STRONGLY NONLINEAR ELLIPTIC VARIATIONAL UNILATERAL PROBLEMS IN ORLICZ SPACE

L. AHAROUCH, A. BENKIRANE, AND M. RHOUDAF

Received 2 March 2005; Accepted 12 April 2005

We will be concerned with the existence result of unilateral problem associated to the equations of the form $Au + g(x,u,\nabla u) = f$, where A is a Leray-Lions operator from its domain $D(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1}E_{\overline{M}}(\Omega)$. On the nonlinear lower order term $g(x,u,\nabla u)$, we assume that it is a Carathéodory function having natural growth with respect to $|\nabla u|$, and satisfies the sign condition. The right-hand side f belongs to $W^{-1}E_{\overline{M}}(\Omega)$.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with segment property. Let us consider the following nonlinear Dirichlet problem:

$$-\operatorname{div}\big(a(x,u,\nabla u)\big) + g(x,u,\nabla u) = f \tag{1.1}$$

$Au = -\operatorname{div} a(x,u,\nabla u)$ is a Leray-Lions operator defined on its domain $\mathfrak{D}(A) \subset W_0^1 L_M(\Omega)$, with M an N-function and where g is a nonlinearity with the "natural" growth condition

$$\big|g(x,s,\xi)\big| \leq b\big(|s|\big)\big(h(x) + M\big(|\xi|\big)\big) \tag{1.2}$$

and which satisfies the classical sign condition

$$g(x,s,\xi)\cdot s \geq 0. \tag{1.3}$$

The right-hand side f belongs to $W^{-1}E_{\overline{M}}(\Omega)$.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2006, Article ID 46867, Pages 1–20
DOI 10.1155/AAA/2006/46867

An existence theorem has been proved in [15] where the nonlinearity g depend only on x and u, and in [3] where g depends also on the ∇u but the author's suppose the Δ_2-condition, while in [8] the author's were concerned of the above problem without assuming a Δ_2-condition on M.

It is our purpose, in this paper, to prove an existence result for unilateral problems associated to (1.1) without assuming the Δ_2-condition.

In our paper, the mean difficulty is the second and the third steps where we study the a priori estimate. To overcome this difficulty, we have changed the classical coercivity condition by the following one:

$$a(x,s,\zeta)(\zeta-\nabla v_0) \geq \alpha M(|\zeta|)-\delta(x) \quad \text{see } (\mathrm{A}_4) \text{ below}, \tag{1.4}$$

(this idea is inspired from the work [16]).

Note that in the case of the equation, the a priori estimate is easily proved in [8] thanks to the some classical technique (see [16]).

Furthermore, in our work, we have not supposed any regularity assumption on the obstacle. Note that this type of equations can be applied in sciences physics. Non-standard examples of $M(t)$ which occur in the mechanics of solids and fluids are $M(t) = t\log(1+t)$, $M(t) = \int_0^t s^{1-\alpha}(\operatorname{arcsinh} s)^{\alpha} ds$ $(0 \leq \alpha \leq 1)$ and $M(t) = t\log(1+\log(1+t))$ (see [10, 11, 13, 12] for more details).

This paper is organized as follows. Section 2 contains some preliminaries and some technical lemmas. Section 3 is concerned with basic assumptions and the main result which is proved in Section 4.

2. Preliminaries

2.1. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, that is, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$.

Equivalently, M admits the representation: $M(t) = \int_0^t a(s)ds$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M} = \int_0^t \bar{a}(s)ds$, where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \leq t\}$.

The N-function M is said to satisfy the Δ_2-condition if, for some k

$$M(2t) \leq kM(t) \quad \forall t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq$ some $t_0 > 0$, then M is said to satisfy the Δ_2-condition near infinity.

We will extend these N-functions even functions on all $\mathbb{R}$.

Moreover, we have the following Young's inequality:

$$\forall s,t \geq 0, \quad st \leq M(t) + \overline{M}(s). \tag{2.2}$$

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\epsilon > 0$, $P(t)/Q(\epsilon t) \to 0$ as $t \to \infty$.

This is the case if and only if $\lim_{t\to\infty}(Q^{-1}(t)/P^{-1}(t)) = 0$.

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_{\Omega} M(u(x))\,dx < +\infty \quad \left(\text{resp., } \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right). \tag{2.3}$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0, \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx \leq 1\right\} \tag{2.4}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_{\Omega} uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

2.3. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives of order 1 lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} ||D^{\alpha}u||_M. \tag{2.5}$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp., $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp., $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm (for more details see [1]).

We recall some lemmas introduced in [3] which will be used later.

Lemma 2.1. *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\dfrac{\partial}{\partial x_i}u & \text{a.e. in } \{x\in\Omega : u(x)\notin D\}, \\ 0 & \text{a.e. in } \{x\in\Omega : u(x)\in D\}. \end{cases} \tag{2.6}$$

Lemma 2.2. *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. We assume that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F: W^1L_M(\Omega)\to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

We give now the following lemma which concerns operators of Nemytskii type in Orlicz spaces (see [3]).

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure.*

Let M, P and Q be N-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:

$$|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2 |s|), \tag{2.7}$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$.

Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x,u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), 1/k_2) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < 1/k_2\}$ into $E_Q(\Omega)$.

Below, we will use the following technical lemma.

Lemma 2.4. *Let (f_n), $f \in L^1(\Omega)$ such that*

(i) *$f_n \geq 0$ a.e. in Ω,*

(ii) *$f_n \to f$ a.e. in Ω,*

(iii) *$\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$,*

then $f_n \to f$ strongly in $L^1(\Omega)$.

3. Main results

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property.

Given an obstacle function $\psi : \Omega \to \mathbb{R}$, we consider

$$K_\psi = \{u \in W_0^1 L_M(\Omega);\ u \geq \psi \text{ a.e. in } \Omega\}, \tag{3.1}$$

this convex set is sequentially $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closed in $W_0{}^1 L_M(\Omega)$. (See [16].) We now state conditions on the differential operator

$$Au = -\operatorname{div}(a(x,u,\nabla u)). \tag{3.2}$$

(A_1) $a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.

(A_2) There exist two N-functions M and P with $P \ll M$, function $c(x)$ in $E_{\overline{M}}(\Omega)$, constant k_1, k_2, k_3, k_4 such that, for a.e. x in Ω for all $s, \zeta \in \mathbb{R}$

$$|a(x,s,\zeta)| \leq c(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\zeta|). \tag{3.3}$$

(A_3) $[a(x,s,\zeta) - a(x,s,\zeta')](\zeta - \zeta') > 0$ for a.e. x in Ω, s in $\mathbb{R}$ and ζ' in $\mathbb{R}^N$, with $\zeta \neq \zeta'$.

(A_4) There exists $\delta(x)$ in $L^1(\Omega)$ and a strictly positive constant α such that, for some fixed element v_0 in $K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$.

$$a(x,s,\zeta)(\zeta - Dv_0) \geq \alpha M(|\zeta|) - \delta(x) \tag{3.4}$$

for a.e. x in Ω, $s \in \mathbb{R}$ and all $\zeta \in \mathbb{R}^N$.

(A_5) For each $v \in K_\psi \cap L^\infty(\Omega)$ there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that $v_j \to v$ for the modular convergence.

Furthermore let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ and all $\zeta \in \mathbb{R}^N$.

(G_1) $g(x,s,\zeta)s \geq 0$,

(G_2) $|g(x,s,\zeta)| \leq b(|s|)(h(x)+M(|\zeta|))$,

where $b : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous non decreasing function, h is given nonnegative function in $L^1(\Omega)$.

We define, for s and k in $\mathbb{R}$, $k \geq 0 T_k(s) = \max(-k, \min(k,s))$.

Consider the following Dirichlet problem:

$$A(u) + g(x,u,\nabla u) = f \quad \text{in } \Omega. \tag{3.5}$$

Remark 3.1. Remark that the condition (A_5) is holds if the one of the following conditions is verified.

(i) There exist $\overline{\psi} \in K_\psi$ such that $\psi - \overline{\psi}$ is continuous in Ω (see [16, Proposition 9]).

(ii) $\psi \in W_0^1 E_M(\Omega)$ (see [16, Proposition 10]).

We will prove the following existence theorem.

Theorem 3.2. *Assume that (A_1)–(A_5), (G_1) and (G_2) hold and $f \in W^{-1}E_{\overline{M}}(\Omega)$. Then there exists at least one solution of the following unilateral problem:*

$$\begin{gathered} u \in K_\psi(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega),\ g(x,u,\nabla u)u \in L^1(\Omega), \\ \int_\Omega a(x,u,\nabla u)\nabla(u-v)\,dx + \int_\Omega g(x,u,\nabla u)(u-v)\,dx \\ \leq \langle f, u-v\rangle, \quad \forall v \in K_\psi \cap L^\infty(\Omega). \end{gathered} \tag{P}$$

Remark 3.3. We remark that the statement of the previous theorem does not exists in the case of Sobolev space. But, some existence result in this sense have been proved under the regularity assumption $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see [7]).

Remark 3.4. We recall that, differently from the methods used in [7, 9], we do not introduce the function ψ^+ in the test function used in the step of a priori estimate.

4. Proof of Theorem 3.2

To prove the existence theorem, we proceed by steps.

Step 1. Approximate unilateral problems.

Let us define

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+(1/n)\,|g(x,s,\xi)|} \tag{4.1}$$

and let us consider the approximate unilateral problems:

$$\begin{gathered} u_n \in K_\psi \cap D(A), \\ \langle Au_n, u_n - v\rangle + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\,dx \leq \langle f, u_n - v\rangle, \\ \forall v \in K_\psi. \end{gathered} \tag{P_n}$$

From Gossez and Mustonen [16, Proposition 5], the problem (P_n) has at least one solution.

Step 2. A priori estimates.

Let $k \geq \|v_0\|_\infty$ and let $\varphi_k(s) = se^{\gamma s^2}$, where $\gamma = (2b(k)/\alpha)^2$.

It is well known that

$$\varphi_k'(s) - \frac{b(k)}{\alpha}\left|\varphi_k(s)\right| \geq \frac{1}{2}, \quad \forall s \in \mathbb{R} \text{ (see [7])}. \tag{4.2}$$

Since $f \in W^{-1}E_{\overline{M}}(\Omega)$ then f can be written as follows:

$$f = f_0 - \operatorname{div} F \quad \text{with } f_0 \in E_{\overline{M}}(\Omega),\ F \in \left(E_{\overline{M}}(\Omega)\right)^N. \tag{4.3}$$

Taking $u_n - \eta\varphi_k(T_l(u_n - v_0))$ as test function in (P_n), where $l = k + \|v_0\|_\infty$, we obtain

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_l(u_n - v_0)\varphi_k'(T_l(u_n - v_0))\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n - v_0))\,dx \\ \leq \int_\Omega f_0\varphi_k(T_l(u_n - v_0))\,dx + \int_\Omega F\nabla T_l(u_n - v_0)\varphi_k'(T_l(u_n - v_0))\,dx.\end{aligned} \tag{4.4}$$

Since $g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n - v_0)) \geq 0$ on the subset $\{x \in \Omega : |u_n(x)| > k\}$, then

$$\begin{aligned}&\int_{\{|u_n - v_0| \leq l\}} a(x,u_n,\nabla u_n)\nabla(u_n - v_0)\varphi_k'(T_l(u_n - v_0))\,dx \\ &\quad\leq \int_{\{|u_n| \leq k\}} \left|g_n(x,u_n,\nabla u_n)\right|\left|\varphi_k(T_l(u_n - v_0))\right|dx + \int_\Omega f_0\varphi_k(T_l(u_n - v_0))\,dx \\ &\qquad+ \int_{\{|u_n - v_0| \leq l\}} F\nabla u_n\varphi_k'(T_l(u_n - v_0))\,dx - \int_{\{|u_n - v_0| \leq l\}} F\nabla v_0\varphi_k'(T_l(u_n - v_0))\,dx,\end{aligned} \tag{4.5}$$

by using (A_4), (G_1) and Young's inequality, we have

$$\begin{aligned}&\alpha\int_{\{|u_n - v_0| \leq l\}} M(|\nabla u_n|)\varphi_k'(T_l(u_n - v_0))\,dx \\ &\quad\leq b(|k|)\int_\Omega \left(h(x) + M(|\nabla T_k(u_n)|)\right)\left|\varphi_k(T_l(u_n - v_0))\right|dx \\ &\qquad+ \int_\Omega \delta(x)\varphi_k'(T_l(u_n - v_0))\,dx + \int_\Omega f_0\varphi_k(T_l(u_n - v_0))\,dx \\ &\qquad+ \frac{\alpha}{2}\int_{\{|u_n - v_0| \leq l\}} M(|\nabla u_n|)\varphi_k'(T_l(u_n - v_0))\,dx + C_1(k),\end{aligned} \tag{4.6}$$

which implies that

$$\frac{\alpha}{2}\int_{\{|u_n-v_0|\le l\}} M(|\nabla u_n|)\varphi_k'(T_l(u_n-v_0))\,dx \\ \le b(|k|)\int_\Omega (h(x)+M(|\nabla T_k(u_n)|))|\varphi_k(T_l(u_n-v_0))|\,dx \\ +\int_\Omega \delta(x)\varphi_k'(T_l(u_n-v_0))\,dx+\int_\Omega f_0\varphi_k(T_l(u_n-v_0))\,dx+C_1(k). \tag{4.7}$$

Since $\{x\in\Omega,\ |u_n(x)|\le k\}\subseteq\{x\in\Omega: |u_n-v_0|\le l\}$ and the fact that h, δ and $f_0\in L^1(\Omega)$, then

$$\int_\Omega M(|\nabla T_k(u_n)|)\varphi_k'(T_l(u_n-v_0))\,dx \\ \le \frac{b(k)}{\alpha}\int_\Omega M(|\nabla T_k(u_n)|)|\varphi_k(T_l(u_n-v_0))|\,dx+C_2(k), \tag{4.8}$$

which implies that

$$\int_\Omega M(|\nabla T_k(u_n)|)\Big[\varphi_k'(T_l(u_n-v_0))-\frac{b(k)}{\beta}|\varphi_k(T_l(u_n-v_0))|\Big]dx\le C_2(k). \tag{4.9}$$

By using (4.2), we deduce

$$\int_\Omega M(|\nabla T_k(u_n)|)\,dx\le C_3(k). \tag{4.10}$$

On the other side, taking $v=v_0$ as test function in (P_n), we get

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)(u_n-v_0)\,dx \\ \le\int_\Omega f_0(u_n-v_0)\,dx+\int_\Omega F\nabla(u_n-v_0)\,dx. \tag{4.11}$$

Let $k>\|v_0\|_\infty$, since $g_n(x,u_n,\nabla u_n)(u_n-v_0)\ge 0$ in the subset $\{x\in\Omega;\ |u_n(x)|\ge k\}$, we deduce

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx+\int_{\{|u_n(x)|\le k\}} g_n(x,u_n,\nabla u_n)(u_n-v_0)\,dx \\ \le\int_\Omega f_0(u_n-v_0)\,dx+\int_\Omega F\nabla u_n\,dx-\int_\Omega F\nabla v_0\,dx, \tag{4.12}$$

thus, implies that, by using (4.10) and (G_2)

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx\le\int_\Omega f_0u_n\,dx+\int_\Omega F\nabla u_n\,dx+C_4(k). \tag{4.13}$$

By using [14, Lemma 5.7] and Young's inequality, we deduce

$$\begin{aligned} \int_\Omega f_0 u_n dx &\le C + \frac{\alpha}{4}\int_\Omega M(|\nabla u_n|)dx, \\ \int_\Omega F\nabla u_n dx &\le C' + \frac{\alpha}{4}\int_\Omega M(|\nabla u_n|)dx. \end{aligned} \tag{4.14}$$

Combining (4.13), (4.14), we get

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n - \nabla v_0)dx \le \frac{\alpha}{4}\int_\Omega M(|\nabla u_n|)dx + \frac{\alpha}{4}\int_\Omega M(|\nabla u_n|)dx + C_5(k), \tag{4.15}$$

which implies, by using (A_4)

$$\alpha\int_\Omega M(|\nabla u_n|)dx \le \frac{\alpha}{2}\int_\Omega M(|\nabla u_n|)dx + C_6(k) \tag{4.16}$$

hence

$$\int_\Omega M(|\nabla u_n|)dx \le C_7(k). \tag{4.17}$$

Hence u_n is bounded in $W_0^1L_M(\Omega)$. So there exists some $u \in W_0^1L_M(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \\ u_n &\longrightarrow u \quad \text{strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{4.18}$$

Step 3. Boundedness of $(a(x,u_n,\nabla u_n))_n$ in $(L_{\overline{M}}(\Omega))^N$.

Let $w \in (E_M(\Omega))^N$ be arbitrary, by (A_3) we have

$$(a(x,u_n,\nabla u_n) - a(x,u_n,w))(\nabla u_n - w) \ge 0, \tag{4.19}$$

this implies that

$$a(x,u_n,\nabla u_n)(w - \nabla v_0) \le a(x,u_n,\nabla u_n)(\nabla u_n - \nabla v_0) - a(x,u_n,w)(\nabla u_n - w) \tag{4.20}$$

hence,

$$\begin{aligned} \int_\Omega a(x,u_n,\nabla u_n)(w - \nabla v_0)dx \le \int_\Omega & a(x,u_n,\nabla u_n)(\nabla u_n - \nabla v_0)dx \\ &+ \int_\Omega a(x,u_n,w)(w - \nabla u_n)dx. \end{aligned} \tag{4.21}$$

We claim that

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n - v_0)dx \le C, \tag{4.22}$$

with C is positive constant.

Indeed, if we take $v = v_0$ as test function in (P_n), we get

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n - \nabla v_0)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v_0)\,dx \le \langle f, u_n - v_0\rangle. \tag{4.23}$$

Since $g_n(x,u_n,\nabla u_n)(u_n - v_0) \ge 0$ on the subset $\{x \in \Omega,\ |u_n| \ge \|v_0\|_\infty\}$, which implies

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n - \nabla v_0)\,dx \\ &\quad \le b\big(\|v_0\|_\infty\big)\int_\Omega h(x)\,dx + b\big(\|v_0\|_\infty\big)\int_\Omega M(|\nabla u_n|)\,dx + \langle f, u_n - v_0\rangle.\end{aligned} \tag{4.24}$$

Combining (4.17) and (4.24), we deduce (4.22).

On the other hand, there exists an N-function Q such that $M \ll Q$ and the space $W_0^1L_M(\Omega)$ is continuously embedded in $L_Q(\Omega)$. Since the sequence $\{\nabla u_n\}$ is bounded in $L_M(\Omega)$, we can choose an $\varepsilon > 0$ such that $\int_\Omega M(\varepsilon\nabla u_n)\,dx \le C_1$ and $\int_\Omega Q(\varepsilon u_n)\,dx \le C_2$. We have by (A_2) $|a(x,u_n,w)| \le c(x) + k_1\overline{M}^{-1}Q(\varepsilon u_n) + k_3\overline{M}^{-1}M(k_4 w) + C_\varepsilon$. When λ large enough we obtain

$$\begin{aligned}\int_\Omega \overline{M}\left(\frac{|a(x,u_n,w)|}{\lambda}\right)dx &\le \frac{1}{\lambda}\int_\Omega \overline{M}(c(x))\,dx + \frac{k_1}{\lambda}\int_\Omega Q(\varepsilon u_n)\,dx \\ &\quad + \frac{k_3}{\lambda}\int_\Omega M(k_4 w)\,dx + \frac{\overline{M}(C_\varepsilon)}{\lambda} \le C_3,\end{aligned} \tag{4.25}$$

thus implies that $\int_\Omega a(x,u_n,w)(w - \nabla u_n)\,dx$ is bounded, therefore by using (4.21) and (4.22), we get

$$\int_\Omega a(x,u_n,\nabla u_n)(w - \nabla v_0)\,dx \le C_4. \tag{4.26}$$

Since w is arbitrary, we deduce $\int_\Omega a(x,u_n,\nabla u_n)w\,dx \le C_5$.

Finally by theorem of Banach-Steinhaus, the sequence $a(x,u_n,\nabla u_n)$ remains bounded in $L_{\overline{M}}(\Omega)$.

Step 4. Almost everywhere convergence of the gradient.

We fix $k > \|v_0\|_\infty$. Let $\Omega_r = \{x \in \Omega,\ |\nabla T_k(u(x))| \le r\}$ and denote by χ_r the characteristic function of Ω_r. Clearly, $\Omega_r \subset \Omega_{r+1}$ and $\operatorname{meas}(\Omega\backslash\Omega_r) \to 0$ as $r \to \infty$.

Fix r and let $s \ge r$, we have

$$\begin{aligned}0 &\le \int_{\Omega_r}[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\ &\le \int_{\Omega_s}[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\ &= \int_{\Omega_s}[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx \\ &\le \int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx.\end{aligned} \tag{4.27}$$

By the condition (A_5) there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular converge in $W_0^1 L_M(\Omega)$.

Here, we define $w_{n,j} = T_k(u_n) - T_k(v_j)$, $w_j = T_k(u) - T_k(v_j)$.

For $\eta = \exp(-4\gamma k^2)$, we defined the following function as

$$v_{n,j} = u_n - \eta \varphi_k(w_{n,j}). \tag{4.28}$$

By taking $v_{n,j}$ as test functions in (P_n), we get

$$\langle A(u_n), \eta\varphi_k(w_{n,j})\rangle + \int_\Omega g_n(x,u_n,\nabla u_n)\eta\varphi_k(w_{n,j})\,dx \le \langle f, \eta\varphi_k(w_{n,j})\rangle. \tag{4.29}$$

Since η is nonnegative, then

$$\langle A(u_n), \varphi_k(w_{n,j})\rangle + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,j})\,dx \le \langle f, \varphi_k(w_{n,j})\rangle. \tag{4.30}$$

It follows that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,j}\varphi_k'(w_{n,j})\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,j})\,dx \le \langle f, \varphi_k(w_{n,j})\rangle. \tag{4.31}$$

Denoting by $\epsilon(n,j)$ any quantity such that

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j) = 0. \tag{4.32}$$

We get, by (4.31),

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j))\varphi_k'(w_{n,j})\,dx&\\ + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,j})\,dx \le \langle f, \varphi_k(w_{n,j})\rangle.&\end{aligned} \tag{4.33}$$

In view of (4.18), we have $\varphi_k(w_{n,j}) \to \varphi_k(w_j)$ weakly in $W_0^1 L_M(\Omega)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ as $n \to +\infty$, and then

$$\langle f, \varphi_k(w_{n,j})\rangle \longrightarrow \langle f, \varphi_k(w_j)\rangle \quad \text{as } n \longrightarrow +\infty. \tag{4.34}$$

Again, tends j to infinity, we get

$$\langle f, \varphi_k(w_j)\rangle \longrightarrow 0 \quad \text{as } j \longrightarrow +\infty. \tag{4.35}$$

Therefore,

$$\langle f, \varphi_k(w_{n,j})\rangle = \epsilon(n,j). \tag{4.36}$$

On the set $\{x \in \Omega,\ |u_n(x)| > k\}$, we have $g(x,u_n,\nabla u_n)\varphi_k(w_{n,j}) \geq 0$, so by (4.31)

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j))\varphi_k'(w_{n,j})\,dx \\ + \int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,j})\,dx \leq \epsilon(n,j).\end{aligned} \tag{4.37}$$

Splitting the first integral on the left-hand side of (4.37) where $|u_n| \leq k$ and $|u_n| > k$, we can write

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,j}\varphi_k'(w_{n,j})\,dx \\ &\quad\geq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(v_j)]\varphi_k'(w_{n,j})\,dx \\ &\quad\quad - \varphi_k'(2k)\int_{\{|u_n|>k\}} |a(x,u_n,\nabla u_n) - a(x,T_k(u_n),0)|\,|\nabla T_k(v_j)|\,dx.\end{aligned} \tag{4.38}$$

Since $|a(x,u_n,\nabla u_n) - a(x,T_k(u_n),0)|$ bounded in $L_{\overline{M}}(\Omega)$ there exists a function $h_k \in L_{\overline{M}}(\Omega)$ such that $|a(x,u_n,\nabla u_n) - a(x,T_k(u_n),0)| \rightharpoonup h_k$ for $\sigma(Ł_{\overline{M}},E_M)$ as $n \to +\infty$, while $|\nabla T_k(v_j)|\chi_{\{|u_n|>k\}} \to |\nabla T_k(v_j)|\chi_{\{|u|>k\}}$ strongly in $E_M(\Omega)$, and by the modular convergence of $T_k(v_j)$, we deduce that the second term of the right-hand side of (4.38) tends to 0 as $n \to \infty$ and $j \to \infty$, hence

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,j}\varphi_k'(w_{n,j})\,dx \\ &\quad\geq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(v_j)]\varphi_k'(w_{n,j})\,dx + \varepsilon(n,j),\end{aligned} \tag{4.39}$$

which implies that

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,j}\varphi_k'(w_{n,j})\,dx \\ &\quad\geq \int_\Omega \Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big) - a\Big(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\Big)\Big] \\ &\quad\quad \times \Big[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big]\varphi_k'(w_{n,j})\,dx \\ &\quad\quad + \int_\Omega a\Big(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\Big)\Big[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big]\varphi_k'(w_{n,j})\,dx \\ &\quad\quad - \int_{\Omega\setminus\Omega_s^j} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\varphi_k'(w_{n,j})\,dx + \epsilon(n,j),\end{aligned} \tag{4.40}$$

where χ_s^j denotes the characteristic function of the subset $\Omega_s^j = \{x \in \Omega : |\nabla T_k(v_j)| \leq s\}$.

Let a function $l_k \in (L_{\overline{M}}(\Omega))^N$ such that $a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup l_k$ for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$, since $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi_k'(w_{n,j})$ tends to $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi_k'(w_j)$ strongly in $(E_M(\Omega))^N$, the third term of the right-hand side of (4.40) tends to quantity $\int_{\Omega\setminus\Omega_s^j} l_k\nabla T_k(v_j)\varphi_k'(w_j)dx$ as n tend to infinity.

Letting now j to infinity, by using the modular convergence of v_j and

$$l_k \chi_{\Omega\setminus\Omega_s^j} \chi_{\{|v_j|\le k\}} \varphi_k'(w_j), \tag{4.41}$$

we have

$$\int_\Omega l_k \nabla T_k(v_j) \chi_{\Omega\setminus\Omega_s^j} \varphi_k'(w_j)\,dx \longrightarrow \int_{\Omega\setminus\Omega_s} l_k \nabla T_k(u) \varphi_k'(0)\,dx \tag{4.42}$$

as j tend to infinity.

Finally

$$-\int_{\Omega\setminus\Omega_s^j} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \varphi_k'(w_{n,j})\,dx = -\int_{\Omega\setminus\Omega_s} l_k \nabla T_k(u) \varphi_k'(0)\,dx + \epsilon(n,j). \tag{4.43}$$

Concerning the second term of the right-hand side of (4.40), since

$$a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j) \varphi_k'(w_{n,j}) \longrightarrow a(x, T_k(u), \nabla T_k(v_j)\chi_s^j) \varphi_k'(w_j) \tag{4.44}$$

as $n \to \infty$ in $(E_{\overline{M}}(\Omega))^N$ by Lemma 2.3 and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(\Omega))^N$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.

Consequently, the second term of the right-hand side of (4.40) tends to quantity $\int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j]\varphi_k'(w_j)\,dx$ as $n \to \infty$, moreover letting j to infinity it is easy to see that

$$\begin{aligned}
&\int_\Omega a\Big(x, T_k(u), \nabla T_k(v_j)\chi_s^j\Big)\Big[\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j\Big]\varphi_k'(w_j)\,dx \\
&\quad \longrightarrow \int_\Omega a(x, T_k(u), \nabla T_k(u)\chi_s)[\nabla T_k(u) - \nabla T_k(u)\chi_s]\varphi_k'(0)\,dx \\
&\quad = \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi_k'(0)\,dx.
\end{aligned} \tag{4.45}$$

Combining (4.40), (4.43) and (4.45), we get

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n) \nabla w_{n,j} \varphi_k'(w_{n,j})\,dx \\
&\quad \ge \int_\Omega \Big[a(x, T_k(u_n), \nabla T_k(u_n)) - a\Big(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j\Big)\Big] \\
&\qquad \times \Big[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big]\varphi_k'(w_{n,j})\,dx \\
&\qquad + \int_{\Omega\setminus\Omega_s} l_k \nabla T_k(u) \varphi_k'(0)\,dx \\
&\qquad + \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi_k'(0)\,dx + \epsilon(n,j).
\end{aligned} \tag{4.46}$$

We now return to the second term of the left-hand side of (4.37), we have, by using (A_4) and (G_2)

$$
\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,j})\,dx\right| \\
&\quad\le b(k)\int_\Omega \left(h(x)+M(|\nabla T_k(u_n)|)\right)|\varphi_k(w_{n,j})|\,dx \\
&\quad\le b(k)\int_\Omega h(x)|\varphi_k(w_{n,j})|\,dx+\frac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_{n,j})| \\
&\qquad+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_{n,j})|\,dx \\
&\qquad-\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla v_0|\varphi_k(w_{n,j})|\,dx \\
&\quad\le \epsilon(n,j)+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_{n,j})|\,dx.
\end{aligned}
\tag{4.47}
$$

The last term of the last side of this inequality write as

$$
\begin{aligned}
&\frac{b(k)}{\alpha}\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\right] \\
&\qquad\times\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]|\varphi_k(w_{n,j})|\,dx \\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]|\varphi_k(w_{n,j})|\,dx \\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_s^j|\varphi_k(w_{n,j})|\,dx
\end{aligned}
\tag{4.48}
$$

and reasoning as above, it is easy to see that

$$
\begin{aligned}
&\frac{b(k)}{\alpha}\int_\Omega a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]|\varphi_k(w_{n,j})|\,dx=\epsilon(n,j), \\
&\qquad-\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_s^j|\varphi_k(w_{n,j})|\,dx=\epsilon(n,j).
\end{aligned}
\tag{4.49}
$$

So that by (4.47) and (4.48) we deduce that

$$
\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,j})\,dx\right| \\
&\quad\le\frac{b(k)}{\alpha}\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n))-a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\right] \\
&\qquad\qquad\times\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]|\varphi_k(w_{n,j})|\,dx+\epsilon(n,j).
\end{aligned}
\tag{4.50}
$$

Combining (4.37), (4.46) and (4.50), we obtain

$$\begin{aligned}&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a\big(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\big)\Big]\\&\qquad\times\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\Big]\varphi_k'(w_{n_i}) - \frac{b(k)}{\alpha}\,|\,\varphi_k(w_{n,j})\,|\,dx\\&\qquad\le \int_{\Omega\setminus\Omega_s} l_k\nabla T_k(u)\varphi_k'(0)dx + \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx + \epsilon(n,j),\end{aligned}\tag{4.51}$$

which implies that, by using (4.2)

$$\begin{aligned}&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a\big(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\big)\Big]\times\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\Big]dx\\&\qquad\le 2\int_{\Omega\setminus\Omega_s} l_k\nabla T_k(u)\varphi_k'(0)dx + 2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx + \epsilon(n,j).\end{aligned}\tag{4.52}$$

Now, remark that

$$\begin{aligned}&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]dx\\&\quad= \int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a\big(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\big)\Big]\\&\qquad\times\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\Big]dx\\&\qquad+\int_\Omega a\big(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\big)\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\Big]dx\\&\qquad-\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\chi_s\big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]dx\\&\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\Big[\nabla T_k(v_j)\chi_s^j-\nabla T_k(u)\chi_s\Big]dx.\end{aligned}\tag{4.53}$$

We will pass to the limit in n and j in the last three terms of the right-hand side of the last inequality, we get

$$\begin{aligned}&\int_\Omega a\big(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\big)\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\Big]dx\\&\qquad= \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx + \epsilon(n,j),\\&\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\chi_s\big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]dx\\&\qquad= \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx + \epsilon(n,j),\\&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\Big[\nabla T_k(v_j)\chi_s^j-\nabla T_k(u)\chi_s\Big]dx = \epsilon(n,j),\end{aligned}\tag{4.54}$$

which implies that

$$\begin{aligned}
&\int_{\Omega}\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a\big(x,T_k(u_n),\nabla T_k(u)\chi_s\big)\Big][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]dx\\
&\quad=\int_{\Omega}\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a\big(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\big)\Big]\\
&\qquad\times\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\Big]dx+\epsilon(n,j).
\end{aligned}\tag{4.55}$$

Combining (4.27), (4.52) and (4.55), we have

$$\begin{aligned}
&\int_{\Omega_r}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dx\\
&\quad\leq\int_{\Omega}\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a\big(x,T_k(u_n),\nabla T_k(u)\chi_s\big)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]dx\\
&\quad\leq 2\int_{\Omega\setminus\Omega_s}l_k\nabla T_k(u)\varphi_k'(0)dx+2\int_{\Omega\setminus\Omega_s}a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx+\epsilon(n,j).
\end{aligned}\tag{4.56}$$

By passing to the lim sup over n, and letting j, s tend to infinity, we obtain

$$\lim_{n\to+\infty}\int_{\Omega_r}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dx=0,\tag{4.57}$$

thus implies by the same method used in [4] that

$$\nabla u\longrightarrow\nabla u_n\quad\text{a.e. in }\Omega.\tag{4.58}$$

Step 5. Modular convergence of the truncation:

Thanks to (4.58), we have $l_k=a(x,T_k(u),\nabla T_k(u))$, which implies by using (4.56)

$$\begin{aligned}
&\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\
&\quad\leq\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u)\chi_s-\nabla v_0)+\delta(x)]dx\\
&\qquad+\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)dx\\
&\qquad+2\int_{\Omega\setminus\Omega_s}a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\varphi_k'(0)dx\\
&\qquad+2\int_{\Omega\setminus\Omega_s}a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx+\epsilon(n,j),
\end{aligned}\tag{4.59}$$

which implies, by using the Fatou's lemma

$$\begin{aligned}
&\int_\Omega \Big[a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)\Big]dx\\
&\quad\le \liminf_{n\to+\infty}\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)\Big]dx\\
&\quad\le \limsup_{n\to+\infty}\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)\Big]dx\\
&\quad\le \limsup_{n\to+\infty}\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u)\chi_s-\nabla v_0)+\delta(x)\Big]dx\\
&\qquad+\limsup_{n\to+\infty}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\chi_s\big)\big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big)dx\\
&\qquad+2\int_{\Omega\setminus\Omega_s} l_k\nabla T_k(u)\varphi_k'(0)dx+2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx+\epsilon(n,j).
\end{aligned}\tag{4.60}$$

Reasoning as above, we have

$$\begin{aligned}
&\limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u)\chi_s-\nabla v_0)+\delta(x)]dx\\
&\qquad=\int_\Omega \Big[a(x,T_k(u),\nabla T_k(u))\big(\nabla T_k(u)\chi_s-\nabla v_0\big)+\delta(x)\Big]dx,\\
&\limsup_{n\to+\infty}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\chi_s\big)\big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big)dx\\
&\qquad=\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx,
\end{aligned}\tag{4.61}$$

which implies that

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)]dx\\
&\quad\le \liminf_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\
&\quad\le \limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\
&\quad\le \int_\Omega \Big[a(x,T_k(u),\nabla T_k(u))\big(\nabla T_k(u)\chi_s-\nabla v_0\big)+\delta(x)\Big]dx\\
&\qquad+2\int_{\Omega\setminus\Omega_s} l_k\nabla T_k(u)\varphi_k'(0)dx\\
&\qquad+2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx+\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx.
\end{aligned}\tag{4.62}$$

Using the fact that $[a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)\chi_s - \nabla v_0)+\delta(x)]$, $l_k\nabla T_k(u)\varphi_k'(0)$, $a(x, T_k(u),0)\nabla T_k(u)\varphi_k'(0)$ and $a(x,T_k(u),0)\nabla T_k(u)$ in $L^1(\Omega)$ and letting $s\to+\infty$, we get, since $\operatorname{meas}(\Omega\backslash\Omega_s)\to 0$,

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)]dx\\
&\quad\le \liminf_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\
&\quad\le \limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\
&\quad\le \int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)]dx.
\end{aligned}\tag{4.63}$$

Finally, we have

$$\begin{aligned}
&\lim_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\
&\quad=\int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)]dx
\end{aligned}\tag{4.64}$$

and by using (A_3), one obtains, by Lemma 2.4

$$M(\nabla T_k(u_n))\longrightarrow M(\nabla T_k(u))\quad \text{in } L^1(\Omega).\tag{4.65}$$

Step 6. Equi-integrability of the nonlinearities.

We need to prove that

$$g_n(x,u_n,\nabla u_n)\longrightarrow g(x,u,\nabla u)\quad \text{strongly in } L^1(\Omega),\tag{4.66}$$

in particular it is enough to prove the equi-integrable of $g_n(x,u_n,\nabla u_n)$. To this purpose, we take $u_n - T_1(u_n - v_0 - T_h(u_n - v_0))$ as test function in (P_n), we obtain

$$\begin{aligned}
\int_{\{|u_n-v_0|>h+1\}}|g_n(x,u_n,\nabla u_n)|dx &\le \langle f, T_1(u_n-v_0-T_h(u_n-v_0))\rangle+\int_{\{|u_n-v_0|>h\}}\delta(x)dx\\
&\le \int_{\{|u_n-v_0|>h\}}(|f_0|+\delta)(x)dx + C\|F\chi_{\{|u_n-v_0|>h\}}\|_{\overline{M}}.
\end{aligned}\tag{4.67}$$

Since $|f_0|+\delta\in L^1(\Omega)$, $F\in E_{\overline{M}}(\Omega)$, using [14, Lemma 4.16], for all $\varepsilon>0$, then there exists $h(\varepsilon)\ge 1$ such that

$$\int_{\{|u_n-v_0|>h(\varepsilon)\}}|g(x,u_n,\nabla u_n)|dx<\varepsilon/2.\tag{4.68}$$

For any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}
\int_E |g_n(x,u_n,\nabla u_n)|dx &\le \int_E b(h(\varepsilon))(c(x)+M(\nabla T_{h(\varepsilon)}(u_n)))dx\\
&\quad+\int_{\{|u_n|>h(\varepsilon)\}}|g(x,u_n,\nabla u_n)|dx.
\end{aligned}\tag{4.69}$$

In view of (4.65) there exists $\eta(\varepsilon) > 0$ such that

$$\int_E b(h(\varepsilon))\left(c(x) + M\left(\nabla T_{h(\varepsilon)}(u_n)\right)\right)dx < \varepsilon/2 \quad \forall E \text{ such that } |E| < \eta(\varepsilon). \tag{4.70}$$

Finally, combining (4.75) and (4.76), one easily has

$$\int_E \left| g_n(x, u_n, \nabla u_n) \right| dx < \varepsilon \quad \forall E \text{ such that } |E| < \eta(\varepsilon), \tag{4.71}$$

which implies (4.66).

Moreover, if we take v_0 as function test in (P_n), we get

$$\int_\Omega g_n(x, u_n, \nabla u_n) u_n dx \le \int_\Omega \delta(x) + dx + \int_\Omega g_n(x, u_n, \nabla u_n) v_0 dx + \langle f, u_n - v_0 \rangle, \tag{4.72}$$

hence

$$\int_\Omega g_n(x, u_n, \nabla u_n) u_n dx \le \beta, \tag{4.73}$$

where β is some positive constant, then by using Fatou's lemma, we have

$$g(x, u, \nabla u) u \in L^1(\Omega). \tag{4.74}$$

Step 7. Passing to the limit.

We take $v \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$, in (P_n), we can write

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla (u_n - v) dx + \int_\Omega g(x, u_n, \nabla u_n)(u_n - v) dx \le \langle f, u_n - v \rangle, \tag{4.75}$$

which implies that

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n) \nabla (u_n - v_0) dx &+ \int_\Omega a(x, u_n, \nabla u_n) \nabla (v_0 - v) dx \\ &+ \int_\Omega g(x, u_n, \nabla u_n)(u_n - v) dx \le \langle f, u_n - v \rangle. \end{aligned} \tag{4.76}$$

By Fatou's lemma and the fact that

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \tag{4.77}$$

weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ one easily sees that

$$\begin{aligned} \int_\Omega a(x, u, \nabla u) \nabla (u - v_0) dx &+ \int_\Omega a(x, u, \nabla u) \nabla (v_0 - v) dx \\ &+ \int_\Omega g(x, u, \nabla u)(u - v) dx \le \langle f, u - v \rangle. \end{aligned} \tag{4.78}$$

Hence

$$\int_\Omega a(x, u, \nabla u) \nabla (u - v) dx + \int_\Omega g(x, u, \nabla u)(u - v) dx \le \langle f, u - v \rangle. \tag{4.79}$$

Now, let $v \in K_\psi \cap L^\infty(\Omega)$, by the condition ($A_5$) there exists $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that v_j converges to v modular, let $h > \|v_0\|_\infty$, taking $v = T_h(v_j)$ in (4.79), we have

$$\int_\Omega a(x,u,\nabla u)\nabla(u - T_h(v_j))\,dx + \int_\Omega g(x,u,\nabla u)(u - T_h(v_j))\,dx \le \langle f, u - T_h(v_j)\rangle. \tag{4.80}$$

We can easily pass to the limit as $j \to +\infty$ to get

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\nabla(u - T_h(v))\,dx + \int_\Omega g(x,u,\nabla u)(u - T_h(v))\,dx \\ \le \langle f, u - T_h(v)\rangle \quad \forall v \in K_\psi \cap L^\infty(\Omega),\end{aligned} \tag{4.81}$$

the same, we pass to the limit as $h \to +\infty$, we deduce

$$\int_\Omega a(x,u,\nabla u)\nabla(u - v)\,dx + \int_\Omega g(x,u,\nabla u)(u - v)\,dx \le \langle f, u - v\rangle \quad \forall v \in K_\psi \cap L^\infty(\Omega). \tag{4.82}$$

This completes the proof of the theorem.

References

[1] R. Adams, *Sobolev Espaces*, Academic Press, New York, 1975.

[2] A. Benkirane, *Approximations de type Hedberg dans les espaces $W^mL\log L(\Omega)$ et applications* [*Hedberg-type approximations in the spaces $W^mL\log L(\Omega)$ and applications*], Toulouse. Faculté des Sciences. Annales. Mathématiques. Série 5 **11** (1990), no. 2, 67–78 (French).

[3] A. Benkirane and A. Elmahi, *An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces*, Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods **36** (1999), no. 1, 11–24.

[4] ———, *A strongly nonlinear elliptic equation having natural growth terms and L^1 data*, Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods **39** (2000), no. 4, 403–411.

[5] A. Benkirane, A. Elmahi, and D. Meskine, *An existence theorem for a class of elliptic problems in L^1*, Applicationes Mathematicae **29** (2002), no. 4, 439–457.

[6] A. Benkirane and J.-P. Gossez, *An approximation theorem in higher order Orlicz-Sobolev spaces and applications*, Polska Akademia Nauk. Instytut Matematyczny. Studia Mathematica **92** (1989), no. 3, 231–255.

[7] A. Bensoussan, L. Boccardo, and F. Murat, *On a nonlinear partial differential equation having natural growth terms and unbounded solution*, Annales de l'Institut Henri Poincaré. Analyse Non Linéaire **5** (1988), no. 4, 347–364.

[8] A. Elmahi and D. Meskine, *Existence of solutions for elliptic equations having natural growth terms in Orlicz spaces*, Abstract and Applied Analysis **2004** (2004), no. 12, 1031–1045.

[9] ———, *Unilateral elliptic problems in L^1 with natural growth terms*, Journal of Nonlinear and Convex Analysis. An International Journal **5** (2004), no. 1, 97–112.

[10] M. Fuchs and L. Gongbao, *L^∞-bounds for elliptic equations on Orlicz-Sobolev spaces*, Archiv der Mathematik. Archives of Mathematics. Archives Mathématiques **72** (1999), no. 4, 293–297.

[11] M. Fuchs and G. Seregin, *A regularity theory for variational integrals with $L\ln L$-growth*, Calculus of Variations and Partial Differential Equations **6** (1998), no. 2, 171–187.

[12] ______, *Regurality for solutions of variational problems in the deformation theory of plasticity with logarithmic hardening*, Proceedings of St. Petersburg Mathematical Society, Vol. 5 (Bonn, 1995), 1998, pp. 184–222, English translation in American Mathematical Society Translations, Series 2. preprint no. 421, SFB256.

[13] ______, *Variational methods for fluids of Prandtl-Eyring type and plastic materials with logarithmic hardening*, Mathematical Methods in the Applied Sciences **22** (1999), no. 4, 317–351.

[14] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Transactions of the American Mathematical Society **190** (1974), 163–205.

[15] ______, *A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces*, Nonlinear Functional Analysis and Its Applications (Berkeley, Calif, 1983), Proc. Sympos. Pure Math., vol. 45, American Mathematical Society, Rhode Island, 1986, pp. 455–462.

[16] J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces*, Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods **11** (1987), no. 3, 379–392.

[17] M. A. Krasnosel'skii and Y. B. Rutikii, *Convex Functions and Orlicz Spaces*, P. Noordhoff, Groningen, 1969.

[18] A. Porretta, *Existence for elliptic equations in L^1 having lower order terms with natural growth*, Portugaliae Mathematica **57** (2000), no. 2, 179–190.

L. Aharouch: Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz, Université Sidi Mohamed Ben Abdallah B.P. 1796, Atlas Fès, Morocco
E-mail address: laharouch@fsdmfes.ac.ma

A. Benkirane: Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz, Université Sidi Mohamed Ben Abdallah B.P. 1796, Atlas Fès, Morocco
E-mail address: abdelmoujib@iam.net.ma

M. Rhoudaf: Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz, Université Sidi Mohamed Ben Abdallah B.P. 1796, Atlas Fès, Morocco
E-mail address: mrhoudaf@fsdmfes.ac.ma