Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2007, Article ID 52840, 4 pages
doi:10.1155/2007/52840

Research Article

On Minimal Norms on M_n

Madjid Mirzavaziri and Mohammad Sal Moslehian

Received 18 June 2007; Accepted 19 August 2007

Recommended by Allan C. Peterson

We show that for each minimal norm $N(\cdot)$ on the algebra $\mathcal{M}_n$ of all $n\times n$ complex matrices, there exist norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{C}^n$ such that $N(A)=\max\{\|Ax\|_2 : \|x\|_1 = 1,\ x\in\mathbb{C}^n\}$ for all $A\in\mathcal{M}_n$. This may be regarded as an extension of a known result on characterization of minimal algebra norms.

1. Introduction

Let $\mathcal{M}_n$ denote the algebra of all $n\times n$ complex matrices A with entries in $\mathbb{C}$, together with the usual matrix operations. By an algebra norm (or a matrix norm) we mean a norm $\|\cdot\|$ on $\mathcal{M}_n$ such that $\|AB\|\le\|A\|\|B\|$ for all $A,B\in\mathcal{M}_n$. It is easy to see that the norm $\|A\|_\sigma=\sum_{i,j=1}^n|\alpha_{ij}|$ is an algebra norm, but the norm $\|A\|_m=\max\{|a_{i,j}| : 1\le i,j\le n\}$ is not an algebra norm, (see [1]).

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathbb{C}^n$. Then the norm $\|\cdot\|_{1,2}$ on $\mathcal{M}_n$ defined by $\|A\|_{1,2}:=\max\{\|Ax\|_2 : \|x\|_1=1\}$ is called the generalized induced (or g-ind) norm constructed via $\|\cdot\|_1$ and $\|\cdot\|_2$. If $\|\cdot\|_1=\|\cdot\|_2$, then $\|\cdot\|_{1,1}$ is called an induced norm.

It is known that $\|A\|_C=\max\{\sum_{i=1}^n|\alpha_{i,j}| :\le j\le n\}$, $\|A\|_R=\max\{\sum_{j=1}^n|\alpha_{i,j}| : 1\le i\le n\}$ and the spectral norm $\|A\|_S=\max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\}$ are induced by ℓ_1,ℓ_∞, and ℓ_2, respectively, (cf. [2]). Recall that the ℓ_p-norm $(1\le p\le\infty)$ on $\mathbb{C}^n$ is defined by

$$\ell_p(x)=\ell_p\left(\sum_{i=1}^n x_ie_i\right)=\begin{cases}\left(\displaystyle\sum_{i=1}^n|x_i|^p\right)^{1/p}, & 1\le p<\infty,\\ \max\{|x_1|,\dots,|x_n|\}, & p=\infty.\end{cases}\tag{1.1}$$

It is known that the algebra norm $\|A\| = \max\{\|A\|_C, \|A\|_R\}$ is not induced, and it is not hard to show that it is not g-ind too (cf. Corollary 3.2.6 of [3]).

A norm $N(\cdot)$ on $\mathcal{M}_n$ is called minimal if for any norm $|||\cdot|||$ on $\mathcal{M}_n$ satisfying $|||\cdot||| \le N(\cdot)$, we have $|||\cdot||| = N(\cdot)$. It is known [3, Theorem 3.2.3] that an algebra norm is an induced norm if and only if it is a minimal element in the set of all algebra norms. Note that a generalized induced norm may not be minimal. For instance, put $\|\cdot\|_\alpha = \ell_\infty(\cdot)$, $\|\cdot\|_\beta = 2\ell_2(\cdot)$, and $\|\cdot\|_\gamma = \ell_2(\cdot)$. Then $\|\cdot\|_{\gamma,\beta} \le \|\cdot\|_{\alpha,\beta}$ but $\|\cdot\|_{\gamma,\beta} \ne \|\cdot\|_{\alpha,\beta}$.

In [1], the authors investigate generalized induced norms. In particular, they examine the problem that "for any norm $\|\cdot\|$ on $\mathcal{M}_n$, are there two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{C}^n$ such that $\|A\| = \max\{\|Ax\|_2 : \|x\|_1 = 1\}$ for all $A \in \mathcal{M}_n$?" In this short note, we utilize some ideas of [1] to study the minimal norms on $\mathcal{M}_n$. More precisely, we show that for each minimal norm $N(\cdot)$ on the algebra $\mathcal{M}_n$ of all $n\times n$ complex matrices, there exist norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{C}^n$ such that $N(A) = \max\{\|Ax\|_2 : \|x\|_1 = 1, x \in \mathbb{C}^n\}$ for all $A \in \mathcal{M}_n$. In particular, if $N(\cdot)$ is an algebra norm, then $\|\cdot\|_1 = \|\cdot\|_2$. This may be regarded as an extension of the above known result on characterization of minimal algebra norms.

2. Main result

For $x \in \mathbb{C}^n$ and $1 \le j \le n$, let $C_{x,j} \in \mathcal{M}_n$ be defined by the operator $C_{x,j}(y) = y_j x$. Hence $C_{x,j}$ is the $n\times n$ matrix with x in the j column and 0 elsewhere. Define $C_x \in \mathcal{M}_n$ by $C_x = \sum_{j=1}^n C_{x,j}$. Hence C_x is the $n\times n$ matrix whose all columns are x.

If $\|\cdot\|_{1,2}$ is a generalized induced norm on $\mathcal{M}_n$ obtained via $\|\cdot\|_1$ and $\|\cdot\|_2$ then $\|C_x\|_{1,2} = \alpha\|x\|_2$, where $\alpha = \max\{|\sum_{j=1}^n y_j| : \|(y_1,\dots,y_j,\dots,y_n)\|_1 = 1\}$.

To achieve our goal, we need the following lemmas.

Lemma 2.1 [1, Theorem 2.7]. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathbb{C}^n$. Then $\|\cdot\|_{1,2}$ is an algebra norm on $\mathcal{M}_n$ if and only if $\|\cdot\|_1 \le \|\cdot\|_2$.*

Lemma 2.2 [1, Corollary 2.5]. *$\|\cdot\|_{1,2} = \|\cdot\|_{3,4}$ if and only if there exists $\gamma > 0$ such that $\|\cdot\|_1 = \gamma\|\cdot\|_3$ and $\|\cdot\|_2 = \gamma\|\cdot\|_4$.*

Theorem 2.3. *Let $N(\cdot)$ be a minimal norm on $\mathcal{M}_n$, then $N(\cdot) = \|\cdot\|_{1,2}$ for some $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{C}^n$. Moreover, if $N(\cdot)$ is an algebra norm, then $\|\cdot\|_1 = \|\cdot\|_2$.*

Proof. For $x \in \mathbb{C}^n$, set

$$\|x\|_1 = \max\{N(C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\},$$
$$\|x\|_2 = N(C_x). \tag{2.1}$$

We will show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\mathbb{C}^n$.

To see that $\|\cdot\|_1$ is a norm, let $x \in \mathbb{C}^n$. Then $\|x\|_1 = 0$ if and only if $N(C_{Ax}) = 0$ for all matrix A with $N(A) = 1$, and this holds if and only if $Ax = 0$ for all A, or equivalently $x = 0$.

For $\alpha \in \mathbb{C}^n$ and $x, y \in \mathbb{C}^n$, we have

$$
\begin{aligned}
\|\alpha x\|_1 &= \max\{N(C_{A(\alpha x)}) : N(A) = 1, A \in \mathcal{M}_n\} \\
&= \max\{N(\alpha C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\} \\
&= \max\{|\alpha| N(C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\} \\
&= |\alpha| \max\{N(C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\} \\
&= |\alpha| \|x\|_1, \\
\|x+y\|_1 &= \max\{N(C_{A(x+y)}) : N(A) = 1, A \in \mathcal{M}_n\} \\
&= \max\{N(C_{Ax} + C_{Ay}) : N(A) = 1, A \in \mathcal{M}_n\} \\
&\le \max\{N(C_{Ax}) : N(A) = 1, A \in \mathcal{M}_n\} \\
&\quad + \max\{N(C_{Ay}) : N(A) = 1, A \in \mathcal{M}_n\} \\
&= \|x\|_1 + \|y\|_1.
\end{aligned}
\tag{2.2}
$$

To see that $\|\cdot\|_2$ is a norm, let $x \in \mathbb{C}^n$. Then $\|x\|_2 = 0$ if and only if $C_x = 0$ and this holds if and only if $x = 0$.

For $\alpha \in \mathbb{C}^n$ and $x, y \in \mathbb{C}^n$, we have

$$
\begin{aligned}
\|\alpha x\|_2 = N(C_{\alpha x}) = N(\alpha C_x) = |\alpha| N(C_x) = |\alpha| \|x\|_2, \\
\|x+y\|_2 = N(C_{x+y}) = N(C_x + C_y) \le N(C_x) + N(C_y) = \|x\|_2 + \|y\|_2.
\end{aligned}
\tag{2.3}
$$

Now let $A \in \mathcal{M}_n \backslash \{0\}$. Then $N(A/N(A)) = 1$ so that

$$
\left\| \frac{A}{N(A)}(x) \right\|_2 = N(C_{(A/N(A))(x)}) \le \|x\|_1, \tag{2.4}
$$

whence

$$
\|Ax\|_2 \le N(A)\|x\|_1. \tag{2.5}
$$

Therefore $\|A\|_{1,2} \le N(A)$. Since $N(\cdot)$ is a minimal norm, we conclude that $\|A\|_{1,2} = N(A)$.

If $N(A)$ is an algebra norm, then Lemma 2.1 implies that $\|\cdot\|_1 \le \|\cdot\|_2$.

Next, let $A \in \mathcal{M}_n$. It follows from $\|Ax\|_1 \le \|A\|_{11}\|x\|_1 \le \|A\|_{1,1}\|x\|_2$, $(x \in \mathbb{C}^n)$ that $\|A\|_{2,1} \le \|A\|_{1,1}$. In a similar fashion, one can get

$$
\|\cdot\|_{2,1} \le \|\cdot\|_{k,k} \le \|\cdot\|_{1,2} \quad (k = 1, 2). \tag{2.6}
$$

By the minimality of $\|\cdot\|_{1,2}$, we deduce that $\|\cdot\|_{1,2} = \|\cdot\|_{1,1}$. It then follows from Lemma 2.2 that $\|\cdot\|_1 = \|\cdot\|_2$. □

References

[1] S. Hejazian, M. Mirzavaziri, and M. S. Moslehian, "Generalized induced norms," *Czechoslovak Mathematical Journal*, vol. 57, no. 1, pp. 127–133, 2007.
[2] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1994.
[3] G. R. Belitskiĭ and Yu. I. Lyubich, *Matrix Norms and Their Applications*, vol. 36 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 1988.

Madjid Mirzavaziri: Department of Mathematics, Ferdowsi University, P.O. Box 1159, Mashhad 91775, Iran; Banach Mathematical Research Group (BMRG), Mashhad, Iran
Email address: madjid@mirzavaziri.com

Mohammad Sal Moslehian: Department of Mathematics, Ferdowsi University, P.O. Box 1159, Mashhad 91775, Iran; Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University, Iran
Email address: moslehian@ferdowsi.um.ac.ir