# AN ORIENTED COINCIDENCE INDEX FOR NONLINEAR FREDHOLM INCLUSIONS WITH NONCONVEX-VALUED PERTURBATIONS

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We suggest the construction of an oriented coincidence index for nonlinear Fredholm operators of zero index and approximable multivalued maps of compact and condensing type. We describe the main properties of this characteristic, including applications to coincidence points. An example arising in the study of a mixed system, consisting of a first-order implicit differential equation and a differential inclusion, is given.



## 1. Introduction

The necessity of studying coincidence points of Fredholm operators and nonlinear (compact and condensing) maps of various classes arises in the investigation of many problems in the theory of partial differential equations and optimal control theory (see, e.g., [3, 4, 7, 13, 17, 18, 20–22]). The use of topological characteristics of coincidence degree type is a very effective tool for solving such type of problems. For inclusions with linear Fredholm operators, a number of such topological invariants was studied in the works [7, 8, 13, 18, 19]. In the present paper, we suggest the general construction of an oriented coincidence index for nonlinear Fredholm operators of zero index and approximable multivalued maps of compact and condensing type. A nonoriented analogue of such index was described earlier in the authors work [17].

The paper is organized in the following way. In Section 2, we give some preliminaries. In Section 3, we present the construction of the oriented coincidence index, first for a finite-dimensional case, and later, on that base, we develop the construction in the case of a compact triplet. In Section 4, using the technique of fundamental sets, we give the most general construction of the oriented index for a condensing triplet and describe its main properties, including its application to the existence of coincidence points. In Section 5, we consider an example of a condensing triplet arising in the study of a mixed system, consisting of a first-order implicit differential equation and a differential inclusion.

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## 2. Preliminaries

By the symbols $E$, $E'$, we will denote real Banach spaces. Everywhere, by $Y$ we will denote an open bounded set $U \subset E$ (case (i)) or $U_* \subset E \times [0,1]$ (case (ii)). We recall some notions (see, e.g., [3]).

*Definition 2.1.* A $C^1$-map $f : Y \to E'$ is Fredholm of index $k \geq 0$ ($f \in \Phi_k C^1(Y)$) if for every $y \in Y$ the Frechet derivative $f'(y)$ is a linear Fredholm map of index $k$, that is, $\dim \operatorname{Ker} f'(y) < \infty$, $\dim Co\ker f'(y) < \infty$, and

$$\dim \operatorname{Ker} f'(y) - \dim Co\ker f'(y) = k. \tag{2.1}$$

*Definition 2.2.* A map $f : \overline{Y} \to E'$ is proper if $f^{-1}(\mathcal{K})$ is compact for every compact set $\mathcal{K} \subset E'$.

We recall now the notion of oriented Fredholm structure on $Y$.

An atlas $\{(Y_i, \Psi_i)\}$ on $Y$ is said to be *Fredholm* if, for each pair of intersecting charts $(Y_i, \Psi_i)$ and $(Y_j, \Psi_j)$ and every $y \in Y_i \cap Y_j$, it is

$$(\Psi_j \circ \Psi_i^{-1})'(\Psi_i(y)) \in CG(\widetilde{E}), \tag{2.2}$$

where $\widetilde{E}$ is the corresponding model space, and $CG(\widetilde{E})$ denotes the collection of all linear invertible operators in $\widetilde{E}$ of the form $i + k$, where $i$ is the identity map and $k$ is a compact linear operator.

The set $CG(\widetilde{E})$ is divided into two connected components. The component containing the identity map will be denoted by $CG^+(\widetilde{E})$.

Two Fredholm atlases are said to be equivalent if their union is still a Fredholm atlas. The class of equivalent atlases is called a *Fredholm structure.*

A Fredholm structure on $U$ is associated to a $\Phi_0 C^1$-map $f : U \to E'$ if it admits an atlas $\{(Y_i, \Psi_i)\}$ with model space $E'$ for which

$$(f \circ \Psi_i^{-1})'(\Psi_i(y)) \in LC(E') \tag{2.3}$$

at each point $y \in U$, where $LC(E')$ denotes the collection of all linear operators in $E'$ of the form: identity plus a compact map. Let us note that each $\Phi_0 C^1$-map $f : U \to E'$ generates a Fredholm structure on $U$ associated to $f$.

A Fredholm atlas $\{(Y_i, \Psi_i)\}$ on $Y$ is said to be oriented if for each intersecting charts $(Y_i, \Psi_i)$ and $(Y_j, \Psi_j)$ and every $y \in Y_i \cap Y_j$, it is true that

$$(\Psi_j \circ \Psi_i^{-1})'(\Psi_i(y)) \in CG^+(E). \tag{2.4}$$

Two oriented Fredholm atlases are called orientally equivalent if their union is an oriented Fredholm atlas on $Y$. The equivalence class with respect to this relation is said to be the *oriented Fredholm structure on $Y$*.

We will need also the following result (see [3]).

Proposition 2.3. *Let $f \in \Phi_k C^1(Y)$; $K \subset Y$ a compact set. Then there exist an open neighborhood $\mathbb{O}$, $K \subset \mathbb{O} \subset Y$, and a finite-dimensional subspace $E'_n \subset E'$ such that*

$$f^{-1}(E'_n) \cap \mathbb{O} = M^{n+k}, \tag{2.5}$$

*where $M^{n+k}$ is an $n+k$ dimensional manifold. Moreover, the restriction $f_{|\mathbb{O}}$ is transversal to $E'_n$, that is, $f'(x)E + E'_n = E'$ for each $x \in \mathbb{O}$.*

We describe now some notions of the theory of multivalued maps that will be used in the sequel (details can be found, e.g., in [1, 2, 9, 12]).

Let $(X, d_X), (Z, d_Z)$ be metric spaces.

Given a subset $A$ and $\varepsilon > 0$, we denote by $O_\varepsilon(A)$ the $\varepsilon$-neighborhood of $A$. Let $K(Z)$ denote the collection of all nonempty compact subsets of $Z$. Given a multivalued map (multimap) $\Sigma : X \multimap K(Z)$, a continuous map, $\sigma_\varepsilon : X \to Z$, $\varepsilon > 0$, is said to be an *$\varepsilon$-approximation* of $\Sigma$ if for every $x \in X$, there exists $x' \in O_\varepsilon(x)$ such that $\sigma_\varepsilon(x) \in O_\varepsilon(\Sigma(x'))$.

It is clear that the notion can be equivalently expressed saying that

$$\sigma_\varepsilon(x) \in O_\varepsilon(\Sigma(O_\varepsilon(x))) \tag{2.6}$$

for all $x \in X$, or that

$$\Gamma_{\sigma_\varepsilon} \subset (\Gamma_\Sigma), \tag{2.7}$$

where $\Gamma_{\sigma_\varepsilon}, \Gamma_\Sigma$ denote the graphs of $\sigma_\varepsilon$ and $\Sigma$, respectively, while the metric in $X \times Z$ is defined in a natural way as

$$d((x,z),(x',z')) = \max\{d_X(x,x'), d_Z(z,z')\}. \tag{2.8}$$

The fact that $\sigma_\varepsilon$ is an $\varepsilon$-approximation of the multimap $\Sigma$ will be denoted by $\sigma_\varepsilon \in a(\Sigma, \varepsilon)$.

A multimap $\Sigma : X \multimap K(Z)$ is said to be *upper semicontinuous (u.s.c.)* if for every open set $V \subset Z$, the set $\Sigma_+^{-1}(V) = \{x \in X : \Sigma(x) \subset V\}$ is open in $X$.

An u.s.c. multimap $\Sigma : X \multimap K(Z)$ is *closed* if its graph $\Gamma_\Sigma$ is a closed subset of $X \times Z$.

We can summarize some properties of $\varepsilon$-approximations in the following statement (see [9]).

Proposition 2.4. *Let $\Sigma : X \multimap K(Z)$ be an u.s.c. multimap.*

(i) *Let $X_1$ be a compact subset of $X$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma \in a(\Sigma, \delta)$ implies $\sigma_{|X_1} \in a(\Sigma_{|X_1}, \varepsilon)$.*

(ii) *Let $X$ be compact, $Z_1$ a metric space, and $\varphi : Z \to Z_1$ a continuous map. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma \in a(\Sigma, \delta)$ implies $\varphi \circ \sigma \in a(\varphi \circ \Sigma, \varepsilon)$.*

(iii) *Let $X$ be compact, $\Sigma_* : X \times [0,1] \to K(Z)$ an u.s.c. multimap. Then, for every $\lambda \in [0,1]$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma_* \in a(\Sigma_*, \delta)$ implies that $\sigma_*(\cdot, \lambda) \in a(\Sigma_*(\cdot, \lambda), \varepsilon)$.*

(iv) *Let $Z_1$ be a metric space, $\Sigma_1 : X \multimap K(Z_1)$ an u.s.c. multimap. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma \in a(\Sigma, \delta)$ and $\sigma_1 \in a(\Sigma_1, \delta)$ imply that $\sigma \times \sigma_1 \in a(\Sigma \times \Sigma_1, \varepsilon)$, where $(\sigma \times \sigma_1)(x) = \sigma(x) \times \sigma_1(x)$, $(\Sigma \times \Sigma_1)(x) = \Sigma(x) \times \Sigma_1(x)$.*

In the sequel, we will use the following important property of $\varepsilon$-approximations.

Proposition 2.5. *Let $X$, $X'$, $Z$ be metric spaces; $f : X \to X'$ a continuous map; $\Sigma : X \multimap K(Z)$ an u.s.c. multimap; $\varphi : Z \to X'$ a continuous map. Suppose that $X_1 \subseteq X$ is a compact subset such that*

$$\operatorname{Coin}(f,\varphi\circ\Sigma)\cap X_1 = \varnothing, \tag{2.9}$$

*where* $\operatorname{Coin}(f,\varphi\circ\Sigma) = \{x \in X : f(x) \in \varphi\circ\Sigma(x)\}$ *is the coincidence points set. If $\varepsilon > 0$ is sufficiently small and $\sigma_\varepsilon \in a(\Sigma,\varepsilon)$, then*

$$\operatorname{Coin}(f,\varphi\circ\sigma_\varepsilon)\cap X_1 = \varnothing. \tag{2.10}$$

*Proof.* Suppose, to the contrary, that there are sequences $\{x_n\} \subset X_1$ and $\varepsilon_n \to 0$, $\varepsilon_n > 0$, such that

$$f(x_n) = \varphi\sigma_{\varepsilon_n}(x_n) \tag{2.11}$$

for a sequence $\sigma_{\varepsilon_n} \in a(\Sigma,\varepsilon_n)$.

From Proposition 2.4(i) and (ii) we can deduce that, without loss of generality, the maps $\varphi\sigma_{\varepsilon_n|X_1}$ form a sequence of $\delta_n$-approximations of $\varphi\Sigma_{|X_1}$, with $\delta_n \to 0$ and hence

$$(x_n,\varphi\sigma_{\varepsilon_n}(x_n)) \in O_{\delta_n}(\Gamma_{\varphi\Sigma_{|X_1}}). \tag{2.12}$$

The graph of the u.s.c. multimap $\varphi\Sigma_{|X_1}$ is a compact set (see, e.g., [12, Theorem 1.1.7]), hence we can assume, without loss of generality, that

$$(x_n,\varphi\sigma_{\varepsilon_n}(x_n)) \xrightarrow[n\to\infty]{} (x_0,y_0) \in \Gamma_{\varphi\Sigma_{|X_1}}, \tag{2.13}$$

that is, $y_0 \in \varphi\Sigma(x_0)$. Passing to the limit in (2.11), we obtain that $f(x_0) = y_0 \in \varphi\Sigma(x_0)$, that is, $x_0 \in \operatorname{Coin}(f,\varphi\Sigma)$, giving the contradiction. □

To present the class of multimaps which will be considered, we recall some notions.

*Definition 2.6* (see, e.g., [1, 9, 10, 15]). A nonempty compact subset $A$ of a metric space $Z$ is said to be *aspheric* (or $UV^\infty$, or $\infty$*-proximally connected*) if for every $\varepsilon > 0$, there exists $\delta$, $0 < \delta < \varepsilon$, such that for each $n = 0,1,2,\ldots$, every continuous map $g : S^n \to O_\delta(A)$ can be extended to a continuous map $\widetilde{g} : B^{n+1} \to O_\varepsilon(A)$, where $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ and $B^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \le 1\}$.

*Definition 2.7* (see [11]). A nonempty compact space $A$ is said to be an $R_\delta$*-set* if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.

*Definition 2.8* (see [9]). An u.s.c. multimap $\Sigma : X \to K(Z)$ is said to be a $J$*-multimap* ($\Sigma \in J(X,Z)$) if every value $\Sigma(x)$, $x \in X$, is an aspheric set.

We will use the notions of absolute retract (AR-space) and of absolute neighborhood retract (ANR-space) (see, e.g., [5, 9]).

Proposition 2.9 (see [9]). *Let $Z$ be an ANR-space. In each of the following cases, an u.s.c. multimap $\Sigma : X \to K(Z)$ is a $J$-multimap: for each $x \in X$, the value $\Sigma(x)$ is*

(a) *a convex set;*
(b) *a contractible set;*
(c) *an $R_\delta$-set;*
(d) *an AR-space.*

*In particular, every continuous map $\sigma : X \to Z$ is a $J$-multimap.*

The next statement describes the approximation properties of $J$-multimaps.

Proposition 2.10 (see [9, 10, 15]). *Let $X$ be a compact ANR-space; $Z$ a metric space; $\Sigma \in J(X,Z)$. Then*

(i) *the multimap $\Sigma$ is approximable; that is, for every $\varepsilon > 0$, there exists $\sigma_\varepsilon \in a(\Sigma,\varepsilon)$;*
(ii) *for each $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for every $\delta$ $(0 < \delta < \delta_0)$ and for every two $\delta$-approximations $\sigma_\delta, \sigma'_\delta \in a(\Sigma,\delta)$, there exists a continuous map (homotopy) $\sigma_* : X \times [0,1] \to Z$ such that*
  (a) $\sigma_*(\cdot,0) = \sigma_\delta$, $\sigma_*(\cdot,1) = \sigma'_\delta$;
  (b) $\sigma_*(\cdot,\lambda) \in a(\Sigma,\varepsilon)$ *for all* $\lambda \in [0,1]$.

*Definition 2.11.* Denote by $CJ(X,X')$ the collection of all multimaps $G : X \to K(X')$ of the form $G = \varphi \circ \Sigma$, where $\Sigma \in J(X,Z)$ for some metric space $Z$, $\varphi : Z \to X'$ is a continuous map. The composition $\varphi \circ \Sigma$ will be called the representation (or decomposition, cf. [9]) of $G$. Denote $G = (\varphi \circ \Sigma) \in CJ(X,X')$.

It has to be noted that a multimap can admit different representations (see [9]).

## 3. Oriented coincidence index for compact triplets

We will start from the following notion.

*Definition 3.1.* The map $f : \overline{Y} \to E'$, the multimap $G = (\varphi \circ \Sigma) \in CJ(X,X')$, and the space $\overline{Y}$ form a *compact triplet* $(f,G,\overline{Y})_C$ if the following conditions are satisfied:

(h1) $f$ is a continuous proper map, $f_{|Y} \in \Phi_k C^1(Y)$ with $k = 0$ in case (i), $k = 1$ in case (ii), and the Fredholm structure on $Y$ generated by $f$ is oriented;
(h2) $G$ is compact, that is, $G(\overline{Y})$ is a relatively compact subset of $E'$;
(h3) $\mathrm{Coin}(f,G) \cap \partial Y = \emptyset$.

Let us mention that from hypotheses (h1), (h2), it follows that the coincidence points set $Q = \mathrm{Coin}(f,G)$ is compact.

**3.1. The case of a finite-dimensional triplet.** Given a triplet $(f,G,\overline{Y})_C$, from Proposition 2.3 we know that there exist an open neighborhood $\mathbb{O} \subset Y$ of the set $Q = \mathrm{Coin}(f,G)$ and an $n$-dimensional subspace $E'_n \subset E'$ such that $f^{-1}(E'_n) \cap \mathbb{O} = M$, a manifold which is $n$-dimensional in case (i) and $(n+1)$-dimensional in case (ii).

Now, suppose that the multimap $G = \varphi \circ \Sigma$ is finite-dimensional, that is, that there exists a finite-dimensional subspace $E'_m \subset E'$ such that $G(\overline{Y}) \subset E'_m$. We can assume, without loss of generality, that $E'_m \subset E'_n$. Then clearly $Q \subset M$. Let us mention also that the orientation on $Y$ induces the orientation on $M$.

A compact triplet $(f,G,\overline{Y})_C$ such that $G$ is finite-dimensional will be denoted by $(f,G,\overline{Y})_{C_m}$ and will be called *finite-dimensional.*

Lemma 3.2. *For $(f,G=(\varphi\circ\Sigma),\overline{Y})_{C_m}$, let $O_\varkappa$ be a $\varkappa$-neighborhood of $Q$. Then, $\Sigma_{|\overline{O}_\varkappa}$ is approximable provided that $\varkappa>0$ is sufficiently small.*

*Proof.* Consider an open bounded set $N$ satisfying the following conditions:

(i) $Q\subset N\subset\overline{N}\subset M$;

(ii) $\overline{N}$ is a compact ANR-space.

Let us note that as $N$ we can take a union of a finite collection of balls with centers in $Q$.

Let us take $\varkappa>0$ so that $O_\varkappa\subset N$. Then the statement follows from Propositions 2.10(i) and 2.4(i). □

Now, let the neighborhood $O_\varkappa$ be chosen so that $\Sigma$ is approximable on $\overline{O}_\varkappa$. From Proposition 2.5, we know that

$$\operatorname{Coin}(f,\varphi\circ\sigma_\varepsilon)\cap\partial O_\varkappa=\varnothing \tag{3.1}$$

provided that $\sigma_\varepsilon\in a(\Sigma_{|\overline{O}_\varkappa},\varepsilon)$ and $\varepsilon>0$ is sufficiently small.

So, we can consider the following map of pairs of spaces:

$$f-\varphi\circ\sigma_\varepsilon:(\overline{O}_\varkappa,\partial O_\varkappa)\longrightarrow(E_n',E_n'\backslash 0). \tag{3.2}$$

Now we are in position to give the following notion.

*Definition 3.3.* The oriented coincidence index of a finite-dimensional triplet $(f,G=(\varphi\circ\Sigma),\overline{U})_{C_m}$ is defined by the equality

$$(f,G=(\varphi\circ\Sigma),\overline{U})_{C_m}:=\deg(f-\varphi\circ\sigma_\varepsilon,\overline{O}_\varkappa), \tag{3.3}$$

where $\varkappa>0$ and $\varepsilon>0$ are taken small enough and the right-hand part of equality (3.3) denotes the Brouwer topological degree.

Now we will demonstrate that the given definition is consistent, that is, the coincidence index does not depend on the choice of an $\varepsilon$-approximation $\sigma_\varepsilon$ and the neighborhood $O_\varkappa$.

Lemma 3.4. *Let $\sigma_\varepsilon$ and $\sigma_\varepsilon'\in a(\Sigma_{|\overline{O}_{\varkappa'}},\varepsilon)$ be two approximations. Then*

$$\deg(f-\varphi\circ\sigma_\varepsilon,\overline{O}_\varkappa)=\deg(f-\varphi\circ\sigma_\varepsilon',\overline{O}_\varkappa) \tag{3.4}$$

*provided that $\varepsilon>0$ is sufficiently small.*

*Proof.* Let us take any neighborhood $N'$ of $Q$ such that $Q\subset N'\subset\overline{N'}\subset O_\varkappa$ and $\overline{N'}$ is an ANR-space. Then, by Propositions 2.4(i) and 2.5, we know that we can take $\varepsilon>0$ small enough to provide that $\sigma_{\varepsilon|\overline{N'}}$ and $\sigma'_{\varepsilon|\overline{N'}}$ are $\delta_0$-approximations of $\Sigma_{|\overline{N'}}$ and

$$\begin{aligned}\operatorname{Coin}(f,\varphi\circ\sigma_\varepsilon)\cap(\overline{O}_\varkappa\backslash N')&=\varnothing,\\ \operatorname{Coin}(f,\varphi\circ\sigma_\varepsilon')\cap(\overline{O}_\varkappa\backslash N')&=\varnothing.\end{aligned} \tag{3.5}$$

Since $\Sigma_{|\overline{N'}}$ is approximable, we can assume that $\varepsilon > 0$ is chosen so small that there exists a map $\gamma : \overline{N'} \times [0,1] \to Z$ with the following properties:

(i) $\gamma(\cdot,0) = \sigma_{\varepsilon|\overline{N'}}, \gamma(\cdot,1) = \sigma'_{\varepsilon|\overline{N'}}$;

(ii) $\gamma(\lambda,\cdot) \in a(\Sigma_{|\overline{N'}},\delta_1)$ for each $\lambda \in [0,1]$, where $\delta_1$ is arbitrary small;

(iii) $\mathrm{Coin}(f,\varphi \circ \gamma(\cdot,\lambda)) \cap \partial N' = \emptyset$ for all $\lambda \in [0,1]$ (see Propositions 2.10(ii) and 2.5).

Each map $f - \varphi \circ \gamma(\cdot,\lambda)$, $\lambda \in [0,1]$, transforms the pair $(\overline{N'},\partial N')$ into the pair $(E_n, E_n \backslash 0)$ for each $\lambda \in [0,1]$, and by the homotopy property of the Brouwer degree we have $\deg(f - \varphi \circ \sigma_\varepsilon, \overline{N'}) = \deg(f - \varphi \circ \sigma'_\varepsilon, \overline{N'})$. Further from (3.5) and the additive property of the Brouwer degree, we have

$$\begin{aligned} \deg\left(f - \varphi \circ \sigma_\varepsilon, \overline{O}_\varkappa\right) &= \deg\left(f - \varphi \circ \sigma_\varepsilon, \overline{N'}\right), \\ \deg\left(f - \varphi \circ \sigma'_\varepsilon, \overline{O}_\varkappa\right) &= \deg\left(f - \varphi \circ \sigma'_\varepsilon, \overline{N'}\right) \end{aligned} \tag{3.6}$$

proving equality (3.4). □

Now, if $O_{\varkappa'} \subset O_\varkappa$, the equality

$$\deg\left(f - \varphi \circ \sigma_\varepsilon, \overline{O}_{\varkappa'}\right) = \deg\left(f - \varphi \circ \sigma_\varepsilon, \overline{O}_\varkappa\right), \tag{3.7}$$

where $\varepsilon > 0$ is sufficiently small, follows easily from Propositions 2.4(i), 2.5, and the additive property of the Brouwer degree.

At last, let us mention also the independence of the construction on the choice of the transversal subspace $E'_n$. In fact, if we take two subspaces $E'_{n_0}$ and $E'_{n_1}$, we may assume, without loss of generality, that $E'_{n_0} \subset E'_{n_1}$. As earlier, we assume that $G(\overline{U}) \subset E'_m \subset E'_{n_0} \subset E'_{n_1}$. Then, from the construction, we obtain two manifolds $M^{n_0}$, $M^{n_1}$, $M^{n_0} \subset M^{n_1}$ and two neighborhoods $O^{n_0}_\varkappa \subset M^{n_0}$, $O^{n_1}_\varkappa \subset M^{n_1}$, $O^{n_0}_\varkappa \subset O^{n_1}_\varkappa$ for $\varkappa > 0$ sufficiently small. Now, take $\varepsilon > 0$ small enough to provide that the degrees $\deg(f - \varphi \circ \sigma_\varepsilon, \overline{O^{n_1}_\varkappa})$ and $\deg(f - \varphi \circ \sigma_\varepsilon, \overline{O^{n_0}_\varkappa})$ are well defined. Then the equality

$$\deg\left(f - \varphi \circ \sigma_\varepsilon, \overline{O^{n_1}_\varkappa}\right) = \deg\left(f - \varphi \circ \sigma_\varepsilon, \overline{O^{n_0}_\varkappa}\right) \tag{3.8}$$

follows from the map restriction property of Brouwer degree.

Now, let us mention the main properties of the defined characteristic. Directly from Definition 3.3 and Proposition 2.5, we deduce the following statement.

Theorem 3.5 (the coincidence point property). *If* $\mathrm{Ind}(f,G,\overline{U})_{C_m} \neq 0$, *then* $\emptyset \neq \mathrm{Coin}(f, G) \subset U$.

To formulate the topological invariance property of the coincidence index, we will give the following definition.

*Definition 3.6.* Two finite-dimensional triplets $(f_0, G_0 = (\varphi_0 \circ \Sigma_0), \overline{U}_0)_{C_m}$ and $(f_1, G_1 = (\varphi_1 \circ \Sigma_1), \overline{U}_1)_{C_m}$ are said to be homotopic,

$$\left(f_0, G_0 = (\varphi_0 \circ \Sigma_0), \overline{U}_0\right)_{C_m} \sim \left(f_1, G_1 = (\varphi_1 \circ \Sigma_1), \overline{U}_1\right)_{C_m}, \tag{3.9}$$

if there exists a finite-dimensional triplet $(f_*, G_*, \overline{U_*})_{C_m}$, where $U_* \subset E \times [0,1]$ is an open set, such that

(a) $U_i = U_* \cap (E \times \{i\})$, $i = 0,1$;

(b) $f_{*|\overline{U_i}} = f_i$, $i = 0,1$;

(c) $G_*$ has the form

$$G_*(x,\lambda) = \varphi_*(\Sigma_*(x,\lambda),\lambda), \tag{3.10}$$

where $\Sigma_* \in J(\overline{U_*}, Z)$, $\varphi_* : Z \times [0,1] \to E'$, is a continuous map, and

$$\Sigma_{*|\overline{U_i}} = \Sigma_i, \quad \varphi_{*|Z\times\{i\}} = \varphi_i, \quad i = 0,1. \tag{3.11}$$

Theorem 3.7 (the homotopy invariance property). *If*

$$(f_0, G_0, \overline{U}_0)_{C_m} \sim (f_1, G_1, \overline{U}_1)_{C_m}, \tag{3.12}$$

*then*

$$\left| \operatorname{Ind}(f_0, G_0, \overline{U}_0)_{C_m} \right| = \left| \operatorname{Ind}(f_1, G_1, \overline{U}_1)_{C_m} \right|. \tag{3.13}$$

*Proof.* Let $(f_*, G_*, \overline{U_*})_{C_m}$ be a finite-dimensional triplet connecting the triplets $(f_0, G_0, \overline{U}_0)_{C_m}$ and $(f_1, G_1, \overline{U}_1)_{C_m}$. Let $O_{*\varkappa} \subset U_*$ be a $\varkappa$-neighborhood of $Q_* = \operatorname{Coin}(f_*, G_*)$, where $\varkappa > 0$ is sufficiently small.

Take $\sigma_{*\varepsilon} \in a(\Sigma_{*|\overline{O_{*\varkappa}}}, \varepsilon)$ for $\varepsilon > 0$ sufficiently small. Applying Propositions 2.4 and 2.5, we can verify that the map $\varphi_* \circ \sigma_{*\varepsilon} : \overline{O_{*\varkappa}} \to E'$, $\varphi_* \circ \sigma_{*\varepsilon}(x,\lambda) = \varphi_*(\sigma_{*\varepsilon}(x,\lambda),\lambda)$ is a $\delta'$-approximation of $G_{*|\overline{O_{*\varkappa}}}$ for $\delta' > 0$ arbitrary small and, moreover,

$$\operatorname{Coin}(f_*, \varphi_* \circ \sigma_{*\varepsilon}) \cap \partial O_{*\varkappa} = \varnothing \tag{3.14}$$

and $\varphi_* \circ \sigma_{*\varepsilon|\overline{O_{\varkappa i}}}$, for $\overline{O_{\varkappa i}} = \overline{O_{*\varkappa}} \cap U_i$, $i = 0,1$, are $\delta''$-approximations of $G_{i|\overline{O_{\varkappa i}}}$, $i = 0,1$, where $\delta'' > 0$ is arbitrary small.

Denoting $\sigma_{*\varepsilon|\overline{O_{\varkappa i}}} = \sigma_i$, $i = 0,1$, we have

$$\left| \deg(f_0 - \varphi_0 \circ \sigma_0, \overline{O_{\varkappa 0}}) \right| = \left| \deg(f_1 - \varphi_1 \circ \sigma_1, \overline{O_{\varkappa 1}}) \right| \tag{3.15}$$

(see [22]), proving the theorem. □

*Remark 3.8.* If the Fredholm map $f$ is constant under the homotopy, that is, $U_*$ has the form $U_* = U \times [0,1]$, where $U \subset E$ is an open set and $f_*(x,\lambda) = f(x)$ for all $\lambda \in [0,1]$, where $f \in \Phi_0 C^1(U)$, then

$$\deg(f - \varphi_0 \circ \sigma_0, \overline{U}) = \deg(f - \varphi_1 \circ \sigma_1, \overline{U}) \tag{3.16}$$

(see [21, 22]). Hence

$$\operatorname{Ind}(f, G_0, \overline{U})_{C_m} = \operatorname{Ind}(f, G_1, \overline{U})_{C_m}. \tag{3.17}$$

From Definition 3.3 and the additive property of the Brouwer degree, we obtain the following property of the oriented coincidence index.

THEOREM 3.9 (additive dependence on the domain property). *Let $U_0$ and $U_1$ be disjoint open subsets of an open bounded set $U \subset E$ and let $(f, G, \overline{U})_{C_m}$ be a finite-dimensional triplet such that*

$$\operatorname{Coin}(f,G) \cap (\overline{U} \setminus (U_0 \cup U_1)) = \varnothing. \tag{3.18}$$

*Then*

$$\operatorname{Ind}(f,G,\overline{U})_{C_m} = (f,G,\overline{U_0})_{C_m} + (f,G,\overline{U_1})_{C_m}. \tag{3.19}$$

**3.2. The case of a compact triplet.** Now, we want to define the oriented coincidence degree for the general case of a compact triplet $(f, G = (\varphi \circ \Sigma), \overline{U})_C$.

From the properness property of $f$ and the compactness of $G$, one can easily deduce the following statement.

PROPOSITION 3.10. *Let $(f,G,\overline{U})_C$ be a compact triplet; $\Lambda : \overline{Y} \to K(E')$ a multimap defined as*

$$\Lambda(y) = f(y) - G(y). \tag{3.20}$$

*Then, for every closed subset $Y_1 \subset \overline{Y}$, the set $\Lambda(Y_1)$ is closed.*

From the above assertion, it follows that, given a compact triplet $(f,G,\overline{U})_C$, there exists $\delta > 0$ such that

$$B_\delta(0) \cap \Lambda(\partial U) = \varnothing, \tag{3.21}$$

where $B_\delta(0) \subset E'$ is a $\delta$-neighborhood of the origin.

Let us take a continuous map $i_\delta : \overline{G(\overline{U})} \to E_m$, where $E_m \subset E$ is a finite-dimensional subspace, with the property that

$$\|i_\delta(v) - v\| < \delta \tag{3.22}$$

for each $v \in \overline{G(\overline{U})}$. As $i_\delta$, we can choose the Schauder projection (see, e.g., [14]).

Now, if $G$ has the representation $G = \varphi \circ \Sigma$, consider the finite-dimensional multimap $G_m = i_\delta \circ \varphi \circ \Sigma$. From (3.21) and (3.22), it follows that $f$, $G_m$ and $\overline{U}$ form a finite-dimensional triplet $(f, G_m, \overline{U})_{C_m}$.

We can now define the oriented coincidence index for a compact triplet in the following way.

*Definition 3.11.* The oriented coincidence index for a compact triplet $(f, G = (\varphi \circ \Sigma), \overline{U})_C$ is defined by the equality

$$\operatorname{Ind}(f,G,\overline{U})_C := \operatorname{Ind}(f,G_m,\overline{U})_{C_m}, \tag{3.23}$$

where $G_m = i_\delta \circ \varphi \circ \Sigma$ and the map $i_\delta$ satisfies condition (3.22).

To prove the consistency of the given definition, it is sufficient to mention that, given two different maps $i_\delta^0, i_\delta^1 : \overline{G(\overline{U})} \to E'_m$ satisfying property (3.22), we have the homotopy of the corresponding finite-dimensional triplets

$$(f, G_m^0, \overline{U})_{C_m} \sim (f, G_m^1, \overline{U})_{C_m}, \tag{3.24}$$

where $G_m^k = i_\delta^k \circ \varphi \circ \Sigma$, $i = 0, 1$. (It is clear that the finite-dimensional space $E'_m$ can be taken the same for both maps $i_\delta^0, i_\delta^1$.)

In fact, the homotopy is realized by the multimap $G_* : \overline{U} \times [0,1] \to K(E'_m)$, defined as

$$G_*(x,\lambda) = \varphi_*(\Sigma(x,\lambda)), \quad \text{where } \varphi_*(z,\lambda) = (1-\lambda) i_\delta^0 \varphi(z) + \lambda i_\delta^1 \varphi(z). \tag{3.25}$$

So, from Remark 3.8, it follows that

$$\operatorname{Ind}(f, G_m^0, \overline{U})_{C_m} = \operatorname{Ind}(f, G_m^1, \overline{U})_{C_m}. \tag{3.26}$$

Applying Proposition 3.10 and Theorem 3.5, we can deduce the following coincidence point property.

Theorem 3.12. *If* $\operatorname{Ind}(f, G, \overline{U})_C \neq 0$, *then* $\varnothing \neq \operatorname{Coin}(f, G) \subset U$.

The definition of homotopy for compact triplets $(f, G_0, \overline{U})_C \sim (f, G_1, \overline{U})_C$ has the same form as in Definition 3.6 with the only difference that the connected triplet $(f_*, G_*, \overline{U_*})$ is assumed to be compact.

Taking a finite-dimensional approximation of $G_* = \varphi_* \circ \Sigma_*$ as $G_{*m} = i_\delta \circ \varphi_* \circ \Sigma_*$ and applying Theorem 3.7 and Definition 3.11, we obtain the following homotopy invariance property.

Theorem 3.13. *If* $(f, G_0, \overline{U})_C \sim (f, G_1, \overline{U})_C$, *then*

$$|\operatorname{Ind}(f_0, G_0, \overline{U}_0)_C| = |\operatorname{Ind}(f_1, G_1, \overline{U}_1)_C|. \tag{3.27}$$

Again, if $f$ and $U$ are constant, we have the equality

$$\operatorname{Ind}(f, G_0, \overline{U})_C = \operatorname{Ind}(f, G_1, \overline{U})_C. \tag{3.28}$$

An analog of the additive dependence on the domain property (see Theorem 3.9) for compact triplets also holds.

## 4. Oriented coincidence index for condensing triplets

In this section, we extend the notion of the oriented coincidence index to the case of condensing triplets. At first we recall some notions (see, e.g., [12]). Denote by $P(E')$ the collection of all nonempty subsets of a Banach space $E'$. Let $(\mathscr{A}, \geq)$ be a partially ordered set.

*Definition 4.1.* A map $\beta : P(E') \to \mathscr{A}$ is called a *measure of noncompactness* (MNC) in $E'$ if

$$\beta(\overline{co} D) = \beta(D) \quad \text{for every } D \in P(E'). \tag{4.1}$$

An MNC $\beta$ is called

(i) *monotone* if $D_0, D_1 \in P(E')$, $D_0 \subseteq D_1$, implies $\beta(D_0) \leq \beta(D_1)$;

(ii) *nonsingular* if $\beta(\{a\} \cup D) = \beta(D)$ for every $a \in E'$, $D \in P(E')$;

(iii) *real* if $A = \overline{\mathbb{R}_+} = [0, +\infty]$ with the natural ordering, and $\beta(D) < +\infty$ for every bounded set $D \in P(E')$.

Among the known examples of MNC satisfying all the above properties we can consider the *Hausdorff MNC*

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon\text{-net}\}, \tag{4.2}$$

and the *Kuratowski MNC*

$$\alpha(D) = \inf\{d > 0 : D \text{ has a finite partition with sets of diameter less than } d\}. \tag{4.3}$$

Let again $Y = U \subset E$, or $U_* \subset E \times [0,1]$, open bounded sets, $f : \overline{Y} \to E'$ a map; $G : \overline{Y} \to K(E')$ a multimap, $\beta$ an MNC in $E'$.

*Definition 4.2.* Maps $f$, $G$ and the space $\overline{Y}$ form a *$\beta$-condensing triplet* $(f, G, \overline{Y})_\beta$ if they satisfy conditions (h1) and (h3) in Definition 3.1, and (h2$_\beta$) a multimap $G = \varphi \circ \Sigma \in CJ(\overline{Y}, E')$ is $\beta$-condensing with respect to $f$, that is,

$$\beta(G(\Omega)) \not\geq \beta(f(\Omega)) \tag{4.4}$$

for every $\Omega \subseteq \overline{Y}$ such that $G(\Omega)$ is not relatively compact.

Our target is to define the coincidence index for a $\beta$-condensing triplet $(f, G, \overline{U})_\beta$. To this aim, let us recall the following notion (see, e.g., [1, 7, 8, 12, 16]).

*Definition 4.3.* A convex, closed subset $T \subset E'$ is said to be fundamental for a triplet $(f, G, \overline{Y})_\beta$ if

(i) $G(f^{-1}(T)) \subseteq T$;

(ii) for any point $y \in \overline{Y}$, the inclusion $f(y) \in \overline{co}(G(y) \cup T)$ implies that $f(y) \in T$.

The entire space $E'$ and the set $\overline{co}G(\overline{Y})$ are natural examples of fundamental sets for $(f, G, \overline{U})_\beta$.

It is easy to verify the following properties of a fundamental set.

Proposition 4.4. (a) *The set* $\mathrm{Coin}(f, G)$ *is included in* $f^{-1}(T)$ *for each fundamental set* $T$ *of* $(f, G, \overline{U})_\beta$.

(b) *Let* $T$ *be a fundamental set of* $(f, G, \overline{U})_\beta$, *and* $P \subset T$, *then the set* $\widetilde{T} = \overline{co}(G(f^{-1}(T)) \cup P)$ *is also fundamental.*

(c) *Let* $\{T_\alpha\}$ *be a system of fundamental sets of* $(f, G, \overline{U})_\beta$. *The set* $T = \cap_\alpha T_\alpha$ *is also fundamental.*

Proposition 4.5. *Each* $\beta$*-condensing triplet* $(f, G, \overline{U})_\beta$, *where* $\beta$ *is a monotone, nonsingular MNC, admits a nonempty, compact fundamental set* $T$.

*Proof.* Consider the collection $\{T_\alpha\}$ of all fundamental sets of $(f,G,\overline{U})_\beta$ containing an arbitrary point $a \in E'$. This collection is nonempty since it contains $E'$. Then, taking $T = \cap_\alpha T_\alpha \neq \emptyset$, we obviously have

$$T = \overline{co}\big(G\big(f^{-1}(T)\big) \cup \{a\}\big), \tag{4.5}$$

and hence

$$\beta\big(f\big(f^{-1}(T)\big)\big) \leq \beta(T) = \beta\big(G\big(f^{-1}(T)\big)\big), \tag{4.6}$$

so $G(f^{-1}(T))$ is relatively compact and $T$ is compact. □

Everywhere from now on, we assume that the MNC $\beta$ is monotone and nonsingular.

Now, if $T$ is a nonempty compact fundamental set of a $\beta$-condensing triplet $(f,G = (\varphi \circ \Sigma),\overline{Y})_\beta$, let $\rho : E' \to T$ be any retraction. Consider the multimap $\widetilde{G} = \rho \circ \varphi \circ \Sigma \in CJ(\overline{Y},E')$. From Proposition 4.4(a), it follows that

$$\operatorname{Coin}(f,\widetilde{G}) = \operatorname{Coin}(f,G). \tag{4.7}$$

Hence, $f$, $\widetilde{G}$, and $\overline{Y}$ form a compact triplet $(f,G,\overline{Y})_C$. We will say that $(f,G,\overline{Y})_C$ is a *compact approximation* of the triplet $(f,G,\overline{Y})_\beta$.

*Definition 4.6.* The oriented coincidence index of a $\beta$-condensing triplet $(f,G,\overline{U})_\beta$ is defined by the equality

$$\operatorname{Ind}(f,G,\overline{U})_\beta := \operatorname{Ind}\big(f,\widetilde{G},\overline{U}\big)_C, \tag{4.8}$$

where $(f,\widetilde{G},\overline{U})_C$ is a compact approximation of $(f,G,\overline{U})_\beta$.

To prove the consistency of the above definition, consider two nonempty, compact fundamental sets $T_0$ and $T_1$ of the triplet $(f,G = \varphi \circ \Sigma,\overline{U})_\beta$ with retractions $\rho_0 : E' \to T_0$ and $\rho_1 : E' \to T_1$, respectively.

If $T_0 \cap T_1 = \emptyset$, then by Proposition 4.4(a) and (c), $\operatorname{Coin}(f,\widetilde{G}_0) = \operatorname{Coin}(f,\widetilde{G}_1) = \operatorname{Coin}(f,\widetilde{G}) = \emptyset$, where $\widetilde{G}_i = \rho_i \circ \varphi \circ \Sigma$, $i = 0,1$. Hence, by Theorem 3.12, $\operatorname{Ind}(f,\widetilde{G}_0,\overline{U})_C = \operatorname{Ind}(f,\widetilde{G}_1,\overline{U})_C = 0$. Otherwise, we can assume, without loss of generality, that $T_0 \subseteq T_1$. In this case, consider the map $\overline{\varphi} : Z \times [0,1] \to E'$, given by $\overline{\varphi}(z,\lambda) = \rho_1 \circ (\lambda\varphi(z) + (1-\lambda)\rho_0 \circ \varphi(z))$ and the multimap $\overline{G} \in CJ(\overline{U} \times [0,1],E')$, $\overline{G}(x,\lambda) = \overline{\varphi}(\Sigma(x),\lambda)$.

The compact triplet $(f,\overline{G},\overline{U} \times [0,1])_C$ realizes the homotopy

$$\big(f,\widetilde{G}_0,\overline{U}\big)_C \sim \big(f,\widetilde{G}_1,\overline{U}\big)_C. \tag{4.9}$$

Indeed, the only fact that we need to verify is that

$$\operatorname{Coin}(\overline{f},\overline{G}) \cap \big(\partial U \times [0,1]\big) = \emptyset, \tag{4.10}$$

where $\overline{f}(x,\lambda) \equiv f(x)$ is the natural extension.

To the contrary, suppose that there exists $(x,\lambda) \in \partial U \times [0,1]$ such that

$$f(x) = \rho_1 \circ (\lambda\varphi(z) + (1-\lambda)\rho_0 \circ \varphi(z)) \tag{4.11}$$

for some $z \in \Sigma(x)$. But in this case, $x \in f^{-1}(T_1)$ and hence $\varphi(z) \in T_1$. Since also $\rho_0 \circ \varphi(z) \in T_1$, we have

$$\lambda\varphi(z) + (1-\lambda)\rho_0 \circ \varphi(z) \in T_1 \tag{4.12}$$

and so

$$f(x) = \lambda\varphi(z) + (1-\lambda)\rho_0 \circ \varphi(z) \in \overline{co}(G(x) \cup T_0) \tag{4.13}$$

and we obtain $f(x) \in T_0$ and $x \in f^{-1}(T_0)$, implying $\varphi(z) \in T_0$ and $\rho_0 \circ \varphi(z) = \varphi(z)$. We conclude that $f(x) = \varphi(z) \in G(x)$ giving the contradiction.

*Definition 4.7.* Two $\beta$-condensing triplets $(f_0, G_0, \overline{U}_0)_\beta$ and $(f_1, G_1, \overline{U}_1)_\beta$ are said to be homotopic:

$$(f_0, G_0, \overline{U}_0)_\beta \sim (f_1, G_1, \overline{U}_1)_\beta, \tag{4.14}$$

if there exists a $\beta$-condensing triplet $(f_*, G_*, \overline{U}_*)_\beta$ satisfying conditions (a), (b), (c) of Definition 3.6.

Theorem 4.8 (the homotopy invariance property). *If*

$$(f_0, G_0, \overline{U}_0)_\beta \sim (f_1, G_1, \overline{U}_1)_\beta, \tag{4.15}$$

*then*

$$|\operatorname{Ind}(f_0, G_0, \overline{U}_0)_\beta| = |\operatorname{Ind}(f_1, G_1, \overline{U}_1)_\beta|. \tag{4.16}$$

*Proof.* Let $T_*$ be a nonempty compact fundamental set of the triplet $(f_*, G_* = (\varphi_* \circ \Sigma_*), \overline{U_*})$ connecting $(f_0, G_0, \overline{U}_0)_\beta$ with $(f_1, G_1, \overline{U}_1)_\beta$. It is easy to see that $T_*$ is fundamental also for the triplets $(f_k, G_k, \overline{U}_k)_\beta$, $k = 0,1$. Let $\rho_* : E' \to T_*$ be any retraction, and $(f_*, \widetilde{G}_* = \rho_* \circ \varphi_* \circ \sigma_*, \overline{U}_*)_C$ the corresponding compact approximation of $(f_*, G_*, \overline{U}_*)_\beta$. Then $(f_*, \widetilde{G}_*, \overline{U}_*)_C$ realizes a compact homotopy connecting the triplets $(f_k, \rho_* \circ \varphi_k \circ \Sigma_k, \overline{U}_k)_C$, $k = 0,1$ which are compact approximations of $(f_k, G_k, \overline{U}_k)_\beta$, $k = 0,1$, respectively.

By Theorem 3.13, we have

$$|\operatorname{Ind}(f_0, \rho_* \circ \varphi_0 \circ \Sigma_0, \overline{U}_0)_C| = |\operatorname{Ind}(f_1, \rho_* \circ \varphi_1 \circ \Sigma_1, \overline{U}_1)_C| \tag{4.17}$$

giving the desired equality (4.16). □

*Remark 4.9.* Let us mention that in case of invariable $f$ and $\overline{U}$:

$$\begin{gathered} U_* = U \times [0,1] \\ f_*(x,\lambda) \equiv f(x), \quad \forall \lambda \in [0,1], \end{gathered} \tag{4.18}$$

the condition of $\beta$-condensivity for a triplet $(f, G_{*}, \overline{U} \times [0,1])_\beta$ may be weakened: for the existence of a nonempty, compact fundamental set $T$, it is sufficient to demand that

$$\beta(G_{*}(\Omega \times [0,1])) \not\geq \beta(f(\Omega)) \tag{4.19}$$

for every $\Omega \subseteq \overline{U}$ such that $G_{*}(\Omega \times [0,1])$ is not relatively compact.

In fact, it is enough to notice that in this case $f_{*}^{-1}(T) = f^{-1}(T) \times [0,1]$ and to follow the line of reasoning of Proposition 4.5.

Taking into consideration the corresponding property of compact triplets, we can precise the above property of homotopy invariance.

If $(f, G_{*}, \overline{U} \times [0,1])_\beta$ is a $\beta$-condensing triplet, where $G_{*}$ has the form (c) of Definition 3.6, then

$$\operatorname{Ind}(f, G_0, \overline{U})_\beta = \operatorname{Ind}(f, G_1, \overline{U})_\beta, \tag{4.20}$$

where $G_k = G_{*}(\cdot, \{k\})$, $k = 0, 1$.

From relation (4.7) and Theorem 3.12, the following theorem follows immediately.

Theorem 4.10 (coincidence point property). *If* $\operatorname{Ind}(f, G, \overline{U})_\beta \neq 0$, *then* $\emptyset \neq \operatorname{Coin}(f, G) \subset U$.

As an example of application of Theorems 4.8 and 4.10, consider the following coincidence point result.

Theorem 4.11. *Let* $f \in \Phi_0 C^1(E, E')$ *be odd;* $G \in CJ(E, E')$ *$\beta$-condensing with respect to* $f$ *on bounded subsets of* $E$, *that is,* $\beta(G(\Omega)) \not\geq \beta(f(\Omega))$ *for every bounded set* $\Omega \subset E$ *such that* $G(\Omega)$ *is not relatively compact.*

*If the set of solutions of one-parameter family of operator inclusions*

$$f(x) \in \lambda G(x) \tag{4.21}$$

*is a priori bounded, then* $\operatorname{Coin}(f, G) \neq \emptyset$.

*Proof.* From the condition it follows that there exists a ball $\mathcal{B} \subset E$ centered at the origin whose boundary $\partial\mathcal{B}$ does not contain solutions of (4.21). Let $\varphi \circ \Sigma$ be a representation of $G$. If $G_{*} : \overline{\mathcal{B}} \times [0,1] \to K(E')$ has the form

$$G_{*}(x, \lambda) = \varphi_{*}(\Sigma(x), \lambda), \quad \varphi_{*}(z, \lambda) = \lambda\varphi(z), \quad (z, \lambda) \in \overline{\mathcal{B}} \times [0,1], \tag{4.22}$$

then $f, G_{*}$, and $\overline{\mathcal{B}} \times [0,1]$ form a $\beta$-condensing triplet $(f, G_{*}, \overline{\mathcal{B}} \times [0,1])_\beta$.

In fact, suppose that $\beta(G_{*}(\Omega)) \geq \beta(f(\Omega))$ for some $\Omega \subset \overline{\mathcal{B}}$. Since $G_{*}(\Omega \times [0,1]) = \overline{co}(G(\Omega) \cup \{0\})$, we have $\beta(G(\Omega)) \geq \beta(f(\Omega))$ implying that $G(\Omega)$, and hence $G_{*}(\Omega \times [0,1])$, is relatively compact.

So the triplet $(f, G_{*}, \overline{\mathcal{B}} \times [0,1])_\beta$ induces a homotopy connecting the triplets $(f, G, \overline{\mathcal{B}})_\beta$ and $(f, 0, \overline{\mathcal{B}})_\beta$. Since the triplet $(f, 0, \overline{\mathcal{B}})_\beta$ is finite-dimensional, from the odd condition on $f$ and the odd field property of the Brouwer degree, it follows that $(f, 0, \overline{\mathcal{B}})_\beta$ is an odd number.

Then, from the equality $\mathrm{Ind}(f,G,\overline{\mathscr{B}})_\beta = \mathrm{Ind}(f,0,\overline{\mathscr{B}})_\beta$, it follows that $\mathrm{Ind}(f,G,\overline{\mathscr{B}})_\beta \neq 0$ and we can apply the coincidence point property. □

In conclusion of this section, let us formulate the additive dependence on the domain property for $\beta$-condensing triplets.

Theorem 4.12. *Let $U_0$ and $U_1$ be disjoint open subsets of an open bounded set $U \subset E$. If $(f,G,\overline{U})_\beta$ is a $\beta$-condensing triplet such that*

$$\mathrm{Coin}(f,G) \cap \left(\overline{U} \setminus (U_0 \cup U_1)\right) = \varnothing, \tag{4.23}$$

*then,*

$$\mathrm{Ind}(f,G,\overline{U})_\beta = \mathrm{Ind}\left(f,G,\overline{U}_0\right)_\beta + \mathrm{Ind}\left(f,G,\overline{U}_1\right)_\beta. \tag{4.24}$$

## 5. Example

Consider a mixed problem of the following form:

$$A\left(t,x(t),x'(t)\right) = B\left(t,x(t),x'(t),y(t)\right), \tag{5.1}$$

$$y'(t) \in C\left(t,x(t),y(t)\right), \tag{5.2}$$

$$x(0) = x_0, \qquad y(0) = y_0, \tag{5.3}$$

where $A : [0,a] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $B : [0,a] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous maps; $C : [0,a] \times \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^m$ is a multimap, and $x_0 \in \mathbb{R}^n$; $y_0 \in \mathbb{R}^m$.

By a solution of problem (5.1)–(5.3), we mean a pair of functions $(x,y)$, where $x \in C^1([0,a];\mathbb{R}^n)$, $y \in AC([0,a];\mathbb{R}^m)$ satisfy initial conditions (5.1), (5.3) for all $t \in [0,a]$ and inclusion (5.2) for a.a. $t \in [0,a]$.

It should be noted that problem (5.1)–(5.3) may be treated as the law of evolution of a system $x(t)$, whose dynamics is described by the implicit differential equation (5.1) and the control $y(t)$ is the subject of the feedback relation (5.2). Our aim is to show that, under appropriate conditions, the problem of solving problem (5.1)–(5.3) can be reduced to the study of a condensing triplet of the above-mentioned form (see Section 4).

Consider the following condition:

(A) for each $(t,u,v) \in [0,a] \times \mathbb{R}^n \times \mathbb{R}^n$, there exist continuous partial derivatives $A'_u(t,u,v)$, $A'_v(t,u,v)$, and moreover, $\det A'_v(t,u,v) \neq 0$.

Proposition 5.1. *Under condition (A), a map $f : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n) \times \mathbb{R}^n$ defined as*

$$f(x)(t) = \left(A\left(t,x(t),x'(t)\right),x(0)\right) \tag{5.4}$$

*is a Fredholm map of index zero, whose restriction to each closed bounded set $D \subset C^1([0,a];\mathbb{R}^n)$ is proper.*

*Proof.* (i) At first, let us prove that $f$ is a Fredholm map of index zero. It is sufficient to show that the map $\widetilde{f} : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$, $\widetilde{f}(x)(t) = A(t,x(t),x'(t))$ is Fredholm of index $n$.

Let us note that $\widetilde{f}$ is a $C^1$ map and, moreover, its derivative can be written explicitly:

$$(\widetilde{f}'(x)h)(t) = A'_u(t,x(t),x'(t))h(t) + A'_v(t,x(t),x'(t))h'(t) \tag{5.5}$$

for $h \in C^1([0,a];\mathbb{R}^n)$. The linear operator $\widetilde{f}'(x) : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ is a Fredholm operator of index $n$. In fact, introducing the auxiliary operators

$$\begin{gathered}
\widetilde{f}'_u(x) : C^1([0,a];\mathbb{R}^n) \longrightarrow C([0,a];\mathbb{R}^n), \\
(\widetilde{f}'_u(x)h)(t) = A'_u(t,x(t),x'(t))h(t), \quad t \in [0,a], \\
\widetilde{f}'_v(x) : C^1([0,a];\mathbb{R}^n) \longrightarrow C([0,a];\mathbb{R}^n), \\
(\widetilde{f}'_v(x)h)(t) = A'_v(t,x(t),x'(t))h'(t), \quad t \in [0,a],
\end{gathered} \tag{5.6}$$

we can write

$$\widetilde{f}'(x)h = \widetilde{f}'_u(x)h + \widetilde{f}'_v(x)h. \tag{5.7}$$

The operator $\widetilde{f}'_u(x)$ is completely continuous since it can be represented as the composition of a completely continuous inclusion map $i : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ and a continuous linear operator $M : C([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ $(Mh)(t) = A'_u(t,x(t),x'(t))h(t)$. Now, it is sufficient to show that the operator $\widetilde{f}'_v(x)$ is a Fredholm operator of index $n$.

Let us represent this operator as the composition of the differentiation operator $d/dt$ : $C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ and the operator $L : C([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$, $(Lz)(t) = A'_v(t,x(t),x'(t))z(t)$. It is well known that the operator $d/dt$ is a Fredholm operator of index $n$. Since the matrix $A'_v(t,x(t),x'(t))$ is invertible, the operator $L$ is invertible too. Hence, the operators $\widetilde{f}'_v(x)$ and, therefore, $\widetilde{f}'(x)$ are Fredholm of index $n$ and $f'(x)$ is a Fredholm map of index zero. So, $f$ is a nonlinear Fredholm map of index zero.

(ii) Now, let $D \subset C^1([0,a];\mathbb{R}^n)$ be a closed bounded set. Denoting the restriction of $\widetilde{f}$ on $D$ by the same symbol, let us demonstrate its properness. Let $\mathcal{K} \subset C([0,a];\mathbb{R}^n)$ be any compact set, and let $\{x_n\}_{n\in\mathbb{N}} \subset \widetilde{f}^{-1}(\mathcal{K})$ be an arbitrary sequence. Without loss of generality, we may assume that $\widetilde{f}(x_n) \to z \in \mathcal{K}$. Since the sequence $\{x_n\}$ is bounded in $C^1([0,a];\mathbb{R}^n)$ we may also assume, without loss of generality, that the sequence $\{x_n\}$ tends, in $C([0,a];\mathbb{R}^n)$, to some $\omega \in C([0,a];\mathbb{R}^n)$. Further, from the representation

$$A(t,\omega(t),x'_n(t)) = A(t,x_n(t),x'_n(t)) + \left[A(t,\omega(t),x'_n(t)) - A(t,x_n(t),x'_n(t))\right] \tag{5.8}$$

it follows that the sequence $z_n = A(\cdot,\omega(\cdot),x'_n(\cdot))$ tends to $z$ in $C([0,a];\mathbb{R}^n)$. From the inverse mapping theorem it follows that $x'_n = \Psi(z_n)$, where $\Psi : C([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ is a continuous map, implying that $x'_n$ tends to $\Psi(z)$ in $C([0,a];\mathbb{R}^n)$. So, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in the space $C^1([0,a];\mathbb{R}^n)$ and, hence, the set $\widetilde{f}^{-1}(\mathcal{K})$ is compact. The properness of $f$ easily follows. $\square$

Now we will describe the assumptions on the map $B$ and the multimap $C$.

Denoting by the symbol $Kv(\mathbb{R}^m)$ the collection of all nonempty compact convex subsets of $\mathbb{R}^m$, we suppose that the multimap $C : [0,a] \times \mathbb{R}^n \times \mathbb{R}^m \to Kv(\mathbb{R}^m)$ satisfies the following conditions:

(C1) the multifunction $C(\cdot,u,w) : [0,a] \to Kv(\mathbb{R}^m)$ has a measurable selection for all $(u,w) \in \mathbb{R}^n \times \mathbb{R}^m$;

(C2) the multimap $C(t,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to Kv(\mathbb{R}^m)$ is upper semicontinuous for a.a. $t \in [0,a]$;

(C3) the multimap $C$ is uniformly continuous in the second argument, in the following sense: for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$C(t,\overline{u},w) \subset W_\varepsilon\big(C(t,u,w)\big) \quad \forall (t,w) \in [0,a] \times \mathbb{R}^m \tag{5.9}$$

whenever $\|\overline{u} - u\| < \delta$ (where $W_\varepsilon$ denotes the $\varepsilon$-neighborhood of a set);

(C4) there exists a function $\gamma \in L^1_+([0,a])$ such that

$$\big|\big|C(t,u,w)\big|\big| := \sup\big\{\|c\| : c \in C(t,u,w)\big\} \leq \gamma(t)\big(1 + \|u\| + \|w\|\big). \tag{5.10}$$

For a given function $x \in C^1([0,a];\mathbb{R}^n)$ consider the multimap $C_x : [0,a] \times \mathbb{R}^m \to Kv(\mathbb{R}^m)$ defined as $C_x(t,w) = C(t,x(t),w)$. From [12, Theorem 1.3.5], it follows that for each $w \in \mathbb{R}^m$ the multifunction $C_x(\cdot,w)$ admits a measurable selection. Furthermore, from (C2) and (C3), it follows that for a.a. $t \in [0,a]$ the multimap $C_x(t,w)$ depends upper semicontinuously on $(x,w)$. Applying known results on existence, topological structure, and continuous dependence of solutions for Carathéodory-type differential inclusions (see, e.g., [2, 6, 12]) we conclude the following.

Proposition 5.2. *For each given $x \in C^1([0,a];\mathbb{R}^n)$, the set $\Pi_x$ of the Carathéodory solutions of the Cauchy problem*

$$\begin{gathered} y'(t) \in C\big(t,x(t),y(t)\big), \\ y(0) = y_0 \end{gathered} \tag{5.11}$$

*is an $R_\delta$-set in $C([0,a];\mathbb{R}^m)$. Moreover, the multimap $\Pi : C^1([0,a];\mathbb{R}^n) \to K(C([0,a];\mathbb{R}^m))$, $\Pi(x) = \Pi_x$ is upper semicontinuous.*

Now, we will assume that the maps $A$ and $B$ satisfy the following Lypschitz-type condition:

(AB) there exists a constant $q$, $0 \leq q < 1$, such that

$$\big|B(t,u,v,w) - B(t,u,\overline{v},w)\big| \leq q\big|A(t,u,v) - A(t,u,\overline{v})\big| \tag{5.12}$$

for all $t \in [0,a]$, $u,v,\overline{v} \in \mathbb{R}^n$, $w \in \mathbb{R}^m$.

Consider the continuous map $\widetilde{\sigma}: C^1([0,a];\mathbb{R}^n)\times C([0,a];\mathbb{R}^m)\to C([0,a];\mathbb{R}^n)$ defined as

$$\widetilde{\sigma}(x,y)(t) = B(t,x(t),x'(t),y(t)) \tag{5.13}$$

and the multimap $\widetilde{\Sigma}: C^1([0,a];\mathbb{R}^n)\to K(C^1([0,a];\mathbb{R}^n)\times C([0,a];\mathbb{R}^m))$, $\widetilde{\Sigma}(x) = \{x\}\times\Pi(x)$.

From Propositions 5.2 and 2.9, it follows that $\widetilde{\Sigma}$ is a $J$-multimap, and hence the composition $\widetilde{G} = \widetilde{\sigma}\circ\widetilde{\Sigma}: C^1([0,a];\mathbb{R}^n)\to K(C([0,a];\mathbb{R}^n))$ is a $CJ$-multimap. It is clear that the set $\widetilde{G}(x)$ consists of all functions of the form $B(t,x(t),x'(t),y(t))$, where $y\in\Pi(x)$.

Define now the $CJ$-multimap $G: C^1([0,a];\mathbb{R}^n)\to K(C([0,a];\mathbb{R}^n)\times\mathbb{R}^n)$ by

$$G(x) = \widetilde{G}(x)\times\{x_0\}. \tag{5.14}$$

The solvability of problem (5.1)–(5.3) is equivalent to the existence of a coincidence point $x\in C^1([0,a];\mathbb{R}^n)$ for the pair $(f,G)$.

If $U\subset C^1([0,a];\mathbb{R}^n)$ is an open bounded set, then to show that $(f,G,\overline{U})$ form a condensing triplet with respect to the Kuratowski MNC, it is sufficient to prove the following statement.

Proposition 5.3. *The triplet $(\widetilde{f},\widetilde{G},\overline{U})$ is $\alpha$-condensing with respect to the Kuratowski MNC $\alpha$ in the space $C([0,a];\mathbb{R}^n)$.*

*Proof.* Take any subset $\Omega\subset\overline{U}$, and let $\alpha(\widetilde{f}(\Omega)) = d$. From the definition of Kuratowski MNC, it follows that taking an arbitrary $\varepsilon>0$ we may find a partition of the set $\widetilde{f}(\Omega)$ into subsets $\widetilde{f}(\Omega_i)$, $i=1,\dots,s$, such that $\operatorname{diam}(\widetilde{f}(\Omega_i))\le d+\varepsilon$. Since the embedding $C^1([0,a];\mathbb{R}^n)\hookrightarrow C([0,a];\mathbb{R}^n)$ is completely continuous, the image $\Omega_C$ of $\Omega$ under this embedding is relatively compact. It is known (see, e.g., [2, 12]) that an u.s.c. compact-valued multimap sends compact sets to compact sets, then we can conclude that the set $\Pi(\Omega)$ is relatively compact. It means that taking a fixed $\delta>0$ and any $\Omega_i$, we may divide the sets $\Omega_{iC}$ and $\Pi(\Omega)$ into a finite number of subsets $\Omega_{ijC}$, $j=1,\dots,p_i$, and balls $D_{ik}(z_{ik})$, $k=1,\dots,r_i$, centered at $z_{ik}\in C([0,a];\mathbb{R}^m)$, respectively, such that for each $t\in[0,a]$; $u_1(\cdot),u_2(\cdot)\in\Omega_{ijC}$, $v\in\mathbb{R}^n$; $w_1(\cdot),w_2(\cdot)\in D_{ik}(z_{ik})$, we have

$$\begin{gathered}|A(t,u_1(t),v)-A(t,u_2(t),v)|<\delta,\\ |B(t,u_1(t),v,w_1(t))-B(t,u_2(t),v,w_2(t))|<\delta.\end{gathered} \tag{5.15}$$

Now, the set $\widetilde{G}(\Omega)$ is covered by a finite number of sets $\Gamma_{ijk}$, $i=1,\dots,s$; $j=1,\dots,p_i$; $k=1,\dots,r_i$ of the form

$$\Gamma_{ijk} = \{B(\cdot,x(\cdot),x'(\cdot),y(\cdot)) : x\in\Omega_{ijC},\ y\in D_{ik}(z_{ik})\}. \tag{5.16}$$

Let us estimate the diameters of these sets. Taking arbitrary $x_1, x_2 \in \Omega_{ijC}$ and $y_1, y_2 \in D_{ik}(z_{ik})$ and applying (5.15), and condition (AB), for any $t \in [0,a]$, we have

$$\begin{aligned} &\left|B(t,x_1(t),x_1'(t),y_1(t)) - B(t,x_2(t),x_2'(t),y_2(t))\right| \\ &\quad < \left|B(t,x_1(t),x_1'(t),z_{ik}(t)) - B(t,x_2(t),x_2'(t),z_{ik}(t))\right| + 2\delta \\ &\quad \le \left|B(t,x_1(t),x_1'(t),z_{ik}(t)) - B(t,x_1(t),x_2'(t),z_{ik}(t))\right| \\ &\qquad + \left|B(t,x_1(t),x_2'(t),z_{ik}(t)) - B(t,x_2(t),x_2'(t),z_{ik}(t))\right| + 2\delta \\ &\quad \le q\left|A(t,x_1(t),x_1'(t)) - A(t,x_1(t),x_2'(t))\right| + 3\delta \\ &\quad \le q\left|A(t,x_1(t),x_1'(t)) - A(t,x_2(t),x_2'(t))\right| \\ &\qquad + q\left|A(t,x_2(t),x_2'(t)) - A(t,x_1(t),x_2'(t))\right| + 3\delta \\ &\quad < q(d+\varepsilon) + q\delta + 3\delta. \end{aligned} \tag{5.17}$$

Now, if $q = 0$, it means, by the arbitrariness of the choice of $\delta > 0$, that $\alpha(\widetilde{G}(\Omega)) = 0$ and then the triplet $(\widetilde{f},\widetilde{G},\overline{U})$, and therefore $(f,G,\overline{U})$, is compact. Otherwise, let us take $\varepsilon > 0$ and $\delta > 0$ so small that

$$q\varepsilon + (q+3)\delta < (1-q)d. \tag{5.18}$$

Then, $q(d+\varepsilon) + q\delta + 3\delta = \mu d$, where $0 < \mu < 1$ and, hence $\operatorname{diam}\Gamma_{ijk} \le \mu d$, implying that

$$\alpha(\widetilde{G}(\Omega)) \le \mu\alpha(\widetilde{f}(\Omega)). \tag{5.19}$$

□

The proved statement implies that the coincidence index theory, developed in the previous sections, can be applied to the study of the solvability of problem (5.1)–(5.3). Moreover, it is easy to see that the coincidence point set $\operatorname{Coin}(f,G)$ of a condensing triplet $(f,G,\overline{U})_\beta$ is a compact set. In case when problem (5.1)–(5.3) is a model for a control system, this approach can be used also to obtain the existence of optimal solutions. As an example, we can consider the following statement.

Proposition 5.4. *Under the above conditions, suppose that the map $A$ is odd: $A(t,-u,-v) = A(t,u,v)$ for all $t \in [0,a]$; $u,v \in \mathbb{R}^n$ and the set of functions $x \in C^1([0,a];\mathbb{R}^n)$ satisfying the family of relations*

$$\begin{aligned} A(t,x(t),x'(t)) &= \lambda B(t,x(t),x'(t),y(t)), \quad \lambda \in [0,1], \\ y'(t) &\in C(t,x(t),y(t)), \\ x(0) = x_0&, \qquad y(0) = y_0 \end{aligned} \tag{5.20}$$

*is a priori bounded. Then, there exists a solution $(x_*,y_*)$ of problem (5.1)–(5.3) minimizing a given lower-semicontinuous functional*

$$l : C^1([0,a];\mathbb{R}^n) \times C([0,a];\mathbb{R}^m) \longrightarrow \mathbb{R}_+. \tag{5.21}$$

*Proof.* The application of Theorem 4.11 yields that the set $Q = \mathrm{Coin}(f,G)$ is nonempty and compact. It remains only to notice that the set of solutions $\{(x,y)\}$ of (5.1)–(5.3) is closed and it is contained in the compact set $Q \times \Pi(Q) \subset C^1([0,a];\mathbb{R}^n) \times C([0,a];\mathbb{R}^m)$. □

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