^{+}+K^{+}](u-w),u-w\rangle \geqslant \min\left(\delta,\frac{b_1}{2}\right)\|u-w\|_Z^2 \quad \forall u,w\in Z. \tag{5.7}$$

Proof of statement (b). Obviously for any $h\in Z$ the equality

$$\lim_{k\to\infty}\langle \Lambda_{t_k}^{\delta}(u_k)-\Lambda_{t_0}^{\delta}(u_0),h\rangle = \lim_{k\to\infty}\langle \Lambda_{t_k}^{\delta}(u_k)-\Lambda_{t_0}^{\delta}(u_k),h\rangle+\lim_{k\to\infty}\langle \Lambda_{t_0}^{\delta}(u_k)-\Lambda_{t_0}^{\delta}(u_0),h\rangle \tag{5.8}$$

holds. The first summand in equality (5.8) tends to zero since $t_k \xrightarrow[k\to\infty]{} t_0$, the second summand tends to zero since all operators in (5.8) are demicontinuous. □

Thus, the maps Λ_0^{δ} and Λ_1^{δ} are homotopic, at the same time the degree $deg(\Lambda_0^{\delta},\Theta_R,0)$ is an odd number, as Λ_0^{δ} is an odd map. Therefore, the solutions of the equation $\Lambda_1^{\delta}(u^{\delta})=0$ exist for sufficiently small δ, as it was required to show.

6. Limiting transition

Take a sequence $\delta_k \xrightarrow[k\to\infty]{} 0$ and assign to every δ_k the solution $u_k\in Z$ of the equation $\Lambda_1^{\delta_k}(u_k)=0$. We have

$$\delta_k^{-1}D^{*}D(u_k)=F-A^{+}(u_k)-A(u_k)-K_{\delta_k}(u_k)-M_{\delta_k}(u_k). \tag{6.1}$$

From the results of [5] it follows that the operator D^{*} carries the out isomorphism of the spaces

$$\begin{gathered}\mathcal{P}_0=\left\{\rho\in L_2(\Omega):\int_{\Omega}\rho(x)dx=0\right\},\\ W_0=\{\mathbf{f}\in Z^{*}:\langle \mathbf{f},u\rangle=0\ \forall u\in W\}.\end{gathered} \tag{6.2}$$

From the a priori estimate and (6.1) it follows that there are elements $u_0\in Z, p_0\in L_2(\Omega)$ such that

$$\begin{gathered}u_k \xrightarrow[k\to\infty]{} u_0 \quad \text{in } Z,\\ u_k \xrightarrow[k\to\infty]{} u_0 \quad \text{in norm of } L_2(\Omega)^n,\ \text{a.e. in } \Omega,\\ u_k \xrightarrow[k\to\infty]{} u_0 \quad \text{in norm of } L_2(S)^n,\ \text{a.e. on } S,\\ D(u_k) \xrightarrow[k\to\infty]{} 0 \quad \text{in } L_2(\Omega),\\ \delta_k^{-1}D(u_k) \xrightarrow[k\to\infty]{} -p_0 \quad \text{in } \mathcal{P}_0.\end{gathered} \tag{6.3}$$

Let

$$\begin{aligned}\Upsilon(u_k,v) = \langle A^+(u_k) + A(u_k) - \mathcal{A}(v,u_k) + K_{\delta_k}(u_k) - K(u_0,p_0) \\ + M_{\delta_k}(u_k) - M(u_0) + \delta_k^{-1}D^*D(u_k) + D^*(p_0), u_k - v\rangle.\end{aligned} \tag{6.4}$$

Taking into account Lemmas 3.1–3.4 and (6.3), we obtain

$$\begin{gathered}\lim_{k\to\infty}\langle A(u_k^v) - \mathcal{A}(v,u_k^v), u_k^v - v\rangle \geqslant 0, \\ \lim_{k\to\infty}\langle K_{\delta_k}(u_k^v) - K(u_0,p_0) + M_{\delta_k}(u_k^v) - M(u_0), u_k^v - v\rangle = 0, \\ \lim_{k\to\infty}\langle \delta_k^{-1}D^*D(u_k^v) + D^*(p_0), u_k^v - v\rangle = 0, \\ \lim_{k\to\infty}\langle A^+(u_k^v), u_k^v - v\rangle = 0.\end{gathered} \tag{6.5}$$

It follows from relations (6.5) that

$$\lim_{k\to\infty}\Upsilon(u_k^v,v) \geqslant 0 \quad \forall v \in Z. \tag{6.6}$$

On the other hand,

$$\begin{aligned}\lim_{k\to\infty}\Upsilon(u_k^v,v) &= \lim_{k\to\infty}\langle A^+(u_k^v), u_k^v - v\rangle \\ &\quad + \lim_{k\to\infty}\langle A(u_k^v) + K_{\delta_k}(u_k^v) + M_{\delta_k}(u_k^v big) + \delta_k^{-1}D^*D(u_k^v), u_k^v - v\rangle \\ &\quad + \lim_{k\to\infty}\langle -\mathcal{A}(v,u_k^v) - K(u_0,p_0) - M(u_0) + D^*(p_0), u_k^v - v\rangle \\ &= \langle F - \mathcal{A}(v,u_0) - K(u_0,p_0) - M(u_0) + D^*(p_0), u_0 - v\rangle \quad \forall v \in Z.\end{aligned} \tag{6.7}$$

Now, take $v = u_0 - \gamma h$, where $\gamma > 0$, $h \in Z$, and pass to the limit as $\gamma \to 0$. From relations (6.6) and (6.7) we get the inequality

$$\langle F - A(u_0) - K(u_0,p_0) - M(u_0) + D^*(p_0), h\rangle \geqslant 0 \quad \forall h \in Z. \tag{6.8}$$

Since h is arbitrary, after replacing h with $-h$, we obtain that $(u = u_0,\ p = p_0)$ is the solution of (2.4).

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