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Research Article

Global Bounds for Cocoercive Variational Inequalities

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By using the strong monotonicity of the perturbed fixed-point map and the normal map associated with cocoercive variational inequalities, we establish two new global bounds measuring the distance between any point and the solution set for cocoercive variational inequalities.

1. Introduction

Throughout this paper, let R^n be a Euclidean space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed convex set in R^n, let $f : R^n \to R^n$ be a continuous function. We consider the variational inequality problem associated with K and f, denoted by VIP(K, f), which is to find a vector $x^* \in K$ such that

$$\langle f(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K. \tag{1.1}$$

Variational inequalities have many applications in different fields such as mathematical programming, game theory, economics, and engineering, see [1–3] and the references mentioned there. Error bounds have played an important role not only in theoretical analysis but also in convergence analysis of iterative algorithms for solving variational inequalities, see [4] for an excellent survey of the theory and application. A few error bounds have been presented for variational inequality, which mainly use the following assumptions on the map f:

(i) strong monotonicity + Lipschitz continuous [5–7];

(ii) strong monotonicity [8, 9].

When the map f is cocoercive, by using the perturbed fixed point and normal maps, and by utilizing Williamson geometric estimation of fixed points of contractive maps, Zhao and Hu [7] established global bounds for $\mathrm{VIP}(K, f)$.

In this paper, by using the strong monotonicity of the perturbed fixed-point and normal maps, we establish two new global bounds measuring the distance between any point and the solution set for cocoercive variational inequalities. We need weaker restriction on the constants involved in the (perturbed) fixed point and normal maps.

2. Preliminaries and notations

The fixed-point and the normal equations for $\mathrm{VIP}(K, f)$ are defined by

$$\pi_\alpha(x) = x - \Pi_K(x - \alpha f(x)) = 0, \tag{2.1}$$

$$\Phi_\alpha(x) = f(\Pi_K(x)) + \alpha(x - \Pi_K(x)) = 0, \tag{2.2}$$

respectively, where α is an arbitrary positive scalar and $\Pi_K(\cdot)$ is the projection operator on the convex set K, that is, $\Pi_K(x) = \min_{z\in K}\|z - x\|$. The projection operator is nonexpansive, that is, for any $x, x' \in R^n$, it holds that

$$||\Pi_K(x) - \Pi_K(x')|| \le \|x - x'\|. \tag{2.3}$$

It is well known that x^* solves $\mathrm{VIP}(K, f)$ if and only if x^* solves the fixed-point equation (2.1); if x^* is a solution of $\mathrm{VIP}(K, f)$, then $x^* - (1/\alpha)f(x^*)$ is a solution of the normal equation (2.2); conversely, if $\Phi_\alpha(y^*) = 0$, then $\Pi_K(y^*)$ is a solution of $\mathrm{VIP}(K, f)$.

In fact, the perturbed fixed-point and normal maps also have been extensively studied, which are defined by

$$\begin{aligned}
\pi_{\alpha,\varepsilon}(x) &= x - \Pi_K(x - \alpha(f(x) + \varepsilon x)),\\
\Phi_{\alpha,\varepsilon}(x) &= f(\Pi_K(x)) + \varepsilon\Pi_K(x) + \alpha(x - \Pi_K(x)),
\end{aligned} \tag{2.4}$$

respectively.

For the map f, we require the following concepts.

Definition 2.1. The map $f : R^n \to R^n$ is said to be

(i) monotone if

$$\langle f(x) - f(y), x - y\rangle \ge 0, \quad \forall x, y \in R^n; \tag{2.5}$$

(ii) strongly monotone with modulus c if there is a scalar $c > 0$ such that

$$\langle f(x) - f(y), x - y\rangle \ge c\|x - y\|^2, \quad \forall x, y \in R^n; \tag{2.6}$$

(iii) Lipschitz with modulus L if there is a constant $L > 0$ such that

$$||f(x) - f(y)|| \le L\|x - y\|, \quad \forall x, y \in R^n. \tag{2.7}$$

If $L < 1$, f is said to be contractive.

Definition 2.2. The map $f : R^n \to R^n$ is said to be cocoercive with modulus β if there exists a constant $\beta > 0$ such that

$$\langle f(x) - f(y), x - y \rangle \geq \beta \|f(x) - f(y)\|^2, \quad \forall x, y \in R^n. \tag{2.8}$$

Remark 2.3. Our analysis in the rest of the paper is based upon the cocoercive condition. Gabay [10] implicitly introduced the strong-f-monotonicity and Tseng [11], using the name cocoercivity, explicitly stated this condition. The cocoercive condition plays an important role in the convergence analysis of algorithms; for more details, see [12, 7, 13–15]. Notice that any cocoercive map with modulus β is monotone and Lipschitz continuous (with modulus $L = 1/\beta$), but it is not necessary to be strongly monotone (e.g., the constant map).

In some cases, the modulus β of the cocoercive map can be determined explicitly, for example, see [14, 15].

Let us introduce more required notations. Let B denote the open unit ball in R^n and SOL(K, f) denotes the solution set of VIP(K, f). Denote dist(x, D) for the distance from the vector x to any set $D \subseteq R^n$, and denote $\pi_\alpha^{-1}(0)$ for the zeros of $\pi_\alpha(x)$.

We state some lemmas, which are crucial in the proof of our main theorems. The first shows us the monotonicity of the (perturbed) fixed-point and normal maps associated with VIP(K, f) under certain conditions.

Lemma 2.4 (Zhao and Li [13]). *Let K be an arbitrary closed convex set in R^n and $K \subseteq S \subseteq R^n$. Let $f : R^n \to R^n$ be a function.*

(i) *If f is cocoercive with modulus $\beta > 0$ on the set S, and if the scalars α and ε satisfy $0 < \alpha < 4\beta$ and $0 < \varepsilon \leq 2(1/\alpha - 1/(4\beta))$, then the perturbed fixed-point map $\pi_{\alpha,\varepsilon}(x)$ is strongly monotone with modulus $\alpha\varepsilon(1 - \alpha\varepsilon/4)$.*

(ii) *If f is cocoercive with modulus $\beta > 0$ on the set S, and if the scalars α and ε satisfy $0 < \varepsilon < \alpha$ and $\alpha > 1/(4\beta)$, then the perturbed normal map $\Phi_{\alpha,\varepsilon}(x)$ is strongly monotone with modulus r, where $r = \min\{\varepsilon, \alpha - 1/4\beta\}$.*

(iii) *If f is strongly monotone with modulus $c > 0$ and f is Lipschitz continuous with constant $L > 0$ on the set S, then for any fixed scalar α satisfying $0 < \alpha < 4c/L^2$, the fixed point map $\pi_\alpha(x)$ is strongly monotone with modulus $\alpha(c - \alpha L^2/4)$ on the set S.*

(iv) *If f is strongly monotone with modulus $c > 0$ and f is Lipschitz continuous with constant $L > 0$ on the set S, then for any α satisfying $\alpha > L^2/(4c)$, the normal map $\Phi_\alpha(x)$ is strongly monotone with modulus r on the set S, where $0 < r < \alpha/2$ and $2r + L^2/4(\alpha - 2r) < c$.*

The upper-semicontinuity theorem concerning weakly univalent maps established by Ravindran and Gowda [16] is as follows.

Lemma 2.5 (Ravindran and Gowda [16]). *Let $g : R^n \to R^n$ be weakly univalent, that is, g is continuous and there exists one-to-one continuous function $g_k : R^n \to R^n$ such that $g_k \to g$ uniformly on every bounded subset of R^n. Suppose that $g^{-1}(0) = \{x \in R^n : g(x) = 0\}$ is nonempty and compact. Then for any given $\gamma > 0$, there exists a scalar $\delta > 0$ such that for any weakly univalent function $h : R^n \to R^n$ with $\sup_{\overline{\Omega}} \|h(x) - g(x)\| < \delta$, one has $\emptyset \neq h^{-1}(0) \subseteq g^{-1}(0) + \gamma B$, where $\overline{\Omega}$ denotes the closure of $\Omega = g^{-1}(0) + \gamma B$.*

The following lemma shows us an important property of strongly monotone maps.

Lemma 2.6. *Let $f : R^n \to R^n$ be strongly monotone with modulus $c > 0$, then the following inequality holds:*

$$\|x - y\| \leq \frac{\|f(x) - f(y)\|}{c}, \quad \forall x, y \in R^n. \tag{2.9}$$

Proof. Since f is strongly monotone with modulus $c > 0$, it holds that

$$\langle f(x) - f(y), x - y \rangle \geq c\|x - y\|^2, \quad \forall x, y \in R^n. \tag{2.10}$$

On the other hand, from the Cauchy-Schwarz inequality, we have

$$\langle f(x) - f(y), x - y \rangle \leq \|f(x) - f(y)\| \|x - y\|. \tag{2.11}$$

Combining (2.10) and (2.11), we obtain

$$\|x - y\| \leq \frac{\|f(x) - f(y)\|}{c}. \tag{2.12}$$

□

3. Main results

In this section, we first establish two global bounds measuring the distance between any point and the solution set for cocoercive $\mathrm{VIP}(K, f)$ by using the strong monotonicity of the perturbed fixed-point and normal maps.

Theorem 3.1. *Let $f : R^n \to R^n$ be cocoercive with modulus $\beta > 0$. Suppose that the solution set of $\mathrm{VIP}(K, f)$ is nonempty and bounded, let α be a constant satisfying $0 < \alpha < 4\beta$. Then there exists a constant $\delta > 0$, and for any ε satisfying $0 < \varepsilon < \min\{\delta/aM^*, 2/\alpha - 1/2\beta\}$, the following result holds for all $x \in R^n$:*

$$\operatorname{dist}(x, \mathrm{SOL}(K, f)) \leq \frac{\|\pi_{\alpha,\varepsilon}(x)\|}{\alpha\varepsilon(1 - \alpha\varepsilon/4)} + \alpha, \tag{3.1}$$

where $M^* \geq \sup_{x \in \overline{\Omega}} \|x\|$, $\Omega := \mathrm{SOL}(K, f) + \alpha B$.

Proof. Let α, ε be constants such that $0 < \alpha < 4\beta$ and $0 < \varepsilon < 2/\alpha - 1/2\beta$, thus by Lemma 2.4(i), the perturbed fixed point map $\pi_{\alpha,\varepsilon}(x)$ is strongly monotone with modulus $\alpha\varepsilon(1 - \alpha\varepsilon/4)$.

Since $\pi_{\alpha,\varepsilon}(x)$ is strongly monotone, we may denote by x^* the unique element of the set $\pi_{\alpha,\varepsilon}^{-1}(0)$. By Lemma 2.6, for any $x \in R^n$, we have

$$\|x - x^*\| \leq \frac{\|\pi_{\alpha,\varepsilon}(x)\|}{\alpha\varepsilon(1 - \alpha\varepsilon/4)}. \tag{3.2}$$

Since any monotone map is weakly univalent, we can replace $h(x)$ with $\pi_{\alpha,\varepsilon}(x)$ in Lemma 2.5. By Lemma 2.5, there exists a constant $\delta > 0$, and then let ε be a constant satisfying $0 < \varepsilon < \min\{\delta/aM^*, 2/\alpha - 1/2\beta\}$ with $M^* \geq \sup_{x \in \overline{\Omega}} \|x\|$ and $\Omega := \mathrm{SOL}(K, f) + \alpha B$

such that

$$\sup_{x\in\overline{\Omega}}\left|\left|\pi_{\alpha,\varepsilon}(x)-\pi_\alpha(x)\right|\right| = \sup_{x\in\overline{\Omega}}\left|\left|\Pi_K(x-\alpha(f(x)+\varepsilon x)-\Pi_K(x-\alpha f(x))\right|\right| \\ \leq \sup_{x\in\overline{\Omega}}\alpha\varepsilon\|x\| \leq \alpha\varepsilon M^* < \delta. \tag{3.3}$$

Thus we have $\varnothing\neq\{x^*\}\subseteq\pi_\alpha^{-1}(0)+\alpha B = \mathrm{SOL}(K,f)+\alpha B$, which yields that

$$\operatorname{dist}(x^*, \mathrm{SOL}(K,f)) \leq \alpha. \tag{3.4}$$

Therefore, for any $x\in R^n$, we obtain

$$\operatorname{dist}(x, \mathrm{SOL}(K,f)) \leq \|x-x^*\| + \operatorname{dist}(x^*,\mathrm{SOL}(K,f)) \leq \frac{\left|\left|\pi_{\alpha,\varepsilon}(x)\right|\right|}{\alpha\varepsilon(1-\alpha\varepsilon/4)}+\alpha. \tag{3.5}$$

□

Remark 3.2. In [7, Theorem 2.1], Zhao and Hu need stronger restriction on α,ε, ensuring that the map $p^\varepsilon : R^n\to R^n$ defined by $p^\varepsilon(x) = \Pi_K(x-\alpha(f(x)+\varepsilon x))$ is contractive, that is, $\|p^\varepsilon(x)-p^\varepsilon(y)\| \leq r\|x-y\|$, where $r=\sqrt{(1-\alpha\varepsilon)^2+2\alpha^2\varepsilon\beta}\in(0,1)$.

On the other hand, if p^ε is a Lipschitz continuous map with modulus $r<1$, it is easy to see that $\pi_{\alpha,\varepsilon} = I-p^\varepsilon$ (I is the identity operator) is strongly monotone with modulus $1-r$. Thus from Lemma 2.6, for any $x\in R^n$, we have

$$\|x-x^*\| \leq \frac{\left|\left|\pi_{\alpha,\varepsilon}(x)\right|\right|}{1-r}. \tag{3.6}$$

Remark 3.3. If the set K is bounded, then the solution set $\mathrm{SOL}(K,f)\subset K$, and we can choose $M^* = \sup_{x\in K}\|x\|+\alpha$.

If the set K is unbounded, it follows from [17, Corollary 1] that the solution set $\mathrm{SOL}(K,f)$ is nonempty and bounded if and only if

$$\exists\rho>0,\ \forall x\in K\setminus K_\rho,\ \exists\, y\in K_\rho,\quad \langle f(x),x-y\rangle>0, \tag{3.7}$$

where $K_\rho = \{x\in K : \|x\|\leq\rho\}$.

If we can find $x_0\in K$ and $\rho>0$ such that

$$\langle f(x),x-x_0\rangle>0,\quad \forall x\in K\setminus K_\rho, \tag{3.8}$$

then $\mathrm{SOL}(K,f)\subset K_\rho$, and we can choose $M^*=\rho+\alpha$.

Theorem 3.4. *Let $f : R^n\to R^n$ be cocoercive with modulus $\beta>0$. Suppose that the solution set of* VIP(K,f) *is nonempty and bounded, and let α be a constant satisfying $\alpha>1/4\beta$. Then there exists a constant $\delta>0$, and for any ε satisfying $0<\varepsilon<\min\{\delta/C^*,\alpha\}$, the following result holds for all $x\in R^n$,*

$$\operatorname{dist}(x,\Phi_\alpha^{-1}(0)) \leq \frac{\left|\left|\Phi_{\alpha,\varepsilon}(x)\right|\right|}{r}+\alpha, \tag{3.9}$$

where $C^* = \sup_{x\in\overline{\Omega}}\|\Pi_K(x)\|$, $r=\min\{\varepsilon,\ \alpha-1/4\beta\}$, $\Omega := \mathrm{SOL}(K,f)+\alpha B$.

Proof. Let α, ε be constants such that $\alpha > 1/4\beta$ and $0 < \varepsilon < \alpha$, thus by Lemma 2.4(ii), the perturbed normal map $\Phi_{\alpha,\varepsilon}(x)$ is strongly monotone with modulus r, where $r = \min\{\varepsilon, \alpha - 1/4\beta\}$. Since $\Phi_{\alpha,\varepsilon}(x)$ is strongly monotone, we may denote by y^* the unique element of the set $\Phi_{\alpha,\varepsilon}^{-1}(0)$.

Since any monotone map is weakly univalent, we can replace $h(x)$ with $\Phi_{\alpha,\varepsilon}(x)$ in Lemma 2.5. Then by Lemma 2.5, there exists a constant $\delta > 0$, and for any ε satisfying $0 < \varepsilon < \min\{\delta/C^*, \alpha\}$ with $C^* = \sup_{x\in\overline{\Omega}}\|\Pi_K(x)\|$ and $\Omega := \mathrm{SOL}(K,f) + \alpha B$, we have

$$\begin{aligned}
\sup_{x\in\overline{\Omega}} &\|\Phi_{\alpha,\varepsilon}(x) - \Phi_\alpha(x)\| \\
&= \sup_{x\in\overline{\Omega}} \|f(\Pi_K(x)) + \varepsilon\Pi_K(x) + \alpha(x - \Pi_K(x)) - (f(\Pi_K(x)) + \alpha(x - \Pi_K(x)))\| \\
&\le \sup_{x\in\overline{\Omega}} \varepsilon\|\Pi_K(x)\| = \varepsilon C^* < \delta.
\end{aligned} \tag{3.10}$$

Thus we obtain that $\varnothing \neq \{x^*\} \subseteq \Phi_\alpha^{-1}(0) + \alpha B = \mathrm{SOL}(K,f) + \alpha B$, which means that

$$\operatorname{dist}(y^*, \Phi_\alpha^{-1}(0)) \le \alpha. \tag{3.11}$$

By Lemma 2.6, for any $x \in R^n$, we have

$$\|x - y^*\| \le \frac{\|\Phi_{\alpha,\varepsilon}(x)\|}{r}, \tag{3.12}$$

where $r = \min\{\varepsilon, \alpha - 1/4\beta\}$.

Combining (3.11) and (3.12), for any $x \in R^n$, we have

$$\operatorname{dist}(x, \Phi_\alpha^{-1}(0)) \le \|x - y^*\| + \operatorname{dist}(y^*, \Phi_\alpha^{-1}(0)) \le \frac{\|\Phi_{\alpha,\varepsilon}(x)\|}{r} + \alpha. \tag{3.13}$$

□

Remark 3.5. In [7, Theorem 2.2], Zhao and Hu need stronger restriction on α, ε, ensuring that the map $q^\varepsilon : R^n \to R^n$ defined by $q^\varepsilon = x - (1/\alpha)\Phi_{\alpha,\varepsilon}(x)$ is contractive, that is, $\|q^\varepsilon(x) - q^\varepsilon(y)\| \le r\|x - y\|$, where $r = \sqrt{(1 - \varepsilon/\alpha)^2 + 2\varepsilon/\alpha^2\beta} \in (0,1)$.

Thus $\Phi_{\alpha,\varepsilon} = \alpha(I - q^\varepsilon)$, where I is the identity operator, and strongly monotone with modulus $\alpha(1 - r)$. By Lemma 2.6, for any $x \in R^n$, we have

$$\|x - x^*\| \le \frac{\|\Phi_{\alpha,\varepsilon}(x)\|}{\alpha(1 - r)}. \tag{3.14}$$

As a direct consequence of Theorem 3.4, we have the following corollary, which shows us a global bound for cocoercive $\mathrm{VIP}(K,f)$.

Corollary 3.6. *Let $f : R^n \to R^n$ be cocoercive with modulus $\beta > 0$. Suppose that the solution set of* VIP(K, f) *is nonempty and bounded, and let α be a constant satisfying $\alpha > 1/4\beta$. Then there exists a constant $\delta > 0$, and for any ε satisfying $0 < \varepsilon < \min\{\delta/C^*, \alpha\}$ such that*

$$\operatorname{dist}(x, \operatorname{SOL}(K, f)) \leq d(x, K) + \frac{||\Phi_{\alpha,\varepsilon}(x)||}{r} + \alpha, \quad \forall x \in R^n, \tag{3.15}$$

where $C^* = \sup_{x \in \overline{\Omega}} \|\Pi_K(x)\|$, $r = \min\{\varepsilon, \alpha - 1/4\beta\}$, $\Omega := \operatorname{SOL}(K, f) + \alpha B$.

Proof. For any $x \in R^n$, by Theorem 3.4, we have

$$\operatorname{dist}(x, \Phi_\alpha^{-1}(0)) \leq \frac{||\Phi_{\alpha,\varepsilon}(x)||}{r} + \alpha. \tag{3.16}$$

This implies that there exists $y^* \in \Phi_\alpha^{-1}(0)$ such that $\|x - y^*\| \leq \|\Phi_{\alpha,\varepsilon}(x)\|/r + \alpha$. Since $y^* \in \Phi_\alpha^{-1}(0)$, thus we have $\Pi_K(y^*) \in \operatorname{SOL}(K, f)$.

Denote $\Pi_K(y^*)$ by x^*, then we have

$$\begin{aligned} \|x - x^*\| &= ||x - \Pi_K(y^*)|| \leq \|x - \Pi_K(x)\| + ||\Pi_K(x) - \Pi_K(y^*)|| \\ &\leq d(x, K) + \|x - y^*\| \leq d(x, K) + \frac{||\Phi_{\alpha,\varepsilon}(x)||}{r} + \alpha. \end{aligned} \tag{3.17}$$

Next, we establish two new error bounds by using the fixed-point and normal maps when f is strongly monotone and Lipschitz continuous. The approaches are different from those in [5, 7]. □

Theorem 3.7. *Let $f : R^n \to R^n$ be strongly monotone with modulus $c > 0$ and let f be Lipschitz continuous with constant $L > 0$. Let α be a fixed scalar such that $0 < \alpha < 4c/L^2$. Denote by x^* the unique solution of* VIP(K, f)*. Then one has*

$$\|x - x^*\| \leq \frac{||\pi_\alpha(x)||}{\alpha(c - \alpha L^2/4)}, \quad \forall x \in R^n. \tag{3.18}$$

Proof. Since f is strongly monotone with modulus c and Lipschitz continuous with constant L, by Lemma 2.4(iii), the fixed-point map $\pi_\alpha(x)$ is strongly monotone with modulus $\alpha(c - \alpha L^2/4)$, where $0 < \alpha < 4c/L^2$.

Since x^* is the unique solution of VIP(K, f), we have $\pi_\alpha(x^*) = 0$. By Lemma 2.6, we have

$$\|x - x^*\| \leq \frac{||\pi_\alpha(x)||}{\alpha(c - \alpha L^2/4)}, \quad \forall x \in R^n. \tag{3.19}$$

□

Theorem 3.8. *Let $f : R^n \to R^n$ be strongly monotone with modulus $c > 0$ and let f be Lipschitz continuous with constant $L > 0$. Let α be a fixed scalar such that $0 < \alpha < L^2/(4c)$. Then* *one has*

$$||x - \Phi_\alpha^{-1}(0)|| \le \frac{||\Phi_\alpha(x)||}{r}, \quad \forall x \in R^n, \tag{3.20}$$

where $0 < r < \alpha/2$ *and* $2r + L^2/4(\alpha - 2r) < c$.

Proof. Since f is strongly monotone with modulus c and Lipschitz continuous with constant L, by Lemma 2.4(iv), the normal map $\Phi_\alpha(x)$ is strongly monotone with modulus r, where $0 < \alpha < L^2/(4c)$, $0 < r < \alpha/2$ and $2r + L^2/4(\alpha - 2r) < c$.

By Lemma 2.6, we have

$$\|x - \Phi_\alpha^{-1}(0)\| \le \frac{||\Phi_\alpha(x)||}{r}, \quad \forall x \in R^n. \tag{3.21}$$

To conclude this section, we present a global bound for cocoercive VIP(K, f) in the term of $\Phi_\alpha(x)$. □

Corollary 3.9. *Let* $f : R^n \to R^n$ *be strongly monotone with modulus* $c > 0$ *and let* f *be Lipschitz continuous with constant* $L > 0$. *Let* α *be a fixed scalar such that* $0 < \alpha < L^2/(4c)$. *Then one has*

$$\operatorname{dist}(x, \mathrm{SOL}(K, f)) \le d(x, K) + \frac{||\Phi_\alpha(x)||}{r}, \quad \forall x \in R^n, \tag{3.22}$$

where $0 < r < \alpha/2$ *and* $2r + L^2/4(\alpha - 2r) < c$.

Proof. For any $x \in R^n$, by Theorem 3.8, we have

$$\|x - \Phi_\alpha^{-1}(0)\| \le \frac{||\Phi_\alpha(x)||}{r}, \tag{3.23}$$

which means that there exists $y^* \in \Phi_\alpha^{-1}(0)$ such that $\|x - y^*\| \le \|\Phi_\alpha(x)\|/r$.

Since $y^* \in \Phi_\alpha^{-1}(0)$, then we have $\Pi_K(y^*) \in \mathrm{SOL}(K, f)$.

Denote $\Pi_K(y^*)$ by x^*, then we obtain

$$\begin{aligned} \|x - x^*\| = ||x - \Pi_K(y^*)|| &\le ||x - \Pi_K(x)|| + ||\Pi_K(x) - \Pi_K(y^*)|| \\ &\le d(x, K) + \|x - y^*\| \le d(x, K) + \frac{||\Phi_\alpha(x)||}{r}. \end{aligned} \tag{3.24}$$

□

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