EXISTENCE OF GLOBAL SOLUTION AND NONTRIVIAL STEADY STATES FOR A SYSTEM MODELING CHEMOTAXIS

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We consider a reaction-diffusion system modeling chemotaxis, which describes the situation of two species of bacteria competing for the same nutrient. We use Moser-Alikakos iteration to prove the global existence of the solution. We also study the existence of nontrivial steady state solutions and their stability.

1. Introduction

Chemotaxis, the oriented movement of cells in response to ambient chemical gradients, is a prominent feature in the organization of many biological populations. Since the pioneer work of Keller and Segel [11] to propose mathematical models for chemotaxis, there has been great interest in modeling chemotaxis and in the mathematical analysis of systems like the Keller-Segel model. In this paper, motivated by the model in [15], we consider a revised model discussed in [16], that is,

$$\begin{gathered}
\frac{\partial N}{\partial t} = \mu N_{xx} - R_1(N)b - R_2(N)B, \quad 0<x<1,\ t>0,\\
\frac{\partial b}{\partial t} = \left(db_x - \alpha b S_1(N)N_x\right)_x + b\left(\rho_1 R_1(N) - b - B\right), \quad 0<x<1,\ t>0,\\
\frac{\partial B}{\partial t} = \left(DB_x - \beta B S_2(N)N_x\right)_x + B\left(\rho_2 R_2(N) - b - B\right), \quad 0<x<1,\ t>0,\\
N_x(0,t)=0, \quad N_x(1,t) = \gamma\left(1-N(1,t)\right), \quad t>0,\\
db_x - \alpha b S_1(N)N_x = 0, \quad x=0,1,\ t>0,\\
DB_x - \beta B S_2(N)N_x = 0, \quad x=0,1,\ t>0,\\
N(x,0)=N_0(x), \quad b(x,0)=b_0(x), \quad B(x,0)=B_0(x), \quad 0<x<1.
\end{gathered} \tag{1.1}$$

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This is the situation of two species of bacteria competing for the same nutrient, where $N(x,t)$ is the concentration of the nutrient and $b(x,t)$, $B(x,t)$ are the densities of two competing species of bacteria. $R_i(N)$, $i = 1,2$, are the consumption rates of the nutrient per cell. $-\mu N_x$, $-db_x$, and $-DB_x$ are the random fluxes of N, b, and B, respectively, while $\alpha b S_1(N)N_x$ and $\beta B S_2(N)N_x$ are the chemotactic fluxes of b and B, where $\mu > 0$, $d > 0$, $D > 0$ and $\alpha \geq 0$, $\beta \geq 0$. For definiteness, we assume that $d < D$. Functions $S_i(N)$, $i = 1,2$, the so-called sensitivity rates, are included to indicate that the sensitivity of cells to the nutrient may vary at different levels of nutrient concentration. When $\alpha = 0$, $\beta = 0$, and $\rho_i = 1$, this model reduces to the model discussed in [16]. But the present model is not a trivial generalization of the model discussed in [16] because of the appearance of the chemotactic fluxes of b and B. Due to the lack of monotone structure on the system, the main tool—the comparison principle—used in [16] does not work here. In [15], the authors considered a similar model and discussed the situations when there is no positive steady state. In this paper, we will give sufficient conditions that guarantee the existence of positive solutions. The method we use here to investigate the existence of steady states is different from that used in [15]. We also consider some special cases in which the sufficient conditions we will derive are not satisfied and the systems have no nontrivial steady states. The boundary conditions represent that the total fluxes of b and B at the boundary points $x = 0$ and $x = 1$ are zero. This is true for N at $x = 0$, but at $x = 1$, N is diffused into the medium. In the adjacent region, $N \equiv 1$, which must also be an upper bound for N inside the medium, and therefore we are only interested in solutions with $0 \leq N \leq 1$. For this reason, we assume that $0 \leq N_0 \leq 1$ throughout the paper.

From biological and technical considerations, we assume that

$$R_i(0) = 0, \quad R_i'(N) > 0, \quad S_i(N) > 0 \quad \text{on } [0,\infty). \tag{1.2}$$

The assumptions about R_i guarantee the nonnegativeness of N, b, and B as long as the initial functions are nonnegative (see [13]). Therefore we will only consider nonnegative solutions of (1.1).

This paper is organized as follows. In Section 2, we will prove the global existence of solutions. In Section 3, we will study the existence of steady states and some special cases.

2. Global existence

By standard existence theory, for example, see [3–5, 12], it is not difficult to establish the local existence of the unique solution $(N(x,t), b(x,t), B(x,t))$ for $0 \leq t < T_{\max}$, where $T_{\max}$ is determined by N_0, b_0, and B_0. It is well known that local existence together with L^∞ a priori bounds ensure the global existence of classical solutions. Therefore, to establish the global existence, we need only to establish a priori estimates for $\|N(\cdot,t)\|_{L^\infty}$, $\|b(\cdot,t)\|_{L^\infty}$, and $\|B(\cdot,t)\|_{L^\infty}$. The boundedness of $\|N(\cdot,t)\|_{L^\infty}$ is trivial because we have $0 \leq N \leq 1$ (in fact, this can be proved directly by using comparison principle). Therefore we need only to establish the boundedness of $\|b(\cdot,t)\|_{L^\infty}$, and $\|B(\cdot,t)\|_{L^\infty}$. This is done by proving several lemmas. The following general imbedding result will be of use to us.

THEOREM 2.1 (see [10, 2]). *Assume that operator $\mathscr{A}$ is sectorial in $X = L^p(\Omega)$, $1 < p < \infty$, where Ω is a bounded smooth domain in $\mathbb{R}^n$ and $X^1 = D(\mathscr{A}) \hookrightarrow W^{m,p}(\Omega)$ (where $\hookrightarrow$ denotes topological as well as set inclusion) for some integer $m \geq 1$. Then for $0 < \theta < 1$,*

(i) $X^\theta \hookrightarrow W^{l,q}(\Omega)$ *if* $l = 0, 1, 2, \ldots, m-1$, $1 \leq q \leq +\infty$, *where* $\theta > l/m$ *and* $1/q > 1/p - (\theta m - l)/n$;

(ii) $X^\theta \hookrightarrow C^\nu(\Omega)$ *if* $0 \leq \nu < \theta m - n/p$.

LEMMA 2.2. *If* $1 \leq p < \infty$, *and* $f \geq 0$, $g \geq 0$, *and* $h \geq 0$, *then*

$$(f+g+h)^p \leq 2^{2p-2}\left(f^p + g^p + h^p\right). \tag{2.1}$$

Proof. It is well known that if $1 \leq p < \infty$, and $a \geq 0$, $b \geq 0$, then (e.g., see [1])

$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right). \tag{2.2}$$

Therefore,

$$\begin{aligned}(f+g+h)^p &\leq 2^{p-1}\left((f+g)^p + h^p\right) \leq 2^{p-1}\left(2^{p-1}\left(f^p+g^p\right)+h^p\right)\\ &\leq 2^{2p-2}\left(f^p+g^p+h^p\right).\end{aligned} \tag{2.3}$$

□

LEMMA 2.3. *There exist positive constants* c_b *and* C_B *such that for* $0 \leq t < T_{\max}$,

$$\begin{aligned}\overline{b}(t) &= \int_0^1 b(x,t)\,dx \leq c_b,\\ \overline{B}(t) &= \int_0^1 B(x,t)\,dx \leq C_B.\end{aligned} \tag{2.4}$$

Proof. Let

$$\Lambda(t) = \int_0^1 b(x,t)\,dx + \int_0^1 B(x,t)\,dx = \overline{b}(t) + \overline{B}(t). \tag{2.5}$$

Obviously, $\overline{b}(t) \geq 0$ and $\overline{B}(t) \geq 0$. Therefore, to prove the lemma, we need only to prove that there exists a constant $M > 0$ such that for $0 \leq t < T_{\max}$,

$$\Lambda(t) \leq M. \tag{2.6}$$

In fact, by adding the b-equation and B-equation in (1.1) and using the boundary conditions, we have

$$\begin{aligned}\Lambda'(t) &= \int_0^1 \left(\rho_1 R_1(N) b(x,t) + \rho_2 R_2(N) B(x,t)\right) dx - \int_0^1 \left(b(x,t)+B(x,t)\right)^2 dx\\ &\leq R\int_0^1 \left(b(x,t)+B(x,t)\right)dx - \left(\int_0^1 \left(b(x,t)+B(x,t)\right)dx\right)^2 = R\Lambda(t) - \left(\Lambda(t)\right)^2,\end{aligned} \tag{2.7}$$

where $R = \max\{\rho_1 R_1(1), \rho_2 R_2(1)\}$. This implies (2.6). □

Lemma 2.4. *For any small $\tau > 0$, there exists a positive constant δ depending on R_i, S_i, b_0, B_0, and $\|N(x,\tau)\|_{H^2(0,1)}$ such that*

$$\begin{aligned}&\max\left\{\int_0^1 b^2(x,t)dx, \int_0^1 B^2(x,t)dx\right\}\\&\quad\le 2\left(\int_0^1 b^2(x,\tau)dx+\int_0^1 B^2(x,\tau)dx\right)+\delta, \quad \tau\le t<T_{\max}.\end{aligned} \tag{2.8}$$

Proof. For $1<p<2$, let $X=L^p(0,1)$. Operator $\mathcal{A}_p=-\mu(d^2/dx^2)$ with domain

$$D(\mathcal{A}_p)=\{u\in W^{2,p}(0,1):u'(0)=0=u'(1)+\gamma u(1)\} \tag{2.9}$$

is sectorial in X and $\sigma(\mathcal{A}_p)\subset\{\mathbf{Z}\in\mathbb{R}:\mathbf{Z}>\lambda_0\}$ for a positive number λ_0 due to the symmetry of $\mathcal{A}_p$, where $\sigma(\mathcal{A}_p)$ is the spectrum of $\mathcal{A}_p$.

Since $\mathcal{A}_p$ is sectorial in X, the operator $-\mathcal{A}_p$ generates an analytic semigroup $\{\mathcal{T}_p(t)\}$ with $\|\mathcal{T}_p(t)\|_X\le ke^{-\lambda_0 t}$ for $t\ge 0$, for a positive constant k.

By Theorem 2.1, we know that, for $1\ge\theta>1/4+1/2p$, fractional space $X^\theta\hookrightarrow H^1(0,1)$. By Theorem 1.4.3 in [10], there exists a constant $k_\theta<\infty$ such that

$$||\mathcal{T}_p(t)||_{X^\theta}\le k_\theta t^{-\theta}e^{-\lambda_0 t}. \tag{2.10}$$

Let $u=1-N$, then u satisfies

$$\begin{gathered}u_t=\mu u_{xx}+R_1(N)b+R_2(N)B, \quad 0<x<1,\ t>0,\\u_x(0,t)=0, \quad u_x(1,t)+\gamma u(1,t)=0, \quad t>0,\\u(x,\tau)=1-N(x,\tau), \quad 0<x<1.\end{gathered} \tag{2.11}$$

Therefore $u\in D(\mathcal{A}_p)$ and for $\tau\le t<T_{\max}$,

$$\begin{aligned}u(\cdot,t)&=\mathcal{T}_p(t-\tau)(1-N(\cdot,\tau))\\&\quad+\int_\tau^t\mathcal{T}_p(t-\xi)(R_1(N(\cdot,\xi))b(\cdot,\xi)+R_2(N(\cdot,\xi))B(\cdot,\xi))d\xi.\end{aligned} \tag{2.12}$$

Now for $\tau\le t<T_{\max}$, we have

$$\begin{aligned}||N_x(\cdot,t)||_{L^2(0,1)}&\le||u(\cdot,t)||_{H^1(0,1)}\le C||u(\cdot,t)||_{X^\theta}\le C||\mathcal{T}_p(t-\tau)(1-N(\cdot,\tau))||_{X^\theta}\\&\quad+C\int_\tau^t||\mathcal{T}_p(t-\xi)(R_1(N(\cdot,\xi))b(\cdot,\xi)+R_2(N(\cdot,\xi))B(\cdot,\xi))||_{X^\theta}d\xi\\&\le C||\mathcal{T}_p(t-\tau)||_X||1-N(\cdot,\tau)||_{X^\theta}\\&\quad+C\int_\tau^t||\mathcal{T}_p(t-\xi)||_{X^\theta}||R_1(N(\cdot,\xi))b(\cdot,\xi)+R_2(N(\cdot,\xi))B(\cdot,\xi)||_Xd\xi\\&\le Ke^{-\lambda_0(t-\tau)}||1-N(\cdot,\tau)||_{X^\theta}+CR_1(1)\int_\tau^t k_\theta(t-\xi)^{-\theta}e^{-\lambda_0(t-\xi)}||b(\cdot,\xi)||_Xd\xi\\&\quad+CR_2(1)\int_\tau^t k_\theta(t-\xi)^{-\theta}e^{-\lambda_0(t-\xi)}||B(\cdot,\xi)||_Xd\xi.\end{aligned} \tag{2.13}$$

Let

$$\omega(t)=\max_{\tau\le\xi<t}\int_0^1 b^2(x,\xi)\,dx,$$
$$\Gamma(t)=\max_{\tau\le\xi<t}\int_0^1 B^2(x,\xi)\,dx,\tag{2.14}$$

then it is easily seen that $\omega(t)$ and $\Gamma(t)$ are nondecreasing functions of t. By Hölder's inequality

$$\|b(\cdot,\xi)\|_X\le \overline{b}^{(2-p)/p}(t)\left(\int_0^1 b^2(x,\xi)\,dx\right)^{(p-1)/p}\le \overline{b}^{(2-p)/p}(t)\left(\omega(t)\right)^{(p-1)/p},$$
$$\|B(\cdot,\xi)\|_X\le \overline{B}^{(2-p)/p}(t)\left(\Gamma(t)\right)^{(p-1)/p},\tag{2.15}$$

where $\overline{b}(t)$ and $\overline{B}(t)$ are defined in Lemma 2.3. Therefore, from (2.13) and Lemma 2.3, we know that there exist constants k_1 depending on $\|N(x,\tau)\|_{H^2(0,1)}$, k_2, and k_3 depending on R_i such that

$$\|N_x(\cdot,t)\|_{L^2(0,1)}\le k_1+k_2\left(\omega(t)\right)^{(p-1)/p}+k_3\left(\Gamma(t)\right)^{(p-1)/p}.\tag{2.16}$$

Now multiplying $b(x,t)$ to the b-equation in (1.1) and integrating by parts on $[0,1]$, we obtain that for $\tau\le t<T_{\max}$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 b^2(x,t)\,dx&\le-\int_0^1\left(db_x-\alpha bS_1(N)N_x\right)b_x\,dx+\int_0^1\rho_1R_1(N)b^2(x,t)\,dx\\&\le -d\int_0^1 b_x^2\,dx+\alpha\int_0^1 S_1(N)N_xbb_x\,dx+\rho_1R_1(1)\int_0^1 b^2(x,t)\,dx\\&\le -d\int_0^1 b_x^2\,dx+c_1\|b(\cdot,t)\|_{L^\infty(0,1)}\left(\int_0^1 N_x^2\,dx\right)^{1/2}\left(\int_0^1 b_x^2\,dx\right)^{1/2}\\&\quad+\rho_1R_1(1)\int_0^1 b^2(x,t)\,dx.\end{aligned}\tag{2.17}$$

From this and the inequality (see [14])

$$\|b(\cdot,t)\|_{L^\infty(0,1)}\le c\left(\overline{b}(t)+\overline{b}^{1/3}(t)\left(\int_0^1 b_x^2(x,t)\,dx\right)^{1/3}\right),\tag{2.18}$$

we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 b^2(x,t)dx \\
&\quad\le -d\int_0^1 b_x^2dx + c_2\left(\overline{b}(t)+\overline{b}^{1/3}(t)\left(\int_0^1 b_x^2dx\right)^{1/3}\right)\left(\int_0^1 N_x^2dx\right)^{1/2}\left(\int_0^1 b_x^2dx\right)^{1/2} \\
&\qquad + \rho_1 R_1(1)\int_0^1 b^2(x,t)dx \\
&\quad= -d\int_0^1 b_x^2dx + c_2\overline{b}(t)\left(\int_0^1 N_x^2dx\right)^{1/2}\left(\int_0^1 b_x^2dx\right)^{1/2} \\
&\qquad + c_2\overline{b}^{1/3}(t)\left(\int_0^1 N_x^2dx\right)^{1/2}\left(\int_0^1 b_x^2dx\right)^{5/6} + \rho_1 R_1(1)\int_0^1 b^2(x,t)dx.
\end{aligned}
\tag{2.19}
$$

From Young's inequality

$$
XY \le \eta X^p + c(\eta)Y^q, \tag{2.20}
$$

(where $1/p+1/q=1$, $c(\eta)=(\eta p)^{-q/p}/q$) with $p=q=2$, $\eta=d/4$ and $p=6/5$, $q=6$, $\eta=d/4$, respectively, we have

$$
\begin{aligned}
c_2\overline{b}(t)\left(\int_0^1 N_x^2dx\right)^{1/2}\left(\int_0^1 b_x^2dx\right)^{1/2} &\le \frac{d}{4}\int_0^1 b_x^2dx + \frac{1}{d}c_2^2\overline{b}^2(t)\int_0^1 N_x^2dx, \\
c_2\overline{b}^{1/3}(t)\left(\int_0^1 N_x^2dx\right)^{1/2}\left(\int_0^1 b_x^2dx\right)^{5/6} &\le \frac{d}{4}\int_0^1 b_x^2dx + c_3\overline{b}^2(t)\left(\int_0^1 N_x^2dx\right)^3.
\end{aligned}
\tag{2.21}
$$

Using (2.21) in (2.19), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 b^2(x,t)dx &\le -d\int_0^1 b_x^2dx + \frac{d}{4}\int_0^1 b_x^2dx + \frac{1}{d}c_2^2\overline{b}^2(t)\int_0^1 N_x^2dx \\
&\quad + \frac{d}{4}\int_0^1 b_x^2dx + c_3\overline{b}^2(t)\left(\int_0^1 N_x^2dx\right)^3 + \rho_1 R_1(1)\int_0^1 b^2(x,t)dx \\
&= -\frac{d}{2}\int_0^1 b_x^2dx + \frac{1}{d}c_2^2\overline{b}^2(t)\int_0^1 N_x^2dx + c_3\overline{b}^2(t)\left(\int_0^1 N_x^2dx\right)^3 \\
&\quad + \rho_1 R_1(1)\int_0^1 b^2(x,t)dx \\
&\le -\frac{d}{2}\int_0^1 b_x^2dx + c_4\overline{b}^2(t) + c_5\overline{b}^2(t)\left(\int_0^1 N_x^2dx\right)^3 + \rho_1 R_1(1)\int_0^1 b^2(x,t)dx.
\end{aligned}
\tag{2.22}
$$

For any $\epsilon>0$, from the following inequality (see [14]),

$$
\int_0^1 b^2(x,t)dx \le \epsilon\int_0^1 b_x^2dx + (c\epsilon^{-1/2}+1)\overline{b}^2(t), \tag{2.23}
$$

we have

$$\int_0^1 b_x^2 dx \geq \frac{1}{\epsilon}\left(\int_0^1 b^2(x,t)dx - (c\epsilon^{-1/2}+1)\overline{b}^2(t)\right). \tag{2.24}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 b^2(x,t)dx \leq & -\frac{d}{2\epsilon}\left(\int_0^1 b^2(x,t)dx - (c\epsilon^{-1/2}+1)\overline{b}^2(t)\right) \\
& + c_4\overline{b}^2(t) + c_5\overline{b}^2(t)\left(\int_0^1 N_x^2 dx\right)^3 + \rho_1 R_1(1)\int_0^1 b^2(x,t)dx \\
\leq & \left(\rho_1 R_1(1) - \frac{d}{2\epsilon}\right)\int_0^1 b^2(x,t)dx + c_6\overline{b}^2(t) + c_5\overline{b}^2(t)\left(\int_0^1 N_x^2 dx\right)^3.
\end{aligned} \tag{2.25}$$

By taking $\epsilon = d/(3\rho_1 R_1(1))$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 b^2(x,t)dx \leq -\frac{\rho_1 R_1(1)}{2}\int_0^1 b^2(x,t)dx + c_6\overline{b}^2(t) + c_5\overline{b}^2(t)\left(\int_0^1 N_x^2 dx\right)^3. \tag{2.26}$$

Then, in view of (2.16) and Lemma 2.2, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 b^2(x,t)dx \leq & -\frac{\rho_1 R_1(1)}{2}\int_0^1 b^2(x,t)dx \\
& + c_6\overline{b}^2(t) + c_5\overline{b}^2(t)\left(k_1 + k_2(\omega(t))^{(p-1)/p} + k_3(\Gamma(t))^{(p-1)/p}\right)^6 \\
\leq & -\frac{\rho_1 R_1(1)}{2}\int_0^1 b^2(x,t)dx + c_7\overline{b}^2(t) + c_8\overline{b}^2(t)(\omega(t))^{6(p-1)/p} \\
& + c_9\overline{b}^2(t)(\Gamma(t))^{6(p-1)/p}.
\end{aligned} \tag{2.27}$$

From this, the monotonicity of $\omega(t)$ and $\Gamma(t)$ and Lemma 2.3, it follows that for $\tau \leq t < T_{\max}$,

$$\int_0^1 b^2(x,t)dx \leq \int_0^1 b^2(x,\tau)dx + c_{10}\left(1 + (\omega(t))^{6(p-1)/p} + (\Gamma(t))^{6(p-1)/p}\right). \tag{2.28}$$

Thus

$$\omega(t) \leq \int_0^1 b^2(x,\tau)dx + c_{10}\left(1 + (\omega(t))^{6(p-1)/p} + (\Gamma(t))^{6(p-1)/p}\right). \tag{2.29}$$

Similarly, we have

$$\Gamma(t) \leq \int_0^1 B^2(x,\tau)dx + c_{11}\Big(1+(\omega(t))^{6(p-1)/p}+(\Gamma(t))^{6(p-1)/p}\Big). \tag{2.30}$$

Therefore,

$$\begin{aligned}\omega(t)+\Gamma(t) &\leq \int_0^1 b^2(x,\tau)dx+\int_0^1 B^2(x,\tau)dx + c_{12}\Big(1+(\omega(t))^{6(p-1)/p}+(\Gamma(t))^{6(p-1)/p}\Big)\\ &\leq \int_0^1 b^2(x,\tau)dx+\int_0^1 B^2(x,\tau)dx + c_{12}+c_{13}(\omega(t))+\Gamma(t))^{6(p-1)/p}.\end{aligned} \tag{2.31}$$

Now we take $6/5 > p > 1$, then $6(p-1)/p < 1$. Therefore there exists a positive constant δ depending on R_i, S_i, ρ_i, b_0, B_0, and $\|N(x,\tau)\|_{H^2(0,1)}$ such that

$$\omega(t)+\Gamma(t) \leq 2\Big(\int_0^1 b^2(x,\tau)dx+\int_0^1 B^2(x,\tau)dx\Big)+\delta, \quad \tau \leq t < T_{\max}. \tag{2.32}$$

From this we know that the lemma is true. □

Lemma 2.5. *For any small $\tau > 0$, there exists a positive constant $L = L(T_{\max})$ such that*

$$||N_x(\cdot,t)||_{L^\infty(0,1)} \leq L, \quad \tau \leq t < T_{\max}. \tag{2.33}$$

Proof. Let $X = L^2(0,1)$. Operator $\mathcal{A}_2 = -\mu(d^2/dx^2)$ with domain

$$D(\mathcal{A}_2) = \{u \in H^2(0,1) : u'(0) = 0 = u'(1)+\gamma u(1)\} \tag{2.34}$$

is sectorial in X and $\sigma(\mathcal{A}_2) \subset \{\mathbf{Z} \in \mathbb{R} : \mathbf{Z} > \lambda_0\}$ for a positive number λ_0 due to the symmetry of $\mathcal{A}_2$, where $\sigma(\mathcal{A}_2)$ is the spectrum of $\mathcal{A}_2$.

Since $\mathcal{A}_2$ is sectorial in X, the operator $-\mathcal{A}_2$ generates an analytic semigroup $\{\mathcal{T}_2(t)\}$ with $\|\mathcal{T}_2(t)\|_X \leq ke^{-\lambda_0 t}$, for some constant k, for $t \geq 0$.

By Theorem 2.1, we know that, for $1 > \theta > 1/4$, fractional space $X^\theta \hookrightarrow C^\nu[0,1]$ for $\nu \in (0, 2\theta - 1/2)$. In particular, we take $\theta > 3/4$, then ν can be taken to be 1 and $X^\theta \hookrightarrow C^1[0,1]$.

Therefore, from (2.12) with $p = 2$, we have, for $\tau \le t < T_{\max}$,

$$
\begin{aligned}
\|N_x(\cdot,t)\|_{C[0,1]} \le \|u(\cdot,t)\|_{C^1[0,1]} \le \|u(\cdot,t)\|_{X^\theta} &\le \|\mathcal{T}_2(t-\tau)(1-N(\cdot,\tau))\|_{X^\theta} \\
&\quad + \int_\tau^t \|\mathcal{T}_2(t-\xi)(R_1(N(\cdot,\xi))b(\cdot,\xi)+R_2(N(\cdot,\xi))B(\cdot,\xi))\|_{X^\theta} d\xi \\
&\le \|\mathcal{T}_2(t-\tau)\|_X \|1-N(\cdot,\tau)\|_{X^\theta} \\
&\quad + \int_\tau^t \|\mathcal{T}_2(t-\xi)\|_{X^\theta} \|(R_1(N(\cdot,\xi))b(\cdot,\xi)+R_2(N(\cdot,\xi))B(\cdot,\xi))\|_X d\xi \\
&\le ke^{-\lambda_0(t-\tau)}\|1-N(\cdot,\tau)\|_{X^\theta} + R\int_\tau^t k_\theta(t-\xi)^{-\theta}e^{-\lambda_0(t-\xi)}\|b(\cdot,\xi)+B(\cdot,\xi)\|_X d\xi \\
&\le ke^{-\lambda_0(t-\tau)}\|1-N(\cdot,\tau)\|_{X^\theta} \\
&\quad + R\int_\tau^t k_\theta(t-\xi)^{-\theta}e^{-\lambda_0(t-\xi)}(\|b(\cdot,\xi)\|_X + \|B(\cdot,\xi)\|_X) d\xi \\
&\le ke^{-\lambda_0(t-\tau)}\|1-N(\cdot,\tau)\|_{X^\theta} \\
&\quad + R\int_\tau^t 2\left(2\left(\int_0^1 b^2(x,\tau)dx + \int_0^1 B^2(x,\tau)dx\right)+\delta\right)^{1/2} k_\theta(t-\xi)^{-\theta}e^{-\lambda_0(t-\xi)} d\xi \\
&= ke^{-\lambda_0(t-\tau)}\|1-N(\cdot,\tau)\|_{X^\theta} \\
&\quad + 2R\left(2\left(\int_0^1 b^2(x,\tau)dx + \int_0^1 B^2(x,\tau)dx\right)+\delta\right)^{1/2} \int_\tau^t k_\theta(t-\xi)^{-\theta}e^{-\lambda_0(t-\xi)} d\xi,
\end{aligned}
\tag{2.35}
$$

where $R = \max\{R_1(1), R_2(1)\}$. This completes the proof. □

Lemma 2.6. *There exists a positive constant $M > 0$ such that for $\tau \le t < T_{\max}$,*

$$\max\{\|b(\cdot,t)\|_{L^\infty}, \|B(\cdot,t)\|_{L^\infty}\} \le M. \tag{2.36}$$

The proof is similar to that of Lemma 4.7 in [14] and therefore is omitted.
Thus we have the following global existence and boundedness theorem.

Theorem 2.7 (global existence and boundedness). *For any N_0, b_0, and $B_0 \in H^1(0,1)$ satisfying $N_0(x) > 0$, $b_0(x) > 0$, and $B_0(x) > 0$ on $[0,1]$, (1.1) has a unique positive global solution (N,b,B) such that*

(i) $(N(x,t), b(x,t), B(x,t)) \in C([0,\infty), H^1(0,1)\times H^1(0,1)\times H^1(0,1))$,

$$(N,b,B) \in C^{2+2\epsilon,1+\epsilon}_{\text{loc}}([0,1]\times(0,\infty)); \tag{2.37}$$

(ii) *$N > 0$, $b > 0$, and $B > 0$ are bounded on $[0,1]\times[0,\infty)$.*

3. Existence of steady states

In this section, we study the existence of steady states of (1.1). Basically, we study the existence of nontrivial steady state solutions of (1.1) in the framework of [9]. But in [9], the author made several assumptions about the reaction terms. Unfortunately, in our model the reaction functions do not satisfy all these assumptions. This fact causes difficulties in using the theory developed in [9]. Therefore, we must do some careful and technical analysis for our model. The steady states of (1.1) satisfy

$$
\begin{gathered}
\mu N'' - R_1(N)b - R_2(N)B = 0, \quad 0 < x < 1, \\
\left(db' - \alpha b S_1(N)N'\right)' + b\left(\rho_1 R_1(N) - b - B\right) = 0, \quad 0 < x < 1, \\
\left(DB' - \beta B S_2(N)N'\right)' + B\left(\rho_2 R_2(N) - b - B\right) = 0, \quad 0 < x < 1, \\
N'(0) = 0, \quad N'(1) = \gamma\left(1 - N(1)\right), \\
db' - \alpha b S_1(N)N' = 0 \quad \text{at } x = 0, 1, \\
DB' - \beta B S_2(N)N' = 0 \quad \text{at } x = 0, 1.
\end{gathered}
\tag{3.1}
$$

Obviously, $(1,0,0)$ is a solution of (3.1), that is, it is a steady state of (1.1). For this, we have the following theorem.

Theorem 3.1. *The trivial steady state solution* $(N,b,B) = (1,0,0)$ *is unstable.*

Proof. To prove this theorem, we use the definition of instability (e.g., see [6]). That is, if O_ϵ is a neighborhood of $(1,0,0)$ consisting of (N,b,B) such that

$$
\|1 - N\|_{L^\infty} + \|b\|_{L^\infty} + \|B\|_{L^\infty} < \epsilon, \tag{3.2}
$$

we can show that for a small $\epsilon > 0$, the solution $(N(x,t), b(x,t), B(x,t))$ always leaves O_ϵ in finite time no matter how close the initial values (N_0, b_0, B_0) are to $(1,0,0)$. In fact, for $\epsilon > 0$ small, we have

$$
\rho_1 R_1(N) - b - B \geq \frac{\rho_1 R_1(1)}{2} > 0, \quad x \in [0,1],\ t > 0. \tag{3.3}
$$

Then by integrating the b-equation in (1.1), we have

$$
\frac{d}{dt}\int_0^1 b(x,t)dx \geq \frac{\rho_1 R_1(1)}{2}\int_0^1 b(x,t)dx, \quad t > 0. \tag{3.4}
$$

It follows that

$$
\int_0^1 b(x,t)\,dx \geq e^{(\rho_1 R_1(1)/2)t}\int_0^1 b_0(x)\,dx \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \tag{3.5}
$$

This implies that $(N(x,t), b(x,t), B(x,t))$ must leave O_ϵ in finite time. □

Lemma 3.2. *If* $(N(x), b(x), B(x))$ *is a solution of (3.1) such that at least one of the functions* $b(x)$ *and* $B(x)$ *is positive, then*

$$
0 < N'(x) < \gamma \quad \textit{for } 0 < x \leq 1, \qquad 0 < N(x) < 1 \quad \textit{for } 0 < x \leq 1, \tag{3.6}
$$

and there exists a positive constant K_1 *such that*

$$
\max\left\{\|b\|_{L^1}, \|B\|_{L^1}\right\} \leq K_1. \tag{3.7}
$$

Also

(i) $0 \le b(x) \le \rho_1 R_1(1) e^{(\alpha/d)\int_0^1 S_1(y)dy}$, *and* $0 \le B(x) \le \rho_2 R_2(1) e^{(\beta/D)\int_0^1 S_2(y)dy}$,

(ii) *there exists a positive constant* K_2 *such that*

$$\max\{|b'(x)|, |B'(x)|\} \le K_2. \tag{3.8}$$

Proof. From the assumptions and the first equation of (3.1), we have $N''(x) > 0$. Therefore $N'(x)$ is increasing. From $N'(0) = 0$, we know that $N'(x) > 0$ for $0 < x \le 1$. In particular, we have $N'(1) > 0$. Then the N-boundary condition at $x = 1$ implies that $N'(1) < \gamma$. Since $N'(x)$ is increasing, we must have $N'(x) < \gamma$ for $0 \le x \le 1$. Also, from the N-boundary condition at $x = 1$, we have $N(1) < 1$. Observe that $N(x)$ is also increasing, hence for $0 \le x \le 1$, $N(x) < 1$. By the comparison principle and the condition $R_i(0) = 0$, we have $N(x) > 0$ for $x > 0$.

Integrating the b-equation in (3.1) from 0 to 1 and using the boundary conditions, we have

$$\int_0^1 b(\rho_1 R_1(N) - b - B)\,dx = 0. \tag{3.9}$$

Therefore,

$$\int_0^1 b\rho_1 R_1(N)dx = \int_0^1 b(b+B)dx = \int_0^1 b^2 dx + \int_0^1 bB\,dx. \tag{3.10}$$

Similarly,

$$\int_0^1 B\rho_2 R_2(N)dx = \int_0^1 B(b+B)dx = \int_0^1 B^2 dx + \int_0^1 bB\,dx. \tag{3.11}$$

Adding (3.10) and (3.11), we obtain

$$\int_0^1 (\rho_1 R_1(N)b + \rho_2 R_2(N)B)\,dx = \int_0^1 (b+B)^2 dx \ge \left(\int_0^1 (b+B)dx\right)^2. \tag{3.12}$$

It follows that

$$\left(\int_0^1 (b+B)dx\right)^2 \le \int_0^1 (\rho_1 R_1(N)b + \rho_2 R_2(N)B)\,dx \le \overline{R}\int_0^1 (b+B)dx, \tag{3.13}$$

where $\overline{R} = \max\{\rho_1 R_1(1), \rho_2 R_2(1)\}$. This implies that there is a constant K such that

$$\int_0^1 (b+B)dx \le K. \tag{3.14}$$

In turn, this implies that (3.7) is true.

Let $G_1(\tau) = \int_0^\tau S_1(y)dy \ge 0$, $G_2(\tau) = \int_0^\tau S_2(y)dy \ge 0$ and $z = e^{-(\alpha/d)G_1(N)}b$, $Z = e^{-(\beta/D)G_2(N)}B$, then we have

$$b' - \frac{\alpha}{d} S_1(N)N'b = e^{(\alpha/d)G_1(N)}z', \tag{3.15}$$

and z satisfies

$$\left(e^{(\alpha/d)G_1(N)}z'\right)' + d^{-1}e^{(\alpha/d)G_1(N)}z\left(\rho_1 R_1(N) - e^{(\alpha/d)G_1(N)}z - e^{(\beta/D)G_2(N)}Z\right) = 0, \quad 0 < x < 1,$$
$$z'(0) = z'(1) = 0. \tag{3.16}$$

Assume that $z(x)$ has its maximum at x_1. Then $z'(x_1) = 0$ and $z''(x_1) \leq 0$. From the above equation, we have, at x_1,

$$\rho_1 R_1(N) - e^{(\alpha/d)G_1(N)}z - e^{(\beta/D)G_2(N)}Z \geq 0. \tag{3.17}$$

Therefore,

$$z(x_1) \leq \rho_1 R_1(N)e^{-(\alpha/d)G_1(N)} \leq \rho_1 R_1(1). \tag{3.18}$$

This implies

$$z(x) \leq \rho_1 R_1(1). \tag{3.19}$$

Thus we have

$$b(x) \leq \rho_1 R_1(1)e^{(\alpha/d)G_1(1)} = \rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy}. \tag{3.20}$$

Similarly,

$$B(x) \leq \rho_2 R_2(1)e^{(\beta/D)\int_0^1 S_2(y)dy}. \tag{3.21}$$

Integrating the b-equation from 0 to x and using the b-boundary condition at $x = 0$, we have

$$\begin{aligned}
db'(x) &= \alpha b S_1(N)N' + \int_0^x b\left(b + B - \rho_1 R_1(N)\right)ds \\
&\leq \alpha b S_1(1)\gamma + \int_0^x b(b+B)ds \leq \alpha b S_1(1)\gamma + \int_0^1 b(b+B)ds \\
&\leq \alpha\rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy}S_1(1)\gamma \\
&\quad + \int_0^1 \rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy}\left(\rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy} + \rho_2 R_2(1)e^{(\beta/D)\int_0^1 S_2(y)dy}\right)ds \\
&= \alpha\rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy}S_1(1)\gamma \\
&\quad + \rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy}\left(\rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy} + \rho_2 R_2(1)e^{(\beta/D)\int_0^1 S_2(y)dy}\right) \\
&\leq \rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy}\left(\alpha S_1(1)\gamma + R\left(e^{(\alpha/d)\int_0^1 S_1(y)dy} + e^{(\beta/D)\int_0^1 S_2(y)dy}\right)\right).
\end{aligned} \tag{3.22}$$

Therefore,

$$b'(x) \leq d^{-1}\left\{\rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy}\left(\alpha S_1(1)\gamma + R\left(e^{(\alpha/d)\int_0^1 S_1(y)dy} + e^{(\beta/D)\int_0^1 S_2(y)dy}\right)\right)\right\}. \tag{3.23}$$

We also have

$$\begin{aligned} db'(x) &= \alpha b S_1(N)N' + \int_0^x b(b+B-\rho_1 R_1(N))\,ds \geq \int_0^x b(-\rho_1 R_1(N))\,ds \\ &\geq \int_0^x \rho_1 R_1(1) e^{(\alpha/d)\int_0^1 S_1(y)dy}(-\rho_1 R_1(1))\,ds \geq -(\rho_1 R_1(1))^2 e^{(\alpha/d)\int_0^1 S_1(y)dy}. \end{aligned} \tag{3.24}$$

Now, let

$$M_1 = d^{-1}\{\rho_1 R_1(1) e^{(\alpha/d)\int_0^1 S_1(y)dy}(\alpha S_1(1)\gamma + R(e^{(\alpha/d)\int_0^1 S_1(y)dy} + e^{(\beta/D)\int_0^1 S_2(y)dy}))\}, \tag{3.25}$$

then

$$|b'(x)| \leq M_1. \tag{3.26}$$

We can prove a similar estimate for $|B'(x)|$. Therefore (3.8) is true. □

Corollary 3.3. *For any $\nu \in (0,1)$, there is a positive constant K such that for any nontrivial solution (N,b,B) of (3.1),*

$$\max\{\|N\|_{C^\nu([0,1])}, \|b\|_{C^\nu([0,1])}, \|B\|_{C^\nu([0,1])}\} \leq K. \tag{3.27}$$

What we are interested in is whether (3.1) has any nontrivial solutions. The case $N \not\equiv 0$ is excluded by the boundary conditions. Therefore we need only to consider the possibilities of the existence of following two types of solutions:

(i) semitrivial solutions: $(N,b,0)$, $(N,0,B)$;

(ii) positive solutions: (N,b,B),

where the components $N > 0$, $b > 0$, $B > 0$. In what follows, we use the theory of fixed point index on cones in a Banach space to study the existence of solutions of these types. First we study the existence of semitrivial solutions.

3.1. Existence of semitrivial solutions. From the symmetry of b and B, we need only to study the existence of solutions of the form $(N,b,0)$. For the convenience of notations, we write N and b as u_0 and u_1, respectively, omit the subscripts of R_1, S_1, and ρ_1, and consider the system

$$\begin{gathered} \mu u_0'' - R(u_0)u_1 = 0, \quad 0 < x < 1, \\ (du_1' - \alpha u_1 S(u_0)u_0')' + u_1(\rho R(u_0) - u_1) = 0, \quad 0 < x < 1, \\ u_0'(0) = 0, \quad u_0'(1) = \gamma(1 - u_0(1)), \\ du_1' - \alpha u_1 S(u_0)u_0' = 0 \quad \text{at } x = 0,1. \end{gathered} \tag{3.28}$$

For $\nu \in (0,1)$, let

$$\begin{gathered} E_i = \{u \in C^\nu([0,1])\}, \quad i = 0,1, \\ C_i = \{u \in C^\nu([0,1]) : u \geq 0 \text{ on } [0,1]\}, \quad i = 0,1, \\ E = E_0 \oplus E_1, \qquad C = C_0 \oplus C_1, \end{gathered} \tag{3.29}$$

then E is an ordered Banach space with positive cone C. For $V=(v_0,v_1)\in C$, let $u_0=A_0(V)$ be the solution of

$$\begin{gathered} u_0''=\mu^{-1}R(u_0)v_1,\quad 0<x<1,\\ u_0'(0)=0,\qquad u_0'(1)+\gamma u_0(1)=\gamma. \end{gathered}\tag{3.30}$$

With $u_0=u_0(v_0,v_1)$ given, define operators Φ_1 and Γ_1 as follows.

$\Phi_1(V):C\to\mathfrak{L}(C^\nu([0,1]))$, the Banach space of bounded linear maps from $C^\nu([0,1])$ to itself, is defined by the following.

For $v\in C^\nu([0,1])$, let $u=\Phi_1(V)v$ be the solution of the problem

$$\begin{gathered} -u''+\left(\frac{\alpha}{d}(S(u_0)u_0'u\right)'+Pu=v,\quad 0<x<1,\\ du'-\alpha S(u_0)u_0'u=0\quad \text{at } x=0,1, \end{gathered}\tag{3.31}$$

and define

$$\Gamma_1(V)=d^{-1}v_1(\rho R(v_0)-v_1)+Pv_1,\tag{3.32}$$

where P is a positive constant such that $d^{-1}(\rho R(v_0)-v_1)+P>0$ for $0\le v_0\le 2$, $0\le v_1\le P_1$, where $P_1=2\rho R(1)e^{(\alpha/d)\int_0^1 S(y)dy}$. Then system (3.28) can be written as a fixed point equation $U=A(U)$, where $U=A(V)$ is given by

$$u_0=A_0(V),\qquad u_1=A_1(V)=\Phi_1(V)\circ\Gamma_1(V).\tag{3.33}$$

It is easily seen that Γ_1 satisfies,

$$\Gamma_1(u_0,0)=0.\tag{3.34}$$

Now we prove the following lemmas.

Lemma 3.4. *The operator $A(V)=(A_0(V),A_1(V)):\overline{\Omega}\to C$ is a well-defined completely continuous operator, where*

$$\Omega=\{(v_0,v_1)\in C:0\le v_0<2,0\le v_1<P_1\}.\tag{3.35}$$

Moreover, fixed points of A in C are nonnegative solutions of (3.28).

Proof. First we show that, for $V=(v_0,v_1)\in\Omega$, (3.30) has a unique nonnegative solution $u_0\not\equiv 0$ and $u_0\in C^{2+\nu}([0,1])$. In fact, it is easily seen that $\underline{u}=0$ and $\overline{u}=1$ is a pair of sub- and supersolutions of (3.30). Therefore (3.30) has a solution $u_0(x)$ satisfying $0\le u_0(x)\le 1$. From the boundary conditions, we have $u_0\not\equiv 0$. From the regularity theory of elliptic equations and the smoothness of R, we have $u_0\in C^{2+\nu}([0,1])$. Now we prove the uniqueness. If there is another solution $u^0(x)$ of (3.30). Let $u = u^0 - u_0$, then u satisfies

$$\begin{aligned} -u'' + \mu^{-1} R'(\widetilde{u}) u v_1 = 0, \quad 0 < x < 1, \\ u'(0) = 0, \qquad u'(1) + \gamma u(1) = 0. \end{aligned} \tag{3.36}$$

From maximum principle, we have $u \equiv 0$.

With u_0 given, u_1 is the solution of the problem

$$\begin{aligned} -\left(u_1' - \frac{\alpha}{d} S(u_0) u_0' u_1\right)' + P u_1 = d^{-1} v_1 (\rho R(v_0) - v_1) + P v_1, \quad 0 < x < 1, \\ d u_1' - \alpha S(u_0) u_0' u_1 = 0 \quad \text{at } x = 0, 1. \end{aligned} \tag{3.37}$$

Let $G(\tau) = \int_0^\tau S(y) dy \geq 0$ and $z = e^{-(\alpha/d)G(u_0)} u_1$, then we have

$$u_1' - \frac{\alpha}{d} S(u_0) u_0' u_1 = e^{(\alpha/d)G(u_0)} z', \tag{3.38}$$

and z satisfies

$$\begin{aligned} -\left(e^{(\alpha/d)G(u_0)} z'\right)' + P e^{(\alpha/d)G(u_0)} z = d^{-1} v_1 (\rho R(v_0) - v_1) + P v_1, \quad 0 < x < 1, \\ z'(0) = z'(1) = 0. \end{aligned} \tag{3.39}$$

Observe that $d^{-1}(\rho R(v_0) - v_1) + P > 0$, from maximum principle, we have $z > 0$. Therefore, $u_1 > 0$. From Schauder's theory for elliptic equations, we have $U = A(V) \in C^{2+\nu}([0,1])$. Therefore A is completely continuous. It is easily seen that the fixed points of A in C are nonnegative solutions of (3.28). $\square$

Lemma 3.5. *There is an $M > 0$ such that*

$$\deg(I - A, B_M, \mathbf{0}) = 1, \tag{3.40}$$

where $B_M = \{U \in C : \|U\|_E < M\}$.

Proof. Consider $H(\eta, U) : [0,1] \times C \to E$ defined by

$$H(\eta, U) = A(\eta U). \tag{3.41}$$

We use the homotopy invariance property to $H(\eta, U)$. For $\eta \in [0,1]$, $V \in C$,

$$U^\eta = (u_0, u_1) = H(\eta, V) = (H_0(\eta, V), H_1(\eta, V)) \tag{3.42}$$

is given by

$$\begin{aligned} u_0'' = \mu^{-1} R(u_0) \eta v_1, \quad 0 < x < 1, \\ u_0'(0) = 0, \qquad u_0'(1) + \gamma u_0(1) = \gamma, \\ -\left(u_1' - \frac{\alpha}{d} S(u_0) u_0' u_1\right)' + P u_1 = d^{-1} \eta v_1 (\rho R(\eta v_0) - \eta v_1) + P \eta v_1, \quad 0 < x < 1, \\ d u_1' - \alpha S(u_0) u_0' u_1 = 0 \quad \text{at } x = 0, 1. \end{aligned} \tag{3.43}$$

It is easy to verify that $H(\eta,U)$ is completely continuous and there is a constant K such that for the solution of $U = H(\eta,U)$, that is, $U = A(\eta U)$, we have $\|U\|_E \le K$. Therefore for $M > K$, $U = A(\eta U)$ has no solution satisfying $\|U\|_E = M$. This implies that for $0 \le \eta \le 1$, $\mathbf{0} \notin (I - H(\eta,\bullet))(\partial B_M)$. Therefore, $\deg(I - H(\eta,\bullet), B_M, \mathbf{0})$ is a constant for $0 \le \eta \le 1$. Thus

$$\deg\left(I - A, B_M, \mathbf{0}\right) = \deg\left(I - H(1,\bullet), B_M, \mathbf{0}\right) = \deg\left(I - H(0,\bullet), B_M, \mathbf{0}\right). \tag{3.44}$$

But $H(0,U) = A(\mathbf{0})$ is a constant map. Therefore $\deg(I - H(0,\bullet), B_M, \mathbf{0}) = 1$. That is, $\deg(I - A, B_M, \mathbf{0}) = 1$. □

Let

$$\Delta_{\{0\}} = \{U = (u_0, 0) \in \Omega : A(U) = U, u_0 > 0\}, \tag{3.45}$$

then we have $\Delta_{\{0\}} = \{(1,0)\}$. For $(v_0,v_1) \in C$, we set $T_1(v_0,v_1) = \Phi_1(v_0,v_1) \circ \partial_1 \Gamma_1(v_0,v_1)$, where $\partial_1 \Gamma_1(v_0,v_1)$ is the partial derivative of $\Gamma_1(v_0,v_1)$ with respect to v_1. An easy computation shows that the operator $T_1((1,0)) : C_1 \to C_1$ is defined by the following for $v \in C_1$, $u = T_1((1,0))v$ is the solution of the boundary value problem

$$\begin{gathered} -u'' + Pu = \left(d^{-1}\rho R(1) + P\right)v, \quad 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{gathered} \tag{3.46}$$

Now we cite the following theorem.

Theorem 3.6 (see [7]). *Let $\mathcal{L}(y) = a_2(x)y'' + a_1(x)y' + a_0(x)y$ be a linear differential operator with no singular points in $[x_1,x_2]$, and suppose that $f(x)$ is continuous on $[x_1,x_2]$. Assume also that $(A_1,A_2) \ne (0,0)$ and $(B_1,B_2) \ne (0,0)$. Then the BVP*

$$\mathcal{L}(y) = f(x); \quad A_1 y(x_1) + A_2 y'(x_1) = 0, \qquad B_1 y(x_2) + B_2 y'(x_2) = 0, \tag{3.47}$$

has a unique solution if and only if the associated homogeneous problem $\mathcal{L}(y) = 0$ with the same boundary conditions has only the trivial solution.

It is easily seen that the homogeneous problem associated with (3.46)

$$\begin{gathered} -u'' + Pu = 0, \quad 0 < x < 1, \\ u'(0) = u'(1) = 0, \end{gathered} \tag{3.48}$$

has only the trivial solution when $P > 0$. Therefore, from the theorem, we know that for any $v \in C_1$, (3.46) has unique solution and by maximum principle, we have $u(x) > 0$ on $[0,1]$. That is, $T_1((1,0))$ is strongly positive. The eigenvalue problem $T_1((1,0))\psi = \lambda\psi$, is equivalent to

$$\begin{gathered} -\psi'' + P\psi = \lambda^{-1}\left(\frac{\rho}{d}R(1) + P\right)\psi, \quad 0 < x < 1, \\ \psi'(0) = \psi'(1) = 0. \end{gathered} \tag{3.49}$$

Obviously, 1 is not an eigenvalue of (3.49) corresponding to a positive eigenfunction. In fact, the eigenvalues of (3.49) are

$$\lambda_n = \frac{(\rho/d)R(1)+P}{P-n^2\pi^2}, \quad n = 0,1,2,\ldots, \tag{3.50}$$

and the associated eigenfunctions are $\psi_n = \cos(n\pi x)$. Therefore we can see that the eigenvalue that corresponds to the positive eigenfunction is $\lambda = ((\rho/d)R(1)+P)/P > 1$. This implies that the spectral radius of $T_1((1,0))$ is greater than 1 and therefore, from [9, Theorem 3.1], we have $\operatorname{ind}(A,\Delta_{\{0\}}) = 0$. From Lemma 3.5, we know that for some $M > 0$, the set of fixed points of A is in B_M. Therefore, from Lemma 3.5, we have

$$\operatorname{ind}(A,\Omega) = \operatorname{ind}\left(A,B_M\right) = 1 \neq 0 = \operatorname{ind}\left(A,\Delta_{\{0\}}\right). \tag{3.51}$$

From [9, Theorem 3.1] mentioned above, we know that (3.28) has at least one positive solution. This implies that (3.1) has solutions of the form $(N,b,0)$ with $N > 0$ and $b > 0$. Similarly, we know that (3.1) has solutions of the form $(N,0,B)$ with $N > 0$ and $B > 0$. Summarizing the analysis above, we have the following theorem.

Theorem 3.7. *System (3.1) has solutions of the form* $(N,b,0)$ *and* $(N,0,B)$ *with* $N > 0$, $b > 0$, *and* $B > 0$.

3.2. Existence of positive solutions. Now we study the existence of positive solutions of (3.1). As before, we write N, b, and B as u_0, u_1, and u_2, respectively, and write system (3.1) in the form of a fixed point equation as follows.

For $\nu \in (0,1)$, let

$$\begin{gathered} E_i = \{u \in C^\nu([0,1])\}, \quad i = 0,1,2, \\ C_i = \{u \in C^\nu([0,1]) : u \geq 0 \text{ on } [0,1]\}, \quad i = 0,1,2, \\ E = E_0 \oplus E_1 \oplus E_2, \qquad C = C_0 \oplus C_1 \oplus C_2. \end{gathered} \tag{3.52}$$

For $V = (v_0,v_1,v_2) \in C$, let $u_0 = A_0(V)$ be the solution of

$$\begin{gathered} u_0'' = \mu^{-1}\left(R_1(u_0)v_1 + R_2(u_0)v_2\right), \quad 0 < x < 1, \\ u_0'(0) = 0, \qquad u_0'(1) + \gamma u_0(1) = \gamma. \end{gathered} \tag{3.53}$$

With u_0 given, define operators Φ_i and Γ_i, $i = 1,2$, as

$$\begin{gathered} \Phi_1(V) = \left\{-\frac{d^2\bullet}{dx^2} + \frac{\alpha}{d}\left(S_1(u_0)u_0'\bullet\right)' + P\bullet,\ d\frac{d\bullet}{dx} - \alpha\bullet S_1(u_0)u_0'\right\}^{-1}, \\ \Phi_2(V) = \left\{-\frac{d^2\bullet}{dx^2} + \frac{\beta}{D}\left(S_2(u_0)u_0'\bullet\right)' + P\bullet,\ D\frac{d\bullet}{dx} - \beta\bullet S_2(u_0)u_0'\right\}^{-1}, \\ \Gamma_1(V) = d^{-1}v_1\left(\rho_1R_1(v_0) - v_1 - v_2\right) + Pv_1, \\ \Gamma_2(V) = D^{-1}v_2\left(\rho_2R_2(v_0) - v_1 - v_2\right) + Pv_2, \end{gathered} \tag{3.54}$$

where P is a positive constant such that $d^{-1}(\rho_1 R_1(v_0) - v_1 - v_2)+P > 0$ and $D^{-1}(\rho_2 R_2(v_0)-v_1 - v_2)+P > 0$ for $0 \le v_0 \le 2$, $0 \le v_1 \le P_1$, and $0 \le v_2 \le P_2$, where $P_1 = 2\rho_1 R_1(1)e^{(\alpha/d)\int_0^1 S_1(y)dy}$ and $P_2 = 2\rho_2 R_2(1)e^{(\beta/D)\int_0^1 S_2(y)dy}$. Then system (3.1) can be written as a fixed point equation $U = A(U)$, where $U = A(V)$ is given by

$$u_0 = A_0(V), \qquad u_i = A_i(V) = \Phi_i(V)\circ\Gamma_i(V), \quad i = 1,2. \tag{3.55}$$

It is easily seen that Γ_i satisfies

$$\Gamma_i(v_0,v_1,v_2) = 0 \quad \text{if } v_i = 0. \tag{3.56}$$

Similar to the proofs of Lemmas 3.4 and 3.5, we can prove the following two lemmas.

Lemma 3.8. *The operator $A(V) = (A_0(V),A_1(V),A_2(V)) : \overline{\Omega} \to C$ is a well-defined completely continuous operator, where*

$$\Omega = \{(v_0,v_1,v_2) \in C : 0 \le v_0 < 2,\ 0 \le v_1 < P_1,\ 0 \le v_2 < P_2\}. \tag{3.57}$$

Moreover, fixed points of A in C are nonnegative solutions of (3.1).

Lemma 3.9. *There is an $M > 0$ such that*

$$\deg(I - A, B_M, \mathbf{0}) = 1, \tag{3.58}$$

where $B_M = \{U \in C : \|U\|_E < M\}$.

For $i = 1,2$, let

$$\Delta_{\{0,i\}} = \{U = (u_0,u_1,u_2) \in \Omega : A(U) = U,\ u_0 > 0,\ u_i > 0,\ u_j = 0 \text{ for } j \ne i \text{ or } 0\}. \tag{3.59}$$

From the analysis in Section 3.1, we know that $\Delta_{\{0,1\}}$ and $\Delta_{\{0,2\}}$ are nonempty. We consider the following two conditions.

($\mathscr{A}_1$) For any $U = (\check{u}_0,\check{u}_1,0) \in \Delta_{\{0,1\}}$, the largest eigenvalue of the eigenvalue problem

$$\begin{gathered} -\phi'' + \frac{\beta}{D}(S_2(\check{u}_0)\check{u}_0'\phi)' + P\phi = \lambda^{-1}(D^{-1}(\rho_2 R_2(\check{u}_0) - \check{u}_1) + P)\phi, \quad 0 < x < 1, \\ D\phi' - \beta S_2(\check{u}_0)\check{u}_0'\phi = 0 \quad \text{at } x = 0,1, \end{gathered} \tag{3.60}$$

is greater than 1 and for any $U = (\hat{u}_0,0,\hat{u}_2) \in \Delta_{\{0,2\}}$, the largest eigenvalue of the eigenvalue problem

$$\begin{gathered} -\psi'' + \frac{\alpha}{d}(S_1(\hat{u}_0)\hat{u}_0'\psi)' + P\psi = \lambda^{-1}(d^{-1}(\rho_1 R_1(\hat{u}_0) - \hat{u}_2) + P)\psi, \quad 0 < x < 1, \\ d\psi' - \alpha S_1(\hat{u}_0)\hat{u}_0'\psi = 0 \quad \text{at } x = 0,1, \end{gathered} \tag{3.61}$$

is greater than 1.

($\mathscr{A}_2$) Both eigenvalues of (3.60) and (3.61) are all less than 1.

We have the following theorem.

Theorem 3.10. *If either ($\mathscr{A}_1$) or ($\mathscr{A}_2$) holds, then (3.1) has at least one positive solution.*

Proof. From Lemmas 3.2 and 3.9, we have $\text{ind}(A,\Omega) = 1$. Let

$$\Delta_{\{0\}} = \{U = (u_0,0,0) \in \Omega : A(U) = U,\ u_0 > 0\}, \tag{3.62}$$

then we have $\Delta_{\{0\}} = \{(1,0,0)\}$ and $\text{ind}(A,\Delta_{\{0\}}) = 0$.

For $i = 1,2$, set

$$T_i(V) = \Phi_i(V) \circ \partial_i \Gamma_i(V), \tag{3.63}$$

where $\partial_i\Gamma_i(V) = \partial_i\Gamma_i(v_0,v_1,v_2)$ is the partial derivative of $\Gamma_i(V)$ with respect to v_i, then an easy calculation shows that, for $U = (\check{u}_0,\check{u}_1,0) \in \Delta_{\{0,1\}}$, the operator $T_2(\check{u}_0,\check{u}_1,0)$ is given by

$$T_2(\check{u}_0,\check{u}_1,0) = \left(-\frac{d^2\bullet}{dx^2} + \frac{\beta}{D}\left(S_2(\check{u}_0)\check{u}_0'\bullet\right)' + P\bullet\right)^{-1}\left(D^{-1}\left(\rho_2R_2(\check{u}_0) - \check{u}_1\right) + P\right) \tag{3.64}$$

with the boundary condition in (3.60), and for $U = (\hat{u}_0,0,\hat{u}_2) \in \Delta_{\{0,2\}}$, the operator $T_1(\hat{u}_0,0,\hat{u}_2)$ is given by

$$T_1(\hat{u}_0,0,\hat{u}_2) = \left(-\frac{d^2\bullet}{dx^2} + \frac{\alpha}{d}\left(S_1(\hat{u}_0)\hat{u}_0'\bullet\right)' + P\bullet\right)^{-1}\left(d^{-1}\left(\rho_1R_1(\hat{u}_0) - \hat{u}_1\right) + P\right) \tag{3.65}$$

with the boundary condition in (3.61). It is easy to verify that $T_2(\check{u}_0,\check{u}_1,0)$ and $T_1(\hat{u}_0,0,\hat{u}_2)$ are all strongly positive. By using the Krein-Rutman theorem, we can see if $(\mathscr{A}_1)$ holds, then the spectral radii of $T_2(\check{u}_0,\check{u}_1,0)$ and $T_1(\hat{u}_0,0,\hat{u}_2)$ are all less than 1 and $\text{ind}(A,\Delta_{\{0,1\}}) = 1$ and $\text{ind}(A,\Delta_{\{0,2\}}) = 1$. Thus

$$\text{ind}\left(A,\Delta_{\{0\}}\right) + \text{ind}\left(A,\Delta_{\{0,1\}}\right) + \text{ind}\left(A,\Delta_{\{0,2\}}\right) = 0+1+1 = 2 \neq \text{ind}(A,\Omega) = 1. \tag{3.66}$$

From [9, Theorem 3.1], we know that (3.1) has positive solutions.

If $(\mathscr{A}_2)$ holds, the spectral radii of $T_2(\check{u}_0,\check{u}_1,0)$ and $T_1(\hat{u}_0,0,\hat{u}_2)$ are all greater than 1 and $\text{ind}(A,\Delta_{\{0,1\}}) = 0$ and $\text{ind}(A,\Delta_{\{0,2\}}) = 0$. Thus

$$\text{ind}\left(A,\Delta_{\{0\}}\right) + \text{ind}\left(A,\Delta_{\{0,1\}}\right) + \text{ind}\left(A,\Delta_{\{0,2\}}\right) = 0+0+0 = 0 \neq \text{ind}(A,\Omega) = 1. \tag{3.67}$$

As before, we conclude that (3.1) has positive solutions. □

3.3. Some special cases. Now we consider some special situations.

(I) First we consider $\alpha = 0, \beta = 0, \rho = 1$, and $R_1 = R_2 = R$, this is the model discussed in [16], that is,

$$\begin{gathered}
\mu N'' - R(N)(b+B) = 0, \quad 0 < x < 1, \\
db'' + b\left(R(N) - b - B\right) = 0, \quad 0 < x < 1, \\
DB'' + B\left(R(N) - b - B\right) = 0, \quad 0 < x < 1, \\
N'(0) = 0, \qquad N'(1) = \gamma(1 - N(1)), \\
b'(0) = b'(1) = 0, \\
B'(0) = B'(1) = 0.
\end{gathered} \tag{3.68}$$

In this case, (3.60) and (3.61) become

$$-\phi'' + P\phi = \lambda^{-1}\big((D^{-1}(R(\check{u}_0) - \check{u}_1) + P\big)\phi, \quad 0 < x < 1,$$
$$\phi'(0) = \phi'(1) = 0, \tag{3.69}$$

$$-\psi'' + P\psi = \lambda^{-1}\big(d^{-1}(R(\hat{u}_0) - \hat{u}_2) + P\big)\psi, \quad 0 < x < 1,$$
$$\psi'(0) = \psi'(1) = 0. \tag{3.70}$$

Let

$$\mathscr{L}_0 = \left(-\frac{d^2}{dx^2}\right)^{-1}\left(\frac{R(u_0) - u_1}{D}\right),$$
$$\mathscr{L}_P = \left(-\frac{d^2}{dx^2} + P\right)^{-1}\left(\frac{R(u_0) - u_1}{D} + P\right). \tag{3.71}$$

We denote the spectral radius of operator $\mathscr{L}$ by $\Upsilon(\mathscr{L})$. It is well known that $\Upsilon(\mathscr{L}_0) > 1$ if and only if $\Upsilon(\mathscr{L}_P) > 1$ for all $P \geq 0$. Therefore we know that the largest eigenvalues of both (3.69) and (3.70) are greater than 1 if and only if the largest eigenvalues of the following two eigenvalue problems are greater than 1:

$$-\phi'' = \lambda^{-1}D^{-1}(R(\check{u}_0) - \check{u}_1)\phi, \quad 0 < x < 1,$$
$$\phi'(0) = \phi'(1) = 0, \tag{3.72}$$

$$-\psi'' = \lambda^{-1}d^{-1}(R(\hat{u}_0) - \hat{u}_2)\psi, \quad 0 < x < 1,$$
$$\psi'(0) = \psi'(1) = 0. \tag{3.73}$$

But it is easily seen that the largest eigenvalue of (3.72) is $\check{\lambda}_1 = d/D < 1$ and the largest eigenvalue of (3.73) is $\hat{\lambda}_1 = D/d > 1$. Therefore neither $(\mathscr{A}_1)$ nor $(\mathscr{A}_2)$ is satisfied. In fact, we can prove that (3.68) has no positive solutions directly. To do this, we first cite the following lemma from [8].

LEMMA 3.11. *Consider the eigenvalue problem*

$$d\triangle\phi + q(x)\phi = \lambda\phi, \quad x \in \Omega,$$
$$\frac{\partial\phi}{\partial\nu} = 0, \quad x \in \partial\Omega, \tag{3.74}$$

where $d > 0$, $q(x) \in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Let $\lambda_1 = \lambda(q,d)$ be the unique "principal eigenvalue". Then $\lambda(q,d)$ is a continuous nonincreasing function of d, and is strictly decreasing if $q(x)$ is not a constant. Furthermore, the following hold:

(a) *$\lambda(q,d) \uparrow Q = \max_{\overline{\Omega}} q(x)$ as $d \to 0$;*
(b) *$\lambda(q,d) \downarrow \omega = (1/|\Omega|)\int_\Omega q(x)dx$ as $d \to \infty$;*
(c) *if $q_1(x) \geq q_2(x)$ for $x \in \Omega$, then $\lambda(q_1,d) \geq \lambda(q_2,d)$ with strict inequality if $q_1(x) \not\equiv q_2(x)$.*

Now we can prove the following theorem.

THEOREM 3.12. *Equation (3.68) has no positive solution.*

Proof. In fact, if (3.68) has a positive solution $U = (N(x), b(x), B(x))$ with $N(x) > 0$, $b(x) > 0$, and $B(x) > 0$, then $b(x) > 0$ and $B(x) > 0$ satisfy

$$\begin{gathered} db'' + (R(N) - b - B)b = 0, \quad 0 < x < 1, \\ DB'' + (R(N) - b - B)B = 0, \quad 0 < x < 1, \\ b'(0) = b'(1) = B'(0) = B'(1) = 0. \end{gathered} \tag{3.75}$$

Because of the positivity of $b(x)$ and $B(x)$, we can consider them as the principal eigenfunctions of the eigenvalue problem

$$\begin{gathered} d\phi'' + q(x)\phi = \lambda\phi, \quad 0 < x < 1, \\ \phi'(0) = \phi'(1) = 0, \end{gathered} \tag{3.76}$$

$$\begin{gathered} B\phi'' + q(x)\phi = \lambda\phi, \quad 0 < x < 1, \\ \phi'(0) = \phi'(1) = 0, \end{gathered} \tag{3.77}$$

with $q(x) = R(N(x)) - b(x) - B(x)$, associated with the principal eigenvalue $\lambda = 0$, respectively. So we have $\lambda(q(x), d) = \lambda(q(x), D)$. Since $q(x) = R(N(x)) - b(x) - B(x) \neq$ constant, by Lemma 3.11, this contradicts the assumption $d < D$. The proof of Theorem 3.12 is complete. □

(II) Now we consider the following model, which was discussed in [15]:

$$\begin{gathered} N'' - R(N)(b + B) = 0, \quad 0 < x < 1, \\ (db' - \alpha bS(N)N')' + b(\rho R(N) - \theta) = 0, \quad 0 < x < 1, \\ (DB' - \beta BS(N)N')' + B(\rho R(N) - \theta) = 0, \quad 0 < x < 1, \\ N'(0) = 0, \qquad N'(1) = \gamma(1 - N(1)), \\ db' - \alpha bS(N)N' = 0 \quad \text{at } x = 0, 1, \\ DB' - \beta BS(N)N' = 0 \quad \text{at } x = 0, 1, \end{gathered} \tag{3.78}$$

where $\theta > 0$ is a constant. In this case, (3.60) and (3.61) become

$$\begin{gathered} -\phi'' + \frac{\beta}{D}(S(\check{u}_0)\check{u}_0'\phi)' + P\phi = \lambda^{-1}(D^{-1}(\rho R(\check{u}_0) - \theta) + P)\phi, \quad 0 < x < 1, \\ D\phi' - \beta S(\check{u}_0)\check{u}_0'\phi = 0 \quad \text{at } x = 0, 1, \end{gathered} \tag{3.79}$$

$$\begin{gathered} -\psi'' + \frac{\alpha}{d}(S(\hat{u}_0)\hat{u}_0'\psi)' + P\psi = \lambda^{-1}(d^{-1}(\rho R(\hat{u}_0) - \theta) + P)\psi, \quad 0 < x < 1, \\ d\psi' - \alpha S(\hat{u}_0)\hat{u}_0'\psi = 0 \quad \text{at } x = 0, 1. \end{gathered} \tag{3.80}$$

The largest eigenvalues of the two eigenvalue problems above are greater than 1 if and only if the largest eigenvalues of the following two eigenvalue problems are greater than 1:

$$\begin{aligned}-\phi'' + \frac{\beta}{D}\big(S(\check{u}_0)\check{u}_0'\phi\big)' &= \lambda^{-1}D^{-1}\big(\rho R(\check{u}_0) - \theta\big)\phi, \quad 0 < x < 1,\\ D\phi' - \beta S(\check{u}_0)\check{u}_0'\phi &= 0 \quad \text{at } x = 0, 1,\end{aligned} \tag{3.81}$$

$$\begin{aligned}-\psi'' + \frac{\alpha}{d}\big(S(\hat{u}_0)\hat{u}_0'\psi\big)' &= \lambda^{-1}d^{-1}\big(\rho R(\hat{u}_0) - \theta\big)\psi, \quad 0 < x < 1,\\ d\psi' - \alpha S(\hat{u}_0)\hat{u}_0'\psi &= 0 \quad \text{at } x = 0, 1.\end{aligned} \tag{3.82}$$

A special case is $\alpha/d = \beta/D$. For this case, it is easily seen that the largest eigenvalue of (3.81) is $\lambda_1 = d/D < 1$ and the associated eigenfunction is $\phi = \check{u}_1$. The largest eigenvalue of (3.82) is $\lambda_1 = D/d > 1$ and the associated eigenfunction is $\psi = \hat{u}_2$. Therefore, neither $(\mathcal{A}_1)$ nor $(\mathcal{A}_2)$ is satisfied. In fact, from [15], we know that (3.78) has no positive solutions for this situation.

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