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Research Article

Multiple Positive Solutions of Nonhomogeneous Elliptic Equations in Unbounded Domains

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We will show that under suitable conditions on f and h, there exists a positive number λ^* such that the nonhomogeneous elliptic equation $-\Delta u + u = \lambda(f(x,u) + h(x))$ in Ω, $u \in H_0^1(\Omega)$, $N \geq 2$, has at least two positive solutions if $\lambda \in (0, \lambda^*)$, a unique positive solution if $\lambda = \lambda^*$, and no positive solution if $\lambda > \lambda^*$, where Ω is the entire space or an exterior domain or an unbounded cylinder domain or the complement in a strip domain of a bounded domain. We also obtain some properties of the set of solutions.

1. Introduction

Let $2^* = 2N/(N-2)$ for $N \geq 3$, $2^* = \infty$ for $N = 2$. In this paper, we study the existence, nonexistence, and multiplicity of solutions of the equation

$$-\Delta u + u = \lambda\big(f(x,u) + h(x)\big) \text{ in } \Omega, \quad u \text{ in } H_0^1(\Omega),\ u > 0 \text{ in } \Omega,\ N \geq 2, \tag{1.1$_\lambda$}$$

where $\lambda > 0$, $N = m + n \geq 2$, $n \geq 1$, $0 \in \omega \subseteq \mathbb{R}^m$ is a smooth bounded domain, $\mathbb{S} = \omega \times \mathbb{R}^n$, D is a smooth bounded domain in $\mathbb{R}^N$ such that $D \subset\subset \mathbb{S}$, $\Omega = \mathbb{S} \backslash \overline{D}$ is the exterior of this domain in the strip.

Associated to $(1.1)_\lambda$, we consider the functional I, for $u \in H_0^1(\Omega)$,

$$I(u) = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 + u^2\big)\,dx - \lambda\int_\Omega F(x,u^+)\,dx - \lambda\int_\Omega h(x)u\,dx, \tag{1.1}$$

where $F(x,t) = \int_0^t f(x,s)ds$.

It is assumed that $h(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 2,3$, $h(x) \geq 0$, $h(x) \not\equiv 0$, and $f(x,t)$ satisfies the following conditions:

($f1$) $f(x,\cdot) \in C^1([0,+\infty),\mathbb{R}^+)$, $f(x,t) \equiv 0$ for $x \in \mathbb{S}$, $t \leq 0$, and $\lim_{t\to 0}(f(x,t)/t) = 0$ uniformly for $x \in \mathbb{S}$;

($f2$) there exists a positive constant C such that for all $x \in \mathbb{S}$ and $t \in \mathbb{R}$,

$$0 < \frac{\partial}{\partial t} f(x,t) \leq C(1 + |t|^{p-2}), \tag{1.2}$$

where $2 < p < 2^*$;

($f3$) there exists a number $\theta \in [1/p, 1)$ such that

$$\theta t \frac{\partial}{\partial t} f(x,t) \geq f(x,t) > 0 \quad \forall x \in \mathbb{S},\ t > 0; \tag{1.3}$$

($f4$) there exists $\overline{f} : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|x|\to\infty} f(x,t) = \overline{f}(t)$ uniformly for bounded $t > 0$, $f(x,t) \geq \overline{f}(t)$, for all $x \in \mathbb{S}$, $t \geq 0$, and $\lim_{t\to\infty}(f(x,t)/t) = \infty$ uniformly for $x \in \mathbb{S}$;

($f5$) $f(x,\cdot) \in C^2(0,+\infty)$ and $(\partial^2/\partial t^2) f(x,t) \geq 0$ for all $x \in \mathbb{S}$, $t \geq 0$.

Given $\varepsilon > 0$, by ($f1$) and ($f2$), there exists a $C_\varepsilon > 0$ such that

$$0 \leq f(x,u) \leq \varepsilon u + C_\varepsilon |u|^{p-1}, \tag{1.4}$$

$$0 \leq F(x,u) \leq \varepsilon u^2 + C_\varepsilon |u|^p. \tag{1.5}$$

If $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N \backslash \overline{D}$ ($m = 0$ in our case), then the homogeneous case of problem $(1.1)_\lambda$ (i.e., the case $h(x) \equiv 0$) has been studied by many authors; see Cao [1] and the references therein. For the nonhomogeneous case ($h(x) \not\equiv 0$), Zhu-Zhou [2] have studied the multiplicity of positive solutions of equations similar to $(1.1)_\lambda$. Recently, Chen [3] showed that there exists a $\lambda^* > 0$ such that $(1.1)_\lambda$ has exactly two positive solutions if $\lambda \in (0,\lambda^*)$, and $(1.1)_\lambda$ has no positive solution when $\lambda \in (\lambda^*,\infty)$. However, her method cannot determine whether λ^* is bounded or infinite (at least for general nonlinearity $f(x,u)$). In this paper, one of our results answers the question (see Theorem 1.1). Now, we state our main results.

Theorem 1.1. *Let $\Omega = \mathbb{S} \backslash \overline{D}$ or $\Omega = \mathbb{R}^N \backslash \overline{D}$ or $\Omega = \mathbb{S}$ or $\Omega = \mathbb{R}^N$. Suppose $h(x) \geq 0$, $h(x) \not\equiv 0$, $h(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 2,3$, and $f(x,t)$ satisfies ($f1$)–($f5$). Then there exists $\lambda^* > 0$, $0 < \lambda^* < \infty$, such that*

(i) *equation $(1.1)_\lambda$ has at least two positive solutions u_λ, U_λ and $u_\lambda < U_\lambda$ if $\lambda \in (0,\lambda^*)$;*

(ii) *equation $(1.1)_{\lambda^*}$ has a unique positive solution u_{λ^*};*

(iii) *equation $(1.1)_\lambda$ has no positive solutions if $\lambda > \lambda^*$,*

where u_λ is the minimal solution of $(1.1)_\lambda$ and U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 4.

Theorem 1.2. *Under the assumptions of Theorem 1.1, then*

(i) *u_λ is strictly increasing with respect to λ, u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in (0,\lambda^*]$ and*

$$u_\lambda \longrightarrow 0 \quad \text{in } L^\infty(\Omega) \cap H_0^1(\Omega) \text{ as } \lambda \longrightarrow 0^+; \tag{1.6}$$

(ii) U_λ *is unbounded in* $L^\infty(\Omega)\cap H_0^1(\Omega)$ *for* $\lambda\in(0,\lambda^*)$, *that is,*

$$\lim_{\lambda\to 0^+}||U_\lambda|| = \lim_{\lambda\to 0^+}||U_\lambda||_\infty = \infty, \tag{1.7}$$

where $\|U_\lambda\| = (\int_\Omega(|\nabla U|^2+U^2)dx)^{1/2}$ *and* $\|U_\lambda\|_\infty = \sup_{x\in\Omega}|U(x)|$.

First of all, we list some properties of $f(x,t)$. The proof can be found in Zhu-Zhou [2, Lemma 2.1].

Lemma 1.3. *Assume* $(f1)$, $(f3)$, *and* $(f5)$ *hold, then*

(i) $tf(x,t)\geq \nu F(x,t)$ *for all* $x\in\mathbb{S}$, $t>0$ *and* $\nu = 1+\theta^{-1}\in(2,p+1]$;

(ii) $t^{-1/\theta}f(x,t)$ *is monotone nondecreasing and* $t^{-1}f(x,t)$ *is strictly monotone increasing for all* $x\in\mathbb{S}$, $t>0$;

(iii) $f(x,t_1+t_2)\geq f(x,t_1)+f(x,t_2)$ *and* $f(x,t_1+t_2)\not\equiv f(x,t_1)+f(x,t_2)$ *for all* $x\in\mathbb{S}$, $t_1,t_2>0$.

2. Asymptotic behavior of solutions

Throughout this paper, let $x=(y,z)$ be the generic point of $\mathbb{R}^N$ with $y\in\mathbb{R}^m$, $z\in\mathbb{R}^n$, $N=m+n\geq 2$, $n\geq 1$. We denote by C and C_i $(i=1,2,\dots)$ universal constants, maybe the constants here should be allowed to depend on n and p, unless some statement is given, and denote $(\partial/\partial t)f(x,t)$ and $(\partial^2/\partial t^2)f(x,t)$ by $f'(x,t)$ and $f''(x,t)$, respectively, in what follows.

We define

$$\begin{aligned}
\|u\| &= \left(\int_\Omega(|\nabla u|^2+u^2)dx\right)^{1/2},\\
\|u\|_p &= \left(\int_\Omega |u|^p dx\right)^{1/p}, \quad 2\leq p<\infty,\\
\|u\|_\infty &= \sup_{x\in\Omega}|u(x)|.
\end{aligned} \tag{2.1}$$

Now, we introduce the equation at infinity associated with $(1.1)_\lambda$ on an unbounded cylinder domain $\mathbb{S}$,

$$\begin{aligned}
-\Delta u+u &= \lambda\overline{f}(u) \quad \text{in } \mathbb{S},\\
u &\in H_0^1(\mathbb{S}), \quad N\geq 2.
\end{aligned} \tag{2.1$_\lambda$}$$

P. L. Lions has studied the following minimization problem closely related to $(2.1)_\lambda$:

$$S^\infty = \inf\{I^\infty(u): u\in H_0^1(\mathbb{S}),\ u\not\equiv 0,\ I^{\infty'}(u)=0\}>0, \tag{2.2}$$

where $I^\infty(u) = (1/2)\int_{\mathbb{S}}(|\nabla u|^2+u^2)dx - \lambda\int_{\mathbb{S}}\overline{F}(u^+)dx$, $\overline{F}(t) = \int_0^t\overline{f}(s)ds$. For this problem, also a minimum exists and is realized by a ground state solution $w>0$ in $\mathbb{S}$ such that

$$S^\infty = I^\infty(w) = \sup_{t\geq 0} I^\infty(tw). \tag{2.3}$$

In order to get the asymptotic behavior of solutions of $(1.1)_\lambda$ and $(2.1)_\lambda$, we need the following Lemmas 2.3 and 2.5. First, we quote two regularity lemmas (see Hsu [4] for the proof). Now, let $\mathbb{X}$ be a $C^{1,1}$ domain in $\mathbb{R}^N$ (typically the domains considered in the introduction).

Lemma 2.1. *Let $f : \mathbb{X}\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \mathbb{X}$, there holds*

$$|f(x,u)| \le C(|u| + |u|^{p-1}) \quad \textit{uniformly in } x \in \mathbb{X}, \tag{2.4}$$

where $2 < p < 2^*$. If $u \in H_0^1(\mathbb{X})$ is a weak solution of equation $-\Delta u = f(x,u) + h(x)$ in $\mathbb{X}$, where $h \in L^{N/2}(\mathbb{X}) \cap L^2(\mathbb{X})$, then $u \in L^q(\mathbb{X})$ for $q \in [2,\infty)$.

Lemma 2.2. *Let $g \in L^2(\mathbb{X}) \cap L^q(\mathbb{X})$ for some $q \in [2,\infty)$ and let $u \in H_0^1(\mathbb{X})$ be a weak solution of the equation $-\Delta u + u = g$ in $\mathbb{X}$. Then $u \in W^{2,q}(\mathbb{X})$ satisfies*

$$\|u\|_{W^{2,q}(\mathbb{X})} \le C(\|u\|_{L^q(\mathbb{X})} + \|g\|_{L^q(\mathbb{X})}), \tag{2.5}$$

where $C = C(N,q,\partial\mathbb{X})$.

By Lemmas 2.1 and 2.2, we obtain the first asymptotic behavior of solution of $(1.1)_\lambda$.

Lemma 2.3 (asymptotic lemma 1). *Let $(f1)$, $(f2)$ hold and let u be a weak solution of $(1.1)_\lambda$, then $u(y,z) \to 0$ as $|z| \to \infty$ uniformly for $y \in \omega$. Moreover, there exist positive constants C_1 and C_2 such that*

$$\|u\|_\infty \le C_1\|u\|_{q_0} + \lambda C_2\Big(\|u\|_{(p-1)q_0}^{p-1} + \|h\|_{q_0}\Big). \tag{2.6}$$

Proof. Suppose that u is a solution of $(1.1)_\lambda$, then $-\Delta u + u = \lambda(f(x,u) + h(x))$ in Ω. Since $h \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \ge 4$, $q_0 = 2$ if $N = 2,3$, this implies $h \in L^2(\Omega) \cap L^{N/2}(\Omega)$ for $N \ge 2$. By (1.4) and Lemma 2.1, we conclude that

$$u \in L^q(\Omega) \quad \text{for } q \in [2,\infty). \tag{2.7}$$

Hence $\lambda(f(x,u) + h(x)) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ and by Lemma 2.2, we have

$$u \in W^{2,2}(\Omega) \cap W^{2,q_0}(\Omega), \qquad q_0 > N/2 \quad \text{if } N \ge 4, \quad q_0 = 2 \quad \text{if } N = 2,3. \tag{2.8}$$

Now, by the Sobolev embedding theorem, we obtain that $u \in C_b(\overline{\Omega})$. It is well known that the Sobolev embedding constants are independent of domains (see Adams [5]). Thus there exists a constant C such that for $R > 0$,

$$\|u\|_{L^\infty(\Omega\backslash B_R)} \le C\|u\|_{W^{2,q_0}(\Omega\backslash B_R)} \quad \text{for } N \ge 2, \tag{2.9}$$

where $B_R = \{x = (y,z) \in \Omega \mid |z| \le R\}$. From this, we conclude that $u(y,z) \to 0$ as $|z| \to \infty$ uniformly for $y \in \omega$. By Lemma 2.2 and (1.4), we also have that

$$\begin{aligned} \|u\|_\infty &\le C\|u\|_{W^{2,q_0}(\Omega)} \\ &\le C\Big(\|u\|_{q_0} + \big\|\lambda f(x,u) + \lambda h(x)\big\|_{q_0}\Big) \\ &\le C_1\|u\|_{q_0} + \lambda C_2\Big(\|u\|_{(p-1)q_0}^{p-1} + \|h\|_{q_0}\Big), \end{aligned} \tag{2.10}$$

where C_1, C_2 are constants independent of λ. □

Remark 2.4. Let w be a positive solution of $(2.1)_\lambda$. If $h(x) \equiv 0$ and $f(x,t) \equiv \overline{f}(t)$ for all $x \in \mathbb{S}$, $t \in \mathbb{R}$, by Lemma 2.3, then we have that $w(y,z) \to 0$ as $|z| \to \infty$ uniformly for $y \in \omega$.

We use Lemma 2.3, and modify the proof in Hsu [6], we obtain a precise asymptotic behavior of solutions of $(2.1)_\lambda$ at infinity and the second asymptotic behavior of solutions of $(1.1)_\lambda$.

Lemma 2.5 (asymptotic lemma 2). *Let w be a positive solution of $(2.1)_\lambda$, let u be a positive solution of $(1.1)_\lambda$ and let φ be the first positive eigenfunction of the Dirichlet problem $-\Delta\varphi = \mu_1\varphi$ in ω, then for any $\varepsilon > 0$ with $0 < \varepsilon < 1 + \mu_1$, there exist constants $C, C_\varepsilon > 0$ such that*

$$\begin{aligned} w(y,z) &\le C_\varepsilon\varphi(y)\exp\Big(-\sqrt{1+\mu_1-\varepsilon}|z|\Big), \\ w(y,z) &\ge C\varphi(y)\exp\Big(-\sqrt{1+\mu_1}|z|\Big)|z|^{-(n-1)/2} \quad \textit{as } |z| \longrightarrow \infty,\ y \in \overline{\omega}, \\ u(y,z) &\ge C\varphi(y)\exp\Big(-\sqrt{1+\mu_1}|z|\Big)|z|^{-(n-1)/2}. \end{aligned} \tag{2.11}$$

Proof. (i) First, we claim that for any $\varepsilon > 0$ with $0 < \varepsilon < 1 + \mu_1$, there exists $C_\varepsilon > 0$ such that

$$w(y,z) \le C_\varepsilon\varphi(y)\exp\Big(-\sqrt{1+\mu_1-\varepsilon}|z|\Big) \quad \text{as } |z| \longrightarrow \infty,\ y \in \overline{\omega}. \tag{2.12}$$

Without loss of generality, we may assume $\varepsilon < 1$. Now given $\varepsilon > 0$, by $(f1)$, $(f4)$, and Remark 2.4, we may choose R_0 large enough such that

$$\lambda\overline{f}(w(y,z)) \le \lambda f(x,w(y,z)) \le \varepsilon w(y,z) \quad \text{for } |z| \ge R_0. \tag{2.13}$$

Let $q = (q_y, q_z)$, $q_y \in \partial\omega$, $|q_z| = R_0$, and B a small ball in Ω such that $q \in \partial B$. Since $\varphi(y) > 0$ for $x = (y,z) \in B$, $\varphi(q_y) = 0$, $w(x) > 0$ for $x \in B$, $w(q) = 0$, by the strong maximum principle $(\partial\varphi/\partial y)(q_y) < 0$, $(\partial w/\partial x)(q) < 0$. Thus

$$\lim_{\substack{x\to q \\ |z|=R_0}} \frac{w(x)}{\varphi(y)} = \frac{(\partial w/\partial x)(q)}{(\partial\varphi/\partial y)(q_y)} > 0. \tag{2.14}$$

Note that $w(x)\varphi^{-1}(y) > 0$ for $x = (y,z)$, $y \in \omega$, $|z| = R_0$. Thus $w(x)\varphi^{-1}(y) > 0$ for $x = (y,z)$, $y \in \overline{\omega}$, $|z| = R_0$. Since $\varphi(y)\exp(-\sqrt{1+\mu_1-\varepsilon}|z|)$ and $w(x)$ belong to $C^1(\overline{\omega \times \partial B_{R_0}(0)})$, if set

$$C_\varepsilon = \sup_{y\in\overline{\omega},\, |z|=R_0} \Big(w(x)\varphi^{-1}(y)\exp\Big(\sqrt{1+\mu_1-\varepsilon}R_0\Big)\Big), \tag{2.15}$$

then $C_\varepsilon > 0$ and

$$C_\varepsilon \varphi(y) \exp\left(-\sqrt{1+\mu_1-\varepsilon} R_0\right) \geq w(x) \quad \text{for } y \in \overline{\omega},\ |z| = R_0. \tag{2.16}$$

Let $\Phi_1(x) = C_\varepsilon \varphi(y) \exp(-\sqrt{1+\mu_1-\varepsilon}|z|)$ for $x \in \overline{\Omega}$. Then for $|z| \geq R_0$, we have

$$\begin{aligned} \Delta(w-\Phi_1)(x) - (w-\Phi_1)(x) &= -\lambda \overline{f}(w(x)) + \left(\varepsilon + \frac{\sqrt{1+\mu_1-\varepsilon}(n-1)}{|z|}\right)\Phi_1(x) \\ &\geq -\varepsilon w(x) + \varepsilon \Phi_1(x) \\ &= \varepsilon(\Phi_1 - w)(x). \end{aligned} \tag{2.17}$$

Hence $\Delta(w-\Phi_1)(x) - (1-\varepsilon)(w-\Phi_1)(x) \geq 0$ for $|z| \geq R_0$.

The strong maximum principle implies that $w(x) - \Phi_1(x) \leq 0$ for $x = (y,z)$, $y \in \overline{\omega}$, $|z| \geq R_0$, and therefore we get this claim.

(ii) Let

$$\Psi(y,z) = \left(1 + \frac{1}{\sqrt{|z|}}\right)\varphi(y) \exp\left(-\sqrt{1+\mu_1}|z|\right)|z|^{-(n-1)/2} \quad \text{for } (y,z) \in \Omega. \tag{2.18}$$

It is very easy to show that

$$-\Delta\Psi + \Psi \leq 0 \quad \text{for } y \in \overline{\omega},\ |z| \text{ large}. \tag{2.19}$$

Therefore, by means of the maximum principle, there exists a constant $C > 0$ such that

$$\begin{aligned} w(y,z) &\geq C\varphi(y) \exp\left(-\sqrt{1+\mu_1}|z|\right)|z|^{-(n-1)/2} \\ u(y,z) &\geq C\varphi(y) \exp\left(-\sqrt{1+\mu_1}|z|\right)|z|^{-(n-1)/2} \end{aligned} \quad \text{as } |z| \longrightarrow \infty,\ y \in \overline{\omega}. \tag{2.20}$$

This completes the proof of Lemma 2.5. □

3. Existence of the minimal solution

We now prove the existence of minimal positive solutions of $(1.1)_\lambda$.

Lemma 3.1. *If $(f1)$ and $(f2)$ hold, then for any given $\rho > 0$, there exists $\lambda_0 > 0$ such that for $\lambda \in (0,\lambda_0)$, one has $I(u) > 0$ for all $u \in S_\rho = \{u \in H_0^1(\Omega) \mid \|u\| = \rho\}$. Moreover, for any $\varepsilon \geq 0$, there exists $\delta > 0$ ($\delta \leq \rho$) such that $I(u) \geq -\varepsilon$ for all $u \in \{u \in H_0^1(\Omega) \mid \rho - \delta \leq \|u\| = \rho\}$.*

Proof. By (1.5), the Sobolev embedding theorem, and the Hölder inequality, we have that, for all $u \in S_\rho$,

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \lambda\int_\Omega F(x,u^+)\,dx - \lambda\int_\Omega hu\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \lambda\int_\Omega (\varepsilon|u|^2 + C_\varepsilon|u|^p)\,dx - \lambda\|h\|_2\|u\| \\ &\geq \frac{1}{2}\|u\|^2 - \lambda C(\|u\|^2 + \|u\|^p)\,dx - \lambda\|h\|_2\|u\| \\ &\geq \rho\left(\frac{1}{2}\rho - \lambda C(\rho + \rho^{p-1}) - \lambda\|h\|_2\right), \end{aligned} \tag{3.1}$$

where $C > 0$ is a constant which is independent of λ, ρ. Hence by (3.1), there exists $\lambda_0 > 0$ such that for $\lambda \in (0,\lambda_0)$, we have $I(u) > 0$ for all $u \in S_\rho$.

Moreover, we can choose $\lambda_0 > 0$ small enough such that

$$\frac{\partial}{\partial\rho}\left(\frac{1}{2}\rho - \lambda C(\rho + \rho^{p-1})\right) = \frac{1}{2} - \lambda\left(1 + (p-1)\rho^{p-2}\right) > 0 \quad \text{for } \lambda \in (0,\lambda_0). \tag{3.2}$$

Then for any $\varepsilon \geq 0$, there exists $\delta > 0$ ($\delta \leq \rho$) such that $I(u) \geq -\varepsilon$ for all $u \in \{u \in H_0^1(\Omega) \mid \rho - \delta \leq \|u\| \leq \rho\}$. □

Lemma 3.2. *Assume $(f1)$ and $(f2)$ hold. If λ_0 is chosen as in Lemma 3.1 and $\lambda \in (0,\lambda_0)$, then there exists a $u_0 \in B_\rho$ such that u_0 is a positive solution of $(1.1)_\lambda$.*

Proof. Since $h \not\equiv 0$ and $h \geq 0$, we can choose a function $\varphi \in H_0^1(\Omega)$ such that $\int_\Omega h\varphi > 0$. For $t \in (0,+\infty)$, then by (1.5),

$$\begin{aligned} I(t\varphi) &= \frac{t^2}{2}\int_\Omega \left(|\nabla\varphi|^2 + \varphi^2\right) - \lambda\int_{\mathbb{R}_+^N} F(x,t\varphi^+) - \lambda t\int_\Omega h\varphi \\ &\leq \frac{t^2}{2}\|\varphi\|^2 + \lambda C t^2\int_\Omega \left(|\varphi|^2 + t^{p-2}|\varphi|^p\right) - \lambda t\int_\Omega h\varphi. \end{aligned} \tag{3.3}$$

Then for t small enough, $I(t\varphi) < 0$. So $\alpha = \inf\{I(u) \mid u \in \overline{B_\rho}\}$. Clearly $\alpha > -\infty$. By Lemma 3.1, there exists ρ' such that $0 < \rho' < \rho$ and $\alpha = \inf\{I(u) \mid u \in \overline{B_{\rho'}}\}$. By Ekeland's variational principle [7], there exists a $(PS)_\alpha$-sequence $\{u_k\} \subset \overline{B_{\rho'}}$, that is, $I(u_k) = \alpha + o(1)$ and $I'(u_k) = o(1)$ strongly in $H^{-1}(\Omega)$ as $k \to \infty$. Then there exists a subsequence $\{u_k\}$ and $u_0 \in H_0^1(\Omega)$ such that $u_k \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, $u_k \to u_0$ strongly in $L^q_{\text{loc}}(\Omega)$ for $2 \leq q < 2^*$ and $u_k \to u_0$ a.e. in Ω. Since $I'(u_k) = o(1)$ strongly in $H^{-1}(\Omega)$ as $k \to \infty$, and by $(f1)$ and $(f2)$, we have $I'(u_0) = 0$ in $H^{-1}(\Omega)$, that is, u_0 is a weak nonnegative solution of $(1.1)_\lambda$; and since $h \not\equiv 0$, by the maximum principle for weak solutions, we have $u_0 > 0$ in Ω. □

By the standard barrier method, we prove the following lemma.

Lemma 3.3. *If $(f1)$ and $(f2)$ hold, then there exists $\lambda^* \in (0,+\infty]$ such that*

(i) *for any $\lambda \in (0,\lambda^*)$, $(1.1)_\lambda$ has a minimal positive solution u_λ and u_λ is strictly increasing in λ;*

(ii) *if $\lambda > \lambda^*$, $(1.1)_\lambda$ has no positive solution.*

Proof. Setting $Q_\lambda = \{0 < \lambda < +\infty \mid (1.1)_\lambda \text{ is solvable}\}$, by Lemma 3.2, we have Q_λ is nonempty. Denoting $\lambda^* = \sup Q_\lambda > 0$, we claim that $(1.1)_\lambda$ has at least one solution for all $\lambda \in (0,\lambda^*)$. In fact, for any $\lambda \in (0,\lambda^*)$, by the definition of λ^*, we know that there exists $\lambda' > 0$ and $0 < \lambda < \lambda' < \lambda^*$ such that $(1.1)_{\lambda'}$ has a solution $u_{\lambda'} > 0$, that is,

$$-\Delta u_{\lambda'} + u_{\lambda'} = \lambda'\left(f(x,u_{\lambda'}) + h(x)\right) \geq \lambda\left(f(x,u_{\lambda'}) + h(x)\right). \tag{3.4}$$

Then $u_{\lambda'}$ is a supersolution of $(1.1)_\lambda$. From $h(x) \geq 0$ and $h(x) \not\equiv 0$, it is easy to see that 0 is a subsolution of $(1.1)_\lambda$. By the standard barrier method, there exists a solution $u_\lambda > 0$ of $(1.1)_\lambda$ such that $0 \leq u_\lambda \leq u_{\lambda'}$. Since 0 is not a solution of $(1.1)_\lambda$ and $\lambda' > \lambda$, the maximum principle implies that $0 < u_\lambda < u_{\lambda'}$. Again using a result of Amann [8, Theorem 9.4], we can choose a minimal positive solution u_λ of $(1.1)_\lambda$. □

Let u_λ be the minimal positive solution of $(1.1)_\lambda$ for $\lambda \in (0,\lambda^*)$, we study the following eigenvalue problem

$$\begin{aligned} -\Delta v + v = \sigma_\lambda f'(x,u_\lambda)v \quad \text{in } \Omega, \\ v \in H_0^1(\Omega), \quad v > 0 \text{ in } \Omega, \end{aligned} \tag{3.5}$$

then we have the following.

Lemma 3.4. *Assume $(f1)$–$(f5)$ hold, and let the first eigenvalue σ_λ of (3.5) be defined by*

$$\sigma_\lambda = \inf\left\{\int_\Omega (|\nabla v|^2 + v^2)\,dx \mid v \in H_0^1(\Omega),\ \int_\Omega f'(x,u_\lambda)v^2 dx = 1\right\}. \tag{3.6}$$

Then

(i) *σ_λ is achieved;*

(ii) *$\sigma_\lambda > \lambda$ and is strictly decreasing in λ, $\lambda \in (0,\lambda^*)$;*

(iii) *$\lambda^* < +\infty$ and $(1.1)_{\lambda^*}$ has a minimal positive solution u_{λ^*}.*

Proof. (i) Indeed, recall assumption $(f3)$, by the definition of σ_λ, we know that $0 < \sigma_\lambda < +\infty$. Let $\{v_k\} \subset H_0^1(\Omega)$ be a minimizing sequence of σ_λ, that is,

$$\int_\Omega f'(x,u_\lambda)v_k^2 dx = 1, \qquad \int_\Omega \left(|\nabla v_k|^2 + v_k^2\right)dx \longrightarrow \sigma_\lambda \quad \text{as } k \longrightarrow \infty. \tag{3.7}$$

This implies that $\{v_k\}$ is bounded in $H_0^1(\Omega)$, then there exists a subsequence, still denoted by $\{v_k\}$ and some $v_0 \in H_0^1(\Omega)$ such that

$$\begin{aligned} &v_k \rightharpoonup v_0 \quad \text{weakly in } H_0^1(\Omega), \\ &v_k \longrightarrow v_0 \quad \text{almost everywhere in } \Omega, \\ &v_k \longrightarrow v_0 \quad \text{strongly in } L^s_{\text{loc}}(\Omega) \quad \text{for } 2 \le s < 2^*. \end{aligned} \tag{3.8}$$

Thus

$$\int_\Omega \left(|\nabla v_0|^2 + v_0^2\right)dx \le \liminf \int_\Omega \left(|\nabla v_k|^2 + v_k^2\right)dx = \sigma_\lambda. \tag{3.9}$$

By Lemma 2.3 and $(f1)$, we have $f'(x,u_\lambda) \to 0$ as $|x| \to \infty$, it is standard to show that v_0 achieves σ_λ. Clearly $|v_0|$ also achieves σ_λ. By (3.5) and the maximum principle, we may assume $v_0 > 0$ in Ω.

(ii) We now prove $\sigma_\lambda > \lambda$. Setting $\lambda' > \lambda > 0$ and $\lambda' \in (0,\lambda^*)$, by Lemma 3.3, $(1.1)_{\lambda'}$ has a positive solution $u_{\lambda'}$. Since u_λ is the minimal positive solution of $(1.1)_\lambda$, then $u_{\lambda'} > u_\lambda$ as $\lambda' > \lambda$. By virtue of $(1.1)_{\lambda'}$ and $(1.1)_\lambda$, we see that

$$-\Delta(u_{\lambda'} - u_\lambda) + (u_{\lambda'} - u_\lambda) = \lambda' f(x,u_{\lambda'}) - \lambda f(x,u_\lambda) + (\lambda' - \lambda)h. \tag{3.10}$$

Applying the Taylor expansion and noting that $\lambda' > \lambda$, $h(x) \geq 0$, and $f''(x,t) \geq 0$, $f(x,t) > 0$ for all $t > 0$, we get

$$\begin{aligned} -\Delta(u_{\lambda'} - u_\lambda) + (u_{\lambda'} - u_\lambda) &\geq (\lambda' - \lambda) f(x,u_\lambda) + \lambda' f'(x,u_\lambda)(u_{\lambda'} - u_\lambda) \\ &> \lambda f'(x,u_\lambda)(u_{\lambda'} - u_\lambda). \end{aligned} \tag{3.11}$$

Let $v_0 \in H_0^1(\Omega)$ and $v_0 > 0$ solves (3.5). Multiplying (3.11) by v_0 and noting (3.5), then we get

$$\sigma_\lambda \int_\Omega f'(x,u_\lambda)(u_{\lambda'} - u_\lambda) v_0 dx > \lambda \int_\Omega f'(x,u_\lambda)(u_{\lambda'} - u_\lambda) v_0 dx, \tag{3.12}$$

hence $\sigma_\lambda > \lambda$. Now, let v_λ be a minimizer of σ_λ, then

$$\int_\Omega f'(x,u_{\lambda'}) v_\lambda^2 dx > \int_\Omega f'(x,u_\lambda) v_\lambda^2 dx = 1, \tag{3.13}$$

and there exists t, with $0 < t < 1$ such that

$$\int_\Omega f'(x,u_{\lambda'})(t v_\lambda)^2 dx = 1. \tag{3.14}$$

Therefore

$$\sigma_{\lambda'} \leq t^2 ||v_\lambda||^2 < ||v_\lambda||^2 = \sigma_\lambda \tag{3.15}$$

showing that σ_λ is strictly decreasing in λ for $\lambda \in (0,\lambda^*)$.

(iii) We show next that $\lambda^* < +\infty$. Let $\lambda_0 \in (0,\lambda^*)$ be fixed. For any $\lambda \geq \lambda_0$, we have $\sigma_\lambda > \lambda$ and by (3.15), then

$$\sigma_{\lambda_0} \geq \sigma_\lambda > \lambda \tag{3.16}$$

for all $\lambda \in [\lambda_0,\lambda^*)$. Thus $\lambda^* < +\infty$.

By (3.5) and $\sigma_\lambda > \lambda$, we have

$$\int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2) dx > \int_\Omega \lambda f'(x,u_\lambda) u_\lambda^2 dx, \tag{3.17}$$

and also we have

$$\int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2) dx - \int_\Omega \lambda f(x,u_\lambda) u_\lambda dx - \int_\Omega \lambda h(x) u_\lambda dx = 0. \tag{3.18}$$

By $(f3)$ and (3.17), we have that

$$\begin{aligned} \int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2) dx &= \int_\Omega \lambda f(x,u_\lambda) u_\lambda dx + \int_\Omega \lambda h(x) u_\lambda dx \\ &\leq \theta \int_\Omega \lambda f'(x,u_\lambda) u_\lambda^2 dx + \lambda \|h\|_2 ||u_\lambda|| \\ &\leq \theta ||u_\lambda||^2 + \lambda \|h\|_2 ||u_\lambda||. \end{aligned} \tag{3.19}$$

This implies that

$$||u_\lambda|| \le \frac{\lambda}{1-\theta}\|h\|_2 \tag{3.20}$$

for all $\lambda \in (0,\lambda^*)$. By Lemma 3.3(i), the solution u_λ is strictly increasing with respect to λ; we may suppose that

$$u_\lambda \rightharpoonup u_{\lambda^*} \quad \text{weakly in } H_0^1(\Omega) \text{ as } \lambda \longrightarrow \lambda^*, \tag{3.21}$$

and by (1.4), we obtain that

$$\begin{aligned} \int_\Omega (\nabla u_\lambda \cdot \nabla\varphi + u_\lambda\varphi)\,dx &\longrightarrow \int_\Omega (\nabla u_{\lambda^*} \cdot \nabla\varphi + u_{\lambda^*}\varphi)\,dx, \\ \lambda\int_\Omega (f(x,u_\lambda)+h)\varphi\,dx &\longrightarrow \lambda^*\int_\Omega (f(x,u_{\lambda^*})+h)\varphi\,dx \end{aligned} \quad \text{as } \lambda \longrightarrow \lambda^* \tag{3.22}$$

for all $\varphi \in H_0^1(\Omega)$. Hence u_{λ^*} is a minimal positive solution of $(1.1)_{\lambda^*}$. This completes the proof of Lemma 3.4. □

4. Existence of second solution

When $\lambda \in (0,\lambda^*)$, we know that $(1.1)_\lambda$ has a minimal positive solution u_λ by Lemma 3.3, then we need only to prove that $(1.1)_\lambda$ has another positive solution in the form of $U_\lambda = u_\lambda + \bar{v}$, where $\bar{v}$ is a solution of the following equation:

$$\begin{gathered} -\Delta v + v = \lambda(f(x,u_\lambda+v) - f(x,u_\lambda)) \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega, \quad v \in H_0^1(\Omega). \end{gathered} \tag{4.1}$$

We define the energy functional $J : H_0^1(\Omega) \to \mathbb{R}$ as follows:

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + v^2)\,dx - \lambda\int_\Omega (F(x,u_\lambda+v^+) - F(x,u_\lambda) - f(x,u_\lambda)v^+)\,dx. \tag{4.2}$$

Using the monotonicity of f and the maximum principle, we know that the nontrivial critical points of energy functional J are the positive solutions of (4.1).

First, we give an inequality about concerning f and u_λ.

Lemma 4.1. *If $(f1)$ and $(f2)$ hold, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$f(x,u_\lambda+s) - f(x,u_\lambda) - f'(x,u_\lambda)s \le \varepsilon s + C_\varepsilon s^{p-1}, \quad s \ge 0, \ \textit{uniformly } \forall x \in \mathbb{S}, \tag{4.3}$$

where $1 < p < 2^* - 1$ and u_λ is the minimal solution of $(1.1)_\lambda$.

Proof. By $(f1)$, $(f2)$, (1.4), and Lemma 2.3, we obtain $u_\lambda \in L^\infty(\Omega)$ and

$$\lim_{s\to 0}\frac{f(x,u_\lambda+s)-f(x,u_\lambda)-f'(x,u_\lambda)s}{s}=0,$$
$$0\le \limsup_{s\to\infty}\frac{f(x,u_\lambda+s)-f(x,u_\lambda)-f'(x,u_\lambda)s}{s^{p-1}}\le C_\varepsilon \tag{4.4}$$

uniformly for all $x\in\mathbb{S}$. Thus, it is clear that Lemma 4.1 holds. □

Lemma 4.2. *If $(f1)$–$(f5)$ hold, then there exist $\rho>0$ and $\alpha>0$ such that*

$$J(v)\ge\alpha>0 \tag{4.5}$$

for all $v\in S_\rho=\{u\in H_0^1(\Omega)\mid \|u\|=\rho\}$.

Proof. By Lemma 3.4, it is easy to see that, for all $v\in H_0^1(\Omega)$,

$$\int_\Omega(|\nabla v|^2+v^2)\,dx\ge\sigma_\lambda\int_\Omega f'(x,u_\lambda)v^2\,dx. \tag{4.6}$$

Again by Lemma 4.1 and the Sobolev embedding theorem, we obtain that

$$\begin{aligned}
J(v)&=\frac12\int_\Omega(|\nabla v|^2+v^2)\,dx-\lambda\int_\Omega\big(F(x,u_\lambda+v^+)-F(x,u_\lambda)-f(x,u_\lambda)v^+\big)\,dx\\
&=\frac12\|v\|^2-\frac{\lambda}{2}\int_\Omega f'(x,u_\lambda)\left|v^+\right|^2dx\\
&\quad-\lambda\int_\Omega\int_0^{v^+}\big(f(x,u_\lambda+s)-f(x,u_\lambda)-f'(x,u_\lambda)s\big)\,ds\,dx\\
&\ge\frac12\|v\|^2-\frac{\lambda}{2}\int_\Omega f'(x,u_\lambda)\left|v^+\right|^2dx-\frac12\lambda\varepsilon\int_\Omega\left|v^+\right|^2dx-\frac1p\lambda C_\varepsilon\int_\Omega\left|v^+\right|^pdx\\
&\ge\frac12\|v\|^2-\frac{\lambda}{2}\sigma^{-1}\|v\|^2-\frac12\lambda\varepsilon\|v\|^2-\lambda C_\varepsilon\|v\|^p\\
&=\frac12\sigma_\lambda^{-1}(\sigma_\lambda-\lambda-\lambda\sigma_\lambda\varepsilon)\|v\|^2-\lambda C_\varepsilon\|v\|^p.
\end{aligned}\tag{4.7}$$

Since $\sigma_\lambda>\lambda$, we may choose $\varepsilon>0$ small enough such that $\sigma_\lambda-\lambda-\lambda\sigma_\lambda\varepsilon>0$. If we take $\varepsilon=(\sigma_\lambda-\lambda)/2\lambda\sigma_\lambda$, then

$$J(v)\ge\frac14\sigma_\lambda^{-1}(\sigma_\lambda-\lambda)\|v\|^2-C\|v\|^p. \tag{4.8}$$

Hence, there exist $\rho>0$ and $\alpha>0$ such that $J(v)\ge\alpha>0$ for all $v\in S_\rho=\{u\in H_0^1(\Omega)\mid \|u\|=\rho\}$. □

Proposition 4.3. *Assume $(f1)$–$(f4)$ hold. Let $\{v_k\}$ be a $(PS)_c$-sequence of J. Then there exists a subsequence (still denoted by $\{v_k\}$) for which the following holds: there exist an integer $l\ge0$, sequences $\{x_k^i\}\subseteq\mathbb{R}^N$, $1\le i\le l$, $k\in\mathbb{N}$, of the form $(0,z_k^i)\in\mathbb{S}$, a solution $\overline{v}$ of (4.1), and solutions u^i of $(2.1)_\lambda$, $1\le i\le l$, such that, for some subsequence $\{v_k\}$, as $k\to\infty$,* *one has*

$$\begin{gathered} v_k \rightharpoonup \overline{v} \quad \text{weakly in } H_0^1(\Omega), \\ J(v_k) \longrightarrow J(\overline{v}) + \sum_{i=1}^{l} I^\infty(u^i), \\ v_k - \left(\overline{v} + \sum_{i=1}^{l} u^i(x - x_k^i)\right) \longrightarrow 0 \quad \text{strong in } H_0^1(\mathbb{S}), \\ |x_k^i| \longrightarrow \infty, \quad |x_k^i - x_k^j| \longrightarrow \infty, \quad 1 \le i \ne j \le l, \end{gathered} \tag{4.9}$$

where one agrees that in the case $l = 0$ the above holds without u^i, x_k^i.

Proof. This result can be derived from the arguments in [9] (see also [10–12]). Here we omit it. □

Now, let δ be small enough, D^δ a δ-tubular neighborhood of D such that $D^\delta \subset\subset \mathbb{S}$. Let $\eta(x) : \mathbb{S} \to [0,1]$ be a C^∞ cutoff function such that $0 \le \eta \le 1$ and

$$\eta(x) = \begin{cases} 0, & \text{if } x \in D; \\ 1, & \text{if } x \in \mathbb{S} \backslash \overline{D}^\delta. \end{cases} \tag{4.10}$$

Let $e_N = (0,0,\dots,0,1) \in \mathbb{R}^N$, denote

$$\begin{aligned} \tau_0 &= 2 \sup_{x \in D^\delta} |x| + 1, \\ w_\tau(x) &= w(x - \tau e_N), \end{aligned} \quad \tau \in [0,\infty), \tag{4.11}$$

where w is a ground state solution of $(2.1)_\lambda$.

Lemma 4.4. *If $(f1)$–$(f5)$ hold, then*

(i) *there exists $t_0 > 0$ such that $J(t\eta w_\tau) < 0$ for $t \ge t_0$, $\tau \ge \tau_0$,*

(ii) *there exists $\tau_* > 0$ such that the following inequality holds for $\tau \ge \tau_*$:*

$$0 < \sup_{t \ge 0} J(t\eta w_\tau) < I^\infty(w) = S^\infty. \tag{4.12}$$

Proof. (i) By the definition of η and Lemma 1.3(iii), we have

$$\begin{aligned} J(t\eta w_\tau) &= \frac{1}{2}\int_\Omega \left(|\nabla(t\eta w_\tau)|^2 + (t\eta w_\tau)^2\right)dx - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(x,u_\lambda + s) - f(x,u_\lambda)\right)ds\,dx \\ &\le \frac{t^2}{2}\int_\Omega\left(|\nabla(\eta w_\tau)|^2 + (\eta w_\tau)^2\right)dx - \lambda\int_{\mathbb{S}\backslash\overline{D}^\delta} F(x,tw_\tau)\,dx. \end{aligned} \tag{4.13}$$

From Lemma 1.3(ii), we have that $F(x,u)/(\nu^{-1}u^\nu)$ is monotone nondecreasing for $u > 0$, where $\nu = 1 + \theta^{-1} > 2$. Thus for any given constant $C > 0$, there exists $u_0 \ge 0$ such that

$$F(x,u) \ge Cu^\nu \quad \forall u \ge u_0. \tag{4.14}$$

Let r_0 be a positive constant such that $B^m(0;r_0) = \{y \mid |y| \le r_0\} \subset\subset \omega$, $B^n(0;1) = \{z \mid |z| \le 1\}$, $\Omega_1 = B^m(0;r_0)\times B^n(0;1)$, and $\Omega_{1\tau} = B^m(0;r_0)\times\{z+\tau e_N \mid |z|\le 1\}$. By the definition of τ_0, we have that $\Omega_{1\tau} \subset\subset \Omega\backslash\overline{D}^{\delta}$ for all $\tau \ge \tau_0$. This also implies that there exists $t_0 \ge 0$, as $t \ge t_0$, we have

$$F(x,tw_\tau) \ge Ct^\nu w^\nu \quad \forall \tau \ge \tau_0,\ \forall x \in \Omega_{1\tau}. \tag{4.15}$$

Therefore as $t > t_0$ and $\tau \ge \tau_0$,

$$\begin{aligned} J(t\eta w_\tau) &\le \frac{t^2}{2}\int_\Omega \left(\left|\nabla(\eta w_\tau)\right|^2 + (\eta w_\tau)^2\right)dx - \lambda C t^\nu \int_{\Omega_{1\tau}} w_\tau^\nu dx \\ &\le \frac{t^2}{2}\|\eta w_\tau\|^2 - \lambda C t^\nu \int_{\Omega_1} w^\nu dx. \end{aligned} \tag{4.16}$$

Since $\nu > 2$, we can choose $t_0 > 0$ large enough such that (i) holds.

(ii) By (i), J is continuous on $H_0^1(\Omega)$, $J(0) = 0$, and Lemma 4.2, we know that there exists t_1 with $0 < t_1 < t_0$ such that

$$\sup_{t\ge 0} J(t\eta w_\tau) = \sup_{t_1 \le t \le t_0} J(t\eta w_\tau) \quad \forall \tau \ge \tau_0. \tag{4.17}$$

Now, we define $\eta_\tau(x) = \eta(x+\tau e_N)$ for all $x \in \mathbb{S}$. For $\tau \ge \tau_0$, $t_1 \le t \le t_0$, by $(f4)$, (1.4), (2.3), Lemmas 1.3 and 2.5, we have

$$\begin{aligned} J(t\eta w_\tau) &= \frac{t^2}{2}\int_\Omega \left(\left|\nabla(\eta w_\tau)\right|^2 + (\eta w_\tau)^2\right)dx - \lambda\int_\Omega F(x,t\eta w_\tau)dx \\ &\quad - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(x,u_\lambda+s) - f(x,u_\lambda) - f(x,s)\right)ds\,dx \\ &\le \frac{t^2}{2}\int_{\mathbb{S}}(-\Delta w + w)(\eta_\tau^2 w)dx + \frac{t^2}{2}\int_{\mathbb{S}}\left|\nabla\eta_\tau\right|^2|w|^2dx - \lambda\int_{\mathbb{S}}F(x,tw_\tau)dx \\ &\quad + \lambda\int_{\mathbb{S}}\int_{t\eta w_\tau}^{tw_\tau} f(x,s)ds\,dx - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(x,u_\lambda+s) - f(x,u_\lambda) - f(x,s)\right)ds\,dx \\ &\le \frac{t^2}{2}\int_{\mathbb{S}}\left(|\nabla w|^2 + w^2\right)dx - \lambda\int_{\mathbb{S}}\overline{F}(tw_\tau)dx + \frac{t_0^2}{2}\int_{D^\delta\backslash D}|\nabla\eta|^2\left|w_\tau\right|^2dx \\ &\quad + \lambda\int_{D^\delta}\int_0^{tw_\tau} f(x,s)ds\,dx - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(x,u_\lambda+s) - f(x,u_\lambda) - f(x,s)\right)ds\,dx \\ &\le S^\infty + C_\varepsilon\exp\left(-2\sqrt{1+\mu_1-\varepsilon}\,\tau\right) + \lambda C\int_{D^\delta}\left[\frac{(tw_\tau)^2}{2} + \frac{(tw_\tau)^p}{p}\right]dx \\ &\quad - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(x,u_\lambda+s) - f(x,u_\lambda) - f(x,s)\right)ds\,dx \\ &\le S^\infty + C_\varepsilon\exp\left(-2\sqrt{1+\mu_1-\varepsilon}\,\tau\right) \\ &\quad - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(x,u_\lambda+s) - f(x,u_\lambda) - f(x,s)\right)ds\,dx, \end{aligned} \tag{4.18}$$

where $0 < \varepsilon < 1+\mu_1$ and C_ε is independent of τ.

It follows from Taylor's expansion that

$$f(x,u_\lambda+s)=f(x,s)+f'(x,s)u_\lambda+\frac{1}{2}f''(x,\xi)u_\lambda^2,\quad \xi\in(s,u_\lambda+s). \tag{4.19}$$

From $(f5)$ and the above formula, for $t_1\le t\le t_0$, we obtain that

$$\begin{aligned}\int_0^{t\eta w_\tau}&\big(f(x,u_\lambda+s)-f(x,u_\lambda)-f(x,s)\big)ds\\ &\ge\int_0^{t_1\eta w_\tau}\big(f'(x,s)u_\lambda-f(x,u_\lambda)\big)ds\\ &=\big[(t_1w_\tau)^{-1}f(x,t_1\eta w_\tau)-\eta u_\lambda^{-1}f(x,u_\lambda)\big]t_1w_\tau u_\lambda.\end{aligned} \tag{4.20}$$

Since $w_\tau>0$ in $\mathbb{S}$, there exists $\gamma_1>0$ such that

$$w_\tau\ge\gamma_1\quad\text{in }\Omega_{1\tau}. \tag{4.21}$$

By the definition of w_τ and $u_\lambda(x)\to 0$ as $|x|\to\infty$, we see that for τ large enough,

$$t_1w_\tau\ge u_\lambda\quad\text{in }\Omega_{1\tau}, \tag{4.22}$$

then Lemma 1.3(ii) implies that there exist $\gamma_2>0$ and $\tau_1>0$ such that, for $\tau\ge\tau_1$,

$$(t_1w_\tau)^{-1}f(x,t_1w_\tau)-u_\lambda^{-1}f(x,u_\lambda)>\gamma_2\quad\text{in }\Omega_{1\tau}. \tag{4.23}$$

Now by Lemma 2.5, for $\tau\ge\max(\tau_0,\tau_1)$ and $t_1\le t\le t_0$, we obtain that

$$\begin{aligned}\int_{\Omega_{1\tau}}\int_0^{t\eta w_\tau}&\big(f(x,u_\lambda+s)-f(x,u_\lambda)-f(x,s)\big)ds\,dx\\ &\ge\int_{\Omega_{1\tau}}\big[(t_1w_\tau)^{-1}f(x,t_1w_\tau)-u_\lambda^{-1}f(x,u_\lambda)\big]t_1w_\tau u_\lambda dx\\ &\ge\gamma_1\gamma_2\int_{\Omega_{1\tau}}t_1u_\lambda dx\\ &\ge C_2\exp\Big(-\sqrt{1+\mu_1}\tau\Big),\end{aligned} \tag{4.24}$$

where C_2 is independent of τ.

Therefore we obtain that

$$J(t\eta w_\tau)\le S^\infty+C_\varepsilon\exp\Big(-2\sqrt{1+\mu_1-\varepsilon}\tau\Big)-\lambda C_2\exp\Big(-\sqrt{1+\mu_1}\tau\Big), \tag{4.25}$$

for $t\in[t_1,t_0]$ and $\tau\ge\max(\tau_0,\tau_1)$.

Now, let $\varepsilon=(1+\mu_1)/2$, then we can find some τ_* large enough such that

$$C_\varepsilon\exp\Big(-\sqrt{2(1+\mu_1)}\tau\Big)-\lambda C_2\exp\Big(-\sqrt{1+\mu_1}\tau\Big)<0, \tag{4.26}$$

for all $\tau\ge\tau_*$ and we complete the proof. □

Theorem 4.5. *If $(f1)$–$(f5)$ hold, then (4.1) has a positive solution $\overline{v}$ if $\lambda\in(0,\lambda^*)$.*

Proof. Now, set

$$\Gamma = \{p \in C([0,1], H_0^1(\Omega)) \mid p(0) = 0,\ p(1) = t_0\eta w_{\tau_*}\},$$
$$c = \inf_{p\in\Gamma} \max_{s\in[0,1]} J(p(s)). \tag{4.27}$$

By Lemmas 4.2 and 4.4, we have

$$0 < \alpha \le c < S^\infty. \tag{4.28}$$

Applying the mountain pass theorem of Ambrosetti-Rabinowitz [13], there exists a $(PS)_c$-sequence $\{v_k\}$, $k \in \mathbb{N}$, such that

$$J(v_k) \longrightarrow c,$$
$$J'(v_k) \longrightarrow 0 \quad \text{strong in } H^{-1}(\Omega). \tag{4.29}$$

By Proposition 4.3, there exist a sequence (still denoted by $\{v_k\}$), an integer $l \ge 0$, sequence $\{x_k^i\}$ in $\mathbb{S}$, $1 \le i \le l$, a solution $\overline{v}$ of (4.1), and solutions u^i of $(2.1)_\lambda$ such that

$$c = J(\overline{v}) + \sum_{i=0}^{l} I^\infty(u^i). \tag{4.30}$$

By the strong maximum principle, to complete the proof, we only need to prove $\overline{v} \not\equiv 0$ in Ω. In fact, we have

$$c = J(\overline{v}) \ge \alpha > 0 \quad \text{if } l = 0, \qquad S^\infty > c \ge J(\overline{v}) + S^\infty \quad \text{if } l \ge 1. \tag{4.31}$$

This implies $\overline{v} \not\equiv 0$ in Ω. □

5. Properties of solutions

Denote by $A = \{(\lambda,u) \mid u \text{ solves problem } (1.1)_\lambda\}$, the set of solutions of $(1.1)_\lambda$, $\lambda \in (0,\lambda^*]$. For each $(\lambda,u) \in A$, let $\sigma_\lambda(u)$ denote the number defined by

$$\sigma_\lambda(u) = \inf\left\{\int_\Omega (|\nabla v|^2 + v^2)dx \mid v \in H_0^1(\Omega),\ \int_\Omega f'(x,u)v^2 dx = 1\right\}, \tag{5.1}$$

which is the smallest eigenvalue of the following problem:

$$-\Delta v + v^2 = \sigma_\lambda(u) f'(x,u)v \quad \text{in } \Omega,$$
$$v > 0, \quad v \in H_0^1(\Omega). \tag{5.2}$$

In this section, we always assume that $(f1)$–$(f5)$ hold. By Lemma 2.3, we have $A \subset \mathbb{R} \times L^\infty(\mathbb{R}^N) \cap H_0^1(\Omega)$.

LEMMA 5.1. *Let u be a solution and u_λ be the minimal solution of $(1.1)_\lambda$ for $\lambda \in (0,\lambda^*)$. Then*

(i) $\sigma_\lambda(u) > \lambda$ *if and only if* $u = u_\lambda$;

(ii) $\sigma_\lambda(U_\lambda) < \lambda$, *where U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 4.*

Proof. Now, let $\psi \geq 0$ and $\psi \in H_0^1(\Omega)$. Since u and u_λ slove $(1.1)_\lambda$, then

$$\begin{aligned}\int_\Omega \nabla\psi\cdot\nabla(u_\lambda-u)\,dx+\int_\Omega \psi(u_\lambda-u)\,dx &= \lambda\int_\Omega\big(f(x,u_\lambda)-f(x,u)\big)\psi\,dx=\lambda\int_\Omega\left(\int_u^{u_\lambda}f'(x,t)dt\right)\psi\,dx\\ &\geq \lambda\int_\Omega f'(x,u)(u_\lambda-u)\psi\,dx.\end{aligned}\tag{5.3}$$

Let $\psi = (u-u_\lambda)^+ \geq 0$ and $\psi \in H_0^1(\Omega)$. If $\psi \not\equiv 0$, then (5.3) implies

$$-\int_\Omega(|\nabla\psi|^2+\psi^2)\,dx \geq -\lambda\int_\Omega f'(x,u)\psi^2dx \tag{5.4}$$

and, therefore, the definition of $\sigma_\lambda(u)$ implies

$$\begin{aligned}\int_\Omega(|\nabla\psi|^2+\psi^2)\,dx &\leq \lambda\int_\Omega f'(x,u)\psi^2dx\\ &< \sigma_\lambda(u)\int_\Omega f'(x,u)\psi^2dx\\ &\leq \int_\Omega(|\nabla\psi|^2+\psi^2)\,dx,\end{aligned}\tag{5.5}$$

which is impossible. Hence $\psi \equiv 0$, and $u = u_\lambda$ in Ω. On the other hand, by Lemma 3.4, we also have that $\sigma_\lambda(u_\lambda) > \lambda$. This completes the proof of (i).

By (i), we get that $\sigma_\lambda(U_\lambda) \leq \lambda$ for $\lambda \in (0,\lambda^*)$. We claim that $\sigma_\lambda(U_\lambda) = \lambda$ cannot occur. We proceed by contradiction. Set $w = U_\lambda - u_\lambda$; we have

$$-\Delta w + w = \lambda[f(x,U_\lambda)-f(x,U_\lambda-w)], \quad w>0 \text{ in } \Omega. \tag{5.6}$$

By $\sigma_\lambda(U_\lambda) = \lambda$, we have that the problem

$$-\Delta\phi+\phi=\lambda f'(x,U_\lambda)\phi, \quad \phi\in H_0^1(\Omega) \tag{5.7}$$

possesses a positive solution ϕ_1.

Multiplying (5.6) by ϕ_1 and (5.7) by w, integrating and subtracting we deduce that

$$\begin{aligned}0 &= \int_\Omega \lambda[f(x,U_\lambda)-f(x,U_\lambda-w)-f'(x,U_\lambda)w]\phi_1dx\\ &= -\frac{1}{2}\int_\Omega \lambda f''(\xi_\lambda)w^2\phi_1dx,\end{aligned}\tag{5.8}$$

where $\xi_\lambda \in (u_\lambda, U_\lambda)$. Thus $w \equiv 0$, that is $U_\lambda = u_\lambda$ for $\lambda \in (0,\lambda^*)$. This is a contradiction. Hence we have that $\sigma_\lambda(U_\lambda) < \lambda$ for $\lambda \in (0,\lambda^*)$. □

THEOREM 5.2. *Suppose u_{λ^*} is a solution of (1.1)$_{\lambda^*}$, then $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$ and the solution u_{λ^*} is unique.*

Proof. Define $\mathcal{F} : \mathbb{R} \times H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$\mathcal{F}(\lambda, u) = \Delta u - u + \lambda\big(f(x,u) + h(x)\big). \tag{5.9}$$

Since $\sigma_\lambda(u_\lambda) > \lambda$ for $\lambda \in (0,\lambda^*)$, we have $\sigma_{\lambda^*}(u_{\lambda^*}) \geq \lambda^*$. If $\sigma_{\lambda^*}(u_{\lambda^*}) > \lambda^*$, the equation $\mathcal{F}_u(\lambda^*, u_{\lambda^*})\phi = 0$ has no nontrivial solution. By the standard argument, we can prove that $\mathcal{F}_u$ maps $\mathbb{R} \times H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Applying the implicit function theorem to $\mathcal{F}$, we can find a neighborhood $(\lambda^* - \delta, \lambda^* + \delta)$ of λ^* such that (1.1)$_\lambda$ possesses a solution u_λ if $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. This is contradictory to the definition of λ^*. Hence we obtain that $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$.

Next, we are going to prove that u_{λ^*} is unique. In fact, suppose (1.1)$_{\lambda^*}$ has another solution $U_{\lambda^*} \geq u_{\lambda^*}$. Set $w = U_{\lambda^*} - u_{\lambda^*}$; we have

$$-\Delta w + w = \lambda^*\left[f(w + u_{\lambda^*}) - f(x, u_{\lambda^*})\right], \quad w > 0 \text{ in } \Omega. \tag{5.10}$$

By $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$, we have that the problem

$$-\Delta\phi + \phi = \lambda^* f'(x, u_{\lambda^*})\phi, \quad \phi \in H_0^1(\Omega) \tag{5.11}$$

possesses a positive solution ϕ_1.

Multiplying (5.10) by ϕ_1 and (5.11) by w, integrating and subtracting we deduce that

$$\begin{aligned} 0 &= \int_\Omega \lambda^*\left[f(w + u_{\lambda^*}) - f(x, u_{\lambda^*}) - f'(x, u_{\lambda^*})w\right]\phi_1 dx \\ &= \frac{1}{2}\int_\Omega \lambda^* f''(\xi_{\lambda^*})w^2\phi_1 dx, \end{aligned} \tag{5.12}$$

where $\xi_{\lambda^*} \in (u_{\lambda^*}, u_{\lambda^*} + w)$. Thus $w \equiv 0$. □

PROPOSITION 5.3. *Let u_λ be the minimal solution of (1.1)$_\lambda$. Then u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in (0, \lambda^*]$, and*

$$u_\lambda \longrightarrow 0 \quad \text{in } L^\infty(\Omega) \cap H_0^1(\Omega) \text{ as } \lambda \longrightarrow 0^+. \tag{5.13}$$

Proof. By (3.20), we have that

$$\|u_\lambda\| \leq \frac{\lambda}{1-\theta}\|h\|_2 \tag{5.14}$$

for $\lambda \in (0, \lambda^*)$, and u_λ is strictly increasing with respect to λ, we can easily deduce that u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for $\lambda \in (0, \lambda^*]$ and $u_\lambda \to 0$ in $H_0^1(\Omega)$ as $\lambda \to 0^+$.

By (2.6) and the fact that u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, we have that

$$\begin{aligned} \|u_\lambda\|_\infty &\leq C_1\|u_\lambda\|_{q_0} + \lambda C_2\left(\|u_\lambda\|_{(p-1)q_0}^{p-1} + \|h\|_{q_0}\right) \\ &\leq C_1\|u_\lambda\|_\infty^{(q_0-2)/q_0}\|u_\lambda\|_2^{2/q_0} + C_3\lambda \\ &\leq C\left(\lambda^{2/q_0} + \lambda\right), \end{aligned} \tag{5.15}$$

where C is independent of λ, and $\lambda \in (0, \lambda^*]$. Hence we obtain that $u_\lambda \to 0$ in $L^\infty(\Omega)$ as $\lambda \to 0^+$. □

Proposition 5.4. *If $\lambda \in (0, \lambda^*)$, then U_λ is unbounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, and*

$$\lim_{\lambda \to 0^+} \|U_\lambda\| = \lim_{\lambda \to 0^+} \|U_\lambda\|_\infty = \infty. \tag{5.16}$$

Proof. Let φ_λ be a minimizer of $\sigma_\lambda(U_\lambda)$ for $\lambda \in (0, \lambda^*)$, that is

$$\int_\Omega f'(x, U_\lambda) \varphi_\lambda^2 = 1, \qquad \|\varphi_\lambda\|^2 = \sigma_\lambda(U_\lambda). \tag{5.17}$$

(i) First, we show that $\{U_\lambda : \lambda \in (0, \lambda_0)\}$ is unbounded in $L^\infty(\Omega)$ for any $\lambda_0 \in (0, \lambda^*)$. We proceed by contradiction. Assume to the contrary that there exists $C_0 > 0$ such that

$$\|U_\lambda\|_\infty \le C_0 < \infty \quad \forall \lambda \in (0, \lambda_0). \tag{5.18}$$

By $(f1)$ and (5.18), there exists a constant M independent of λ, such that $f'(x, U_\lambda(x)) \le M$ for all $\lambda \in (0, \lambda_0)$ and $x \in \Omega$. Hence, by (5.17) and $\sigma_\lambda(U_\lambda) < \lambda$ for all $\lambda \in (0, \lambda_0)$, we obtain that

$$1 = \int_\Omega f'(x, U_\lambda) \varphi_\lambda^2 \le M \|\varphi_\lambda\|^2 = M \sigma_\lambda(U_\lambda) < M\lambda. \tag{5.19}$$

This is a contradiction for all $\lambda < 1/M$. Hence, for any $\lambda_0 \in (0, \lambda^*)$, we have that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $L^\infty(\Omega)$. From this result, it is easy to see that $\lim_{\lambda \to 0^+} \|U_\lambda\|_\infty = \infty$.

(ii) Now, we show that $\{U_\lambda : \lambda \in (0, \lambda_0)\}$ is unbounded in $H_0^1(\Omega)$ for any $\lambda_0 \in (0, \lambda^*)$. If not, then there exists a constant M independent of λ, such that

$$\|U_\lambda\| \le M \quad \forall \lambda \in (0, \lambda_0). \tag{5.20}$$

By (5.17), (5.20), $(f2)$, the Hölder inequality, the Sobolev embedding theorem, and $\sigma_\lambda(U_\lambda) < \lambda$ for all $\lambda \in (0, \lambda^*)$, we have that

$$\begin{aligned} 1 &= \int_\Omega f'(x, U_\lambda) \varphi_\lambda^2 \le C_1 \int_\Omega \left(1 + U_\lambda^{p-2}\right) \varphi_\lambda^2 \le C_1 \|\varphi_\lambda\|^2 + C_1 \|U_\lambda\|_p^{p-2} \|\varphi_\lambda\|_p^2 \\ &\le C_1 \|\varphi_\lambda\|^2 + C_2 \|U_\lambda\|^{p-2} \|\varphi_\lambda\|^2 \le C_3 \|\varphi_\lambda\|^2 = C_3 \sigma_\lambda(U_\lambda) < C_3 \lambda, \end{aligned} \tag{5.21}$$

where C_1, C_2, and C_3 are constants independent of λ. Now, let $\lambda \to 0^+$, then we obtain a contradiction. Hence $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $H_0^1(\Omega)$ and $\lim_{\lambda \to 0^+} \|U_\lambda\| = +\infty$. □

Proof of Theorems 1.1 and 1.2. First, we consider the case $\Omega = \mathbb{S} \backslash \overline{D}$. Theorem 1.1 now follows from Lemmas 3.3, 3.4, and Theorems 4.5, 5.2. Theorem 1.2 follows immediately from Lemma 3.4, and Propositions 5.3, 5.4. □

With the same argument, we also have that Theorems 1.1 and 1.2 hold for $\Omega = \mathbb{R}^N \backslash \overline{D}$ or $\Omega = \mathbb{S}$ or $\Omega = \mathbb{R}^N$.

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