

Research Article

Stabilization for a Periodic Predator-Prey System

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A reaction-diffusion system modelling a predator-prey system in a periodic environment is considered. We are concerned in stabilization to zero of one of the components of the solution, via an internal control acting on a small subdomain, and in the preservation of the nonnegativity of both components.

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1. Introduction

This paper concerns the internal zero stabilization of the predator population of a predator-prey system in a periodic environment. Our starting point is the system describing the evolution of a predator population and a prey population distributed over the habitat Ω :

$$\begin{aligned}h_t - d_1 \Delta h &= r(t)h - k(t)h^2 - f_1(t, h, p)hp, & x \in \Omega, t > 0, \\p_t - d_2 \Delta p &= -a(t)p + f_2(t, h, p)hp, & x \in \Omega, t > 0, \\ \frac{\partial h}{\partial \nu} &= \frac{\partial p}{\partial \nu} = 0, & x \in \partial\Omega, t > 0,\end{aligned}\tag{1.1}$$

$$h(x, 0) = h_0(x), \quad p(x, 0) = p_0(x), \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with a smooth enough boundary $\partial\Omega$. Here $h(x, t)$ is the density of preys at position $x \in \overline{\Omega}$ and time $t \geq 0$ and $p(x, t)$ is the density of predators at position $x \in \overline{\Omega}$ and time $t \geq 0$; h and p are both nonnegative functions. $d_1, d_2 > 0$ are the diffusivity constants of the two populations. $r(t)$ is the intrinsic growth rate of preys in the absence of predators, at the moment $t \geq 0$ (which can be positive, zero, or

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negative) and is T -periodic ($T > 0$). Usually, the period T is of one year. $a(t)$ is the decay rate of predators in the absence of preys, at the moment t , and is also T -periodic. k is a T -periodic and positive function. $k(t)h(x, t)$ represents an additional mortality rate of the preys due to the overpopulation.

Homogeneous Neumann boundary conditions mean that there is no flux of species through the boundary $\partial\Omega$ (this corresponds to isolated populations). h_0 and p_0 are the initial densities of the two populations.

The following cases are well known in the literature.

When $f_1(t, h, p) = \theta_1$ and $f_2(t, h, p) = \theta_2$, where θ_1, θ_2 are positive constants, the standard Lotka-Volterra system is obtained.

For $f_1(t, h, p) = \theta_1/(1 + qh)$ and $f_2(t, h, p) = \theta_2/(1 + qh)$, for every $h, p \geq 0$, where θ_1, θ_2, q are positive constants, we obtain a Holling II functional response to predation.

Finally, in the case $f_1(t, h, p) = \theta_1/(1 + qh + \tilde{q}p)$ and $f_2(t, h, p) = \theta_2/(1 + qh + \tilde{q}p)$, for every $h, p \geq 0$, and $\theta_1, \theta_2, q, \tilde{q}$ positive constants, a Beddington-De Angelis functional response for predation is obtained. For a complete study of the solutions to this model we refer to [1]. For a description of the predator-prey systems and some basic results we refer to [2, 3].

Throughout this paper, the following assumptions will be considered:

(H1) $h_0, p_0 \in L^\infty(\Omega)$, $h_0(x) \geq 0$, $p_0(x) \geq 0$, a.e. $x \in \Omega$,

$$\|h_0(x)\|_{L^\infty(\Omega)}, \quad \|p_0(x)\|_{L^\infty(\Omega)} > 0; \quad (1.2)$$

(H2) $r, k, a \in C([0, +\infty))$ satisfy

$$\begin{aligned} r(t) &= r(t+T), \quad k(t) = k(t+T), \quad a(t) = a(t+T), \quad \forall t \geq 0, \\ k(t) &\geq k_0 > 0, \quad \forall t \geq 0 \text{ (where } k_0 \text{ is a constant),} \\ \int_0^T r(t) dt &> 0, \\ a(t) &\geq a_0 > 0, \quad \forall t \geq 0 \text{ (where } a_0 \text{ is a constant);} \end{aligned} \quad (1.3)$$

(H3) $f_1, f_2 : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions and locally Lipschitz continuous with respect to (h, p) and satisfy

$$\begin{aligned} f_1(t, h, p) &= f_1(t+T, h, p), \quad f_2(t, h, p) = f_2(t+T, h, p), \quad \forall t \geq 0, h \geq 0, p \geq 0, \\ \exists C > 0 \text{ such that } 0 &\leq f_1(t, h, p), \quad f_2(t, h, p) \leq C, \quad \forall t \geq 0, h \geq 0, p \geq 0; \end{aligned} \quad (1.4)$$

(H4) the application $h \mapsto hf_2(t, h, p)$ is nondecreasing on $[0, +\infty)$, $\forall t \geq 0, \forall p \geq 0$;

(H5) the application $p \mapsto f_2(t, h, p)$ is nonincreasing on $[0, +\infty)$, $\forall t \geq 0, \forall h \geq 0$.

Condition $\int_0^T r(t) dt > 0$ is a persistence condition for the preys in the absence of predators. So, if $p_0 \equiv 0$ and $h_0(x) > 0$ a.e. in Ω , then the necessary and sufficient condition for the persistence of preys is the above-mentioned one.

For basic results concerning the solutions of periodic predator-prey systems (without diffusion) we refer to [4].

Let $\omega \subset \mathbb{R}^N$ be a nonempty domain with a smooth-enough boundary $\partial\omega$ and satisfying $\omega \subset \subset \Omega$. We denote by m the characteristic function of ω .

The questions we want to investigate are the following.

(1) Is there any control $u \in L_{loc}^\infty(\bar{\omega} \times [0, \infty))$ such that the solution to the initial-boundary value problem

$$\begin{aligned} h_t - d_1 \Delta h &= r(t)h - k(t)h^2 - f_1(t, h, p)hp, & x \in \Omega, t > 0, \\ p_t - d_2 \Delta p &= -a(t)p + f_2(t, h, p)hp + m(x)u(x, t), & x \in \Omega, t > 0, \\ \frac{\partial h}{\partial \nu} &= \frac{\partial p}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ h(x, 0) &= h_0(x), \quad p(x, 0) = p_0(x), & x \in \Omega, \end{aligned} \tag{1.5}$$

satisfies

$$\begin{aligned} h(x, t) &\geq 0, \quad p(x, t) \geq 0 \quad \text{a.e. } x \in \Omega, \forall t \geq 0, \\ \lim_{t \rightarrow \infty} p(t) &= 0 \quad \text{in } L^\infty(\Omega)? \end{aligned} \tag{1.6}$$

(2) Is there any control $v \in L_{loc}^\infty(\bar{\omega} \times [0, \infty))$ such that the solution to the initial-boundary value problem

$$\begin{aligned} h_t - d_1 \Delta h &= r(t)h - k(t)h^2 - f_1(t, h, p)hp + m(x)v(x, t), & x \in \Omega, t > 0, \\ p_t - d_2 \Delta p &= -a(t)p + f_2(t, h, p)hp, & x \in \Omega, t > 0, \\ \frac{\partial h}{\partial \nu} &= \frac{\partial p}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ h(x, 0) &= h_0(x), \quad p(x, 0) = p_0(x), & x \in \Omega, \end{aligned} \tag{1.7}$$

satisfies (1.6)?

Definition 1.1. Say that the predator population is *p-zero stabilizable* if for any h_0, p_0 satisfying (H1), the answer to the first question is affirmative. *p-zero stabilizable* means that the zero stabilizability holds for controls acting only on the predator population.

Definition 1.2. Say that the predator population is *h-zero stabilizable* if for any h_0, p_0 satisfying (H1), the answer to the second question is affirmative. *h-zero stabilizable* means that the zero stabilizability holds for controls acting only on the prey population.

We are dealing here with some results of zero stabilizability with state constraints.

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First notice that, due to assumption (H3) and to the comparison principle for parabolic equations, the solution (h, p) to (1.1) satisfies

$$0 \leq h(x, t) \leq \bar{h}(x, t) \quad \text{a.e. } x \in \Omega, \forall t \geq 0, \quad (1.8)$$

where \bar{h} is the solution to

$$\begin{aligned} \bar{h}_t - d_1 \Delta \bar{h} &= r(t)\bar{h} - k(t)\bar{h}^2, \quad x \in \Omega, t > 0, \\ \frac{\partial \bar{h}}{\partial \nu} &= 0, \quad x \in \Omega, t > 0, \\ \bar{h}(x, 0) &= h_0(x), \quad x \in \Omega. \end{aligned} \quad (1.9)$$

LEMMA 1.3. *The solution \bar{h} to (1.9) satisfies*

$$\lim_{t \rightarrow \infty} \|\bar{h}(t) - \tilde{h}(t)\|_{L^\infty(\Omega)} = 0, \quad (1.10)$$

where \tilde{h} is the unique nontrivial nonnegative solution to the following problem:

$$\begin{aligned} \tilde{h}_t - d_1 \Delta \tilde{h} &= r(t)\tilde{h} - k(t)\tilde{h}^2, \quad x \in \Omega, t > 0, \\ \frac{\partial \tilde{h}}{\partial \nu} &= 0, \quad x \in \Omega, t > 0, \\ \tilde{h}(x, t) &= \tilde{h}(x, t + T), \quad x \in \Omega, t > 0. \end{aligned} \quad (1.11)$$

Remark 1.4. In fact, we will show that (1.11) has exactly two nonnegative solutions, the trivial one and the unique nontrivial and nonnegative solution to

$$\begin{aligned} g_t &= r(t)g - k(t)g^2, \quad t > 0, \\ g(t) &= g(t + T), \quad t > 0. \end{aligned} \quad (1.12)$$

If $\int_0^T r(t)dt \leq 0$, then (1.12) has a unique nonnegative solution (the trivial one). This follows by a simple calculation and taking into account that the first equation in (1.12) is a Bernoulli equation.

Proof of Lemma 1.3. Since $\|h_0\|_{L^\infty(\Omega)} > 0$, it follows that there exists a positive constant $\rho_1 > 0$ such that

$$\bar{h}(x, T) \geq \rho_1 > 0 \quad \text{a.e. } x \in \Omega \quad (1.13)$$

(this is a consequence of a result in [5]). Therefore, we can assert that

$$\bar{h}(x, t) \geq h^{\rho_1}(t), \quad \text{a.e. } x \in \Omega, \forall t \geq T, \quad (1.14)$$

where $h^{\rho_1}(t)$ is the solution to

$$\begin{aligned} (h^\rho)_t - d_1 \Delta h^\rho &= r(t)h^\rho - k(t)(h^\rho)^2, \quad x \in \Omega, t > T, \\ \frac{\partial h^\rho}{\partial \nu} &= 0, \quad x \in \Omega, t > T, \\ h^\rho(x, T) &= \rho, \quad x \in \Omega, \end{aligned} \quad (1.15)$$

corresponding to $\rho := \rho_1$ (h^{ρ_1} does not depend explicitly on x).

If we choose $\rho_1 > 0$ sufficiently small and taking into account that $\int_0^T r(t)dt > 0$, it follows that

$$h^{\rho_1}(T) < h^{\rho_1}(2T). \quad (1.16)$$

By mathematical induction, we get that

$$h^{\rho_1}(t + T + nT) \leq h^{\rho_1}(t + T + (n+1)T), \quad \forall t \in [0, T], \forall n \in \mathbb{N} \quad (1.17)$$

and consequently

$$h_n^{\rho_1}(t) \leq h_{n+1}^{\rho_1}(t), \quad \text{a.e. } x \in \Omega, \forall t \in [0, T], \quad (1.18)$$

for any $n \in \mathbb{N}$, where $h_n^{\rho_1}(t) = h^{\rho_1}(t + T + nT)$, $\forall t \in [0, T]$. Obviously, $h_n^{\rho_1}$ is the solution of

$$\begin{aligned} (h_n^{\rho_1})_t - d_1 \Delta h_n^{\rho_1} &= r(t)h_n^{\rho_1} - k(t)(h_n^{\rho_1})^2, \quad x \in \Omega, t \in (0, T), \\ \frac{\partial h_n^{\rho_1}}{\partial \nu} &= 0, \quad x \in \Omega, t \in (0, T), \\ h_n^{\rho_1}(x, 0) &= h_{n-1}^{\rho_1}(x, T) = h^{\rho_1}(x, T + nT), \quad x \in \Omega, \end{aligned} \quad (1.19)$$

for any $n \in \mathbb{N}^*$.

In the same manner, taking $\rho_2 > 0$ sufficiently large, we can obtain a nonincreasing bounded sequence $h_n^{\rho_2}$, where $h_n^{\rho_2}(t) = h^{\rho_2}(t + T + nT)$, for all $t \in [0, T]$, for all $n \in \mathbb{N}$ and h^{ρ_2} is the solution to (1.15) corresponding to $\rho := \rho_2$.

Using the comparison result for parabolic equations, we have that

$$h_n^{\rho_1}(t) \leq \bar{h}(x, t + (n+1)T) \leq h_n^{\rho_2}(t), \quad \text{a.e. } x \in \Omega, \forall t \in [0, T], \forall n \in \mathbb{N}. \quad (1.20)$$

Taking into account (1.20), we may pass to the limit in (1.19) and get that

$$h_n^{\rho_1} \longrightarrow \tilde{h}_1, \quad (1.21)$$

in $C([0, T])$, as $n \rightarrow +\infty$, where \tilde{h}_1 is a positive solution (has only positive values) of

$$\begin{aligned} \tilde{h}_t - d_1 \Delta \tilde{h} &= r(t)\tilde{h} - k(t)\tilde{h}^2, \quad x \in \Omega, t \in (0, T), \\ \frac{\partial \tilde{h}}{\partial \nu} &= 0, \quad x \in \partial\Omega, t \in (0, T), \\ \tilde{h}(x, 0) &= \tilde{h}(x, T), \quad x \in \Omega, \end{aligned} \quad (1.22)$$

where \tilde{h}_1 does not depend explicitly on x (because $h_n^{p_1}$ does not). We may extend \tilde{h}_1 by T -periodicity to $[0, +\infty)$ and we deduce that \tilde{h}_1 is a positive solution to (1.11) and to (1.12). Since (1.12) has a unique nontrivial nonnegative solution, we may infer that this one is \tilde{h}_1 . So,

$$\lim_{t \rightarrow +\infty} |h^{p_1}(t) - \tilde{h}_1(t)| = 0. \quad (1.23)$$

In the same manner, it follows that

$$\lim_{t \rightarrow +\infty} |h^{p_2}(t) - \tilde{h}_1(t)| = 0. \quad (1.24)$$

By (1.20) we conclude that

$$\lim_{t \rightarrow \infty} \|\bar{h}(t) - \tilde{h}_1(t)\|_{L^\infty(\Omega)} = 0. \quad (1.25)$$

Let us prove that there is only one nontrivial and nonnegative solution to (1.11).

Let \tilde{h}_2 be a nontrivial and nonnegative solution to (1.11). It follows immediately that there exists $\rho_0 > 0$ (see [5]) such that $\tilde{h}_2(x, T) \geq \rho_0$ a.e. $x \in \Omega$. If we choose ρ_1 and ρ_2 such that $0 < \rho_1 < \rho_0 \leq \tilde{h}_2(x, 0) = \tilde{h}_2(x, T) \leq \rho_2$ a.e. $x \in \Omega$ with ρ_1 small enough and ρ_2 large enough, then it follows as before that $\tilde{h}_2 \equiv \tilde{h}_1$ (because $h_n^{p_1}(t) \leq \tilde{h}_2(x, t) \leq h_n^{p_2}(t)$ a.e. $x \in \Omega$, for all $t \in [0, T]$, for all $n \in \mathbb{N}$) and so we get the conclusion of the lemma. \square

Let us consider now the corresponding equation in p for $h := \tilde{h}$, that is,

$$\begin{aligned} p_t - d_2 \Delta p &= -a(t)p + f_2(t, \tilde{h}(t), p)\tilde{h}(t)p, \quad x \in \Omega, t > 0, \\ \frac{\partial p}{\partial \nu} &= 0, \quad x \in \partial\Omega, t > 0, \\ p(x, 0) &= p_0(x), \quad x \in \Omega. \end{aligned} \quad (1.26)$$

Having in mind (H5), we obtain that

$$f_2(t, h, p) \leq f_2(t, h, 0), \quad \forall t, h, p \geq 0, \quad (1.27)$$

therefore, the solution p to (1.26) satisfies (using the comparison principle for parabolic equations)

$$0 \leq p(x, t) \leq \bar{p}(x, t), \quad \text{a.e. } x \in \Omega, \quad \forall t \geq 0, \quad (1.28)$$

where \bar{p} is a solution to

$$\begin{aligned}\bar{p}_t - d_2 \Delta \bar{p} &= -a(t)\bar{p} + f_2(t, \tilde{h}(t), 0)\tilde{h}(t)\bar{p}, \quad x \in \Omega, t > 0, \\ \frac{\partial \bar{p}}{\partial \nu} &= 0, \quad x \in \partial\Omega, t > 0, \\ \bar{p}(x, 0) &= p_0(x), \quad x \in \Omega.\end{aligned}\tag{1.29}$$

This may be rewritten as

$$\begin{aligned}\bar{p}_t - d_2 \Delta \bar{p} &= l(t)\bar{p}, \quad x \in \Omega, t > 0, \\ \frac{\partial \bar{p}}{\partial \nu} &= 0, \quad x \in \partial\Omega, t > 0, \\ \bar{p}(x, 0) &= p_0(x), \quad x \in \Omega,\end{aligned}\tag{1.30}$$

where

$$l(t) = f_2(t, \tilde{h}(t), 0)\tilde{h}(t) - a(t), \quad \forall t \geq 0.\tag{1.31}$$

Thus, the solution \bar{p} can be written as

$$\bar{p}(x, t) = \exp \left\{ \int_0^t l(\tau) d\tau \right\} f(x, t), \quad x \in \Omega, t \geq 0\tag{1.32}$$

with f solution to

$$\begin{aligned}f_t - d_2 \Delta f &= 0, \quad x \in \Omega, t > 0, \\ \frac{\partial f}{\partial \nu} &= 0, \quad x \in \partial\Omega, t > 0, \\ f(x, 0) &= p_0(x), \quad x \in \Omega.\end{aligned}\tag{1.33}$$

LEMMA 1.5. *There exist a real constant α^* and a T -periodic continuous function $w : [0, +\infty) \rightarrow \mathbb{R}$ such that*

$$\exp \left\{ \int_0^t l(\tau) d\tau \right\} = \exp \{ \alpha^* t \} w(t), \quad \forall t \geq 0.\tag{1.34}$$

Indeed, one can check directly that, due to the periodicity assumptions made on a and f_2 , for $\alpha^* = (1/T) \int_0^T l(\tau) d\tau$, the function

$$w(t) = \exp \left\{ \int_0^t (l(s) - \alpha^*) ds \right\}, \quad \forall t \geq 0,\tag{1.35}$$

is a T -periodic function.

Let us denote by λ_1 the principal eigenvalue of the following eigenvalue problem

$$\begin{aligned} -d_2\Delta\varphi &= \lambda\varphi, & x \in \Omega, \\ \frac{\partial\varphi}{\partial\nu} &= 0, & x \in \partial\Omega. \end{aligned} \tag{1.36}$$

Remark that $\lambda_{-1} = 0$. Now, we notice that if $\lambda_1 > \alpha^*$, then (1.32) and (1.34) imply that the predator population goes to extinction without any control. Therefore, in the rest of this paper we will assume

(H6) $0 < \alpha^*$.

For basic results concerning the solutions to predator-prey systems we refer to [1, 6]. Stabilization of predator-prey systems with r, k, a constants has been investigated in [7, 8]. If in (1.1) the predator is an alien population, then our main goal is to eliminate this population. This problem and its importance have been discussed in [9]. We will investigate next what happens in the cases when we act with a control with support in $\bar{\omega}$.

Section 2 is devoted to the study of p -zero stabilization, while Section 3 concerns the h -zero stabilization. Some remarks are given in Section 4.

2. The p -zero stabilization of the predator population

Denote by $\lambda_1^{\omega,p}$ the principal eigenvalue of the next problem

$$\begin{aligned} -d_2\Delta\varphi &= \lambda\varphi & \text{in } \Omega \setminus \bar{\omega}, \\ \varphi &= 0 & \text{on } \partial\omega, \\ \frac{\partial\varphi}{\partial\nu} &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Then, according to Rayleigh's principle (see [10]), $\lambda_1^{\omega,p}$ satisfies

$$\lambda_1^{\omega,p} = \min \left\{ d_2 \int_{\Omega \setminus \bar{\omega}} |\nabla\varphi|^2 dx; \varphi \in H^1(\Omega \setminus \omega), \varphi = 0 \text{ on } \partial\omega, \|\varphi\|_{L^2(\Omega \setminus \bar{\omega})} = 1 \right\}. \tag{2.2}$$

Here is one of the main results of our paper.

THEOREM 2.1. *If the predator population is p -zero stabilizable, then $\lambda_1^{\omega,p} \geq \alpha^*$, where*

$$\alpha^* = \frac{1}{T} \int_0^T l(s) ds \tag{2.3}$$

and l is defined by (1.31).

Conversely, if $\lambda_1^{\omega,p} > \alpha^*$, then the predator population is p -zero stabilizable and, for $\gamma > 0$ large enough, the feedback control $u := -\gamma p$ realizes (1.6), where (h, p) is the nonnegative solution to (1.5) corresponding to $u := -\gamma p$.

In order to prove Theorem 2.1, we need first to establish two auxiliary results. For any $\gamma \geq 0$ we consider the following problem:

$$\begin{aligned} -d_2 \Delta \varphi + m(x) \gamma \varphi &= \lambda \varphi \quad \text{in } \Omega, \\ \frac{\partial \varphi}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \end{aligned} \quad (2.4)$$

and denote by $\lambda_{1,\gamma}^p$ its principal eigenvalue.

LEMMA 2.2.

$$\lim_{\gamma \rightarrow \infty} \lambda_{1,\gamma}^p = \lambda_1^{\omega,p}. \quad (2.5)$$

Proof of Lemma 2.2. By Rayleigh's principle, one gets

$$\lambda_{1,\gamma}^p = \min \left\{ d_2 \int_{\Omega} |\nabla \varphi|^2 dx + \gamma \int_{\omega} |\varphi|^2 dx; \varphi \in H^1(\Omega), \|\varphi\|_{L^2(\Omega)} = 1 \right\}. \quad (2.6)$$

Hence, for every $0 \leq \gamma_1 \leq \gamma_2$, we have

$$\lambda_{1,\gamma_1}^p \leq \lambda_{1,\gamma_2}^p. \quad (2.7)$$

Now, denoting by φ_1 the corresponding eigenfunction to $\lambda_1^{\omega,p}$ satisfying $\|\varphi_1\|_{L^2(\Omega)} = 1$, $\varphi_1(x) \geq 0$ a.e. $x \in \Omega$, we get that φ_1 is the minimum point for the right-hand side of (2.2).

We extend φ_1 to Ω as follows:

$$\tilde{\varphi}(x) = \begin{cases} \varphi_1(x), & x \in \Omega \setminus \bar{\omega}, \\ 0, & x \in \omega. \end{cases} \quad (2.8)$$

Then

$$\lambda_1^{\omega,p} = d_2 \int_{\Omega} |\nabla \tilde{\varphi}|^2 dx + \gamma \int_{\omega} |\tilde{\varphi}|^2 dx \geq \lambda_{1,\gamma}^p, \quad \forall \gamma \geq 0. \quad (2.9)$$

Thus one obtains

$$\lim_{\gamma \rightarrow \infty} \lambda_{1,\gamma}^p \leq \lambda_1^{\omega,p}. \quad (2.10)$$

To prove the equality, let us consider $\varphi_\gamma \in H^1(\Omega)$ such that $\|\varphi_\gamma\|_{L^2(\Omega)} = 1$ and

$$\lambda_{1,\gamma}^p = d_2 \int_{\Omega} |\nabla \varphi_\gamma|^2 dx + \gamma \int_{\omega} |\varphi_\gamma|^2 dx \leq \lambda_1^{\omega,p}. \quad (2.11)$$

It follows that there exists a constant $M \geq 0$ such that

$$\int_{\Omega} |\nabla \varphi_\gamma|^2 dx \leq M, \quad \gamma \int_{\omega} |\varphi_\gamma|^2 dx \leq M, \quad \forall \gamma \geq 0. \quad (2.12)$$

Therefore, there exists a subsequence (also denoted by $\{\varphi_\gamma\}$), such that

$$\begin{aligned} \varphi_\gamma &\rightharpoonup \varphi^* \quad \text{weakly in } H^1(\Omega), \\ \varphi_\gamma &\rightharpoonup \varphi^* \quad \text{in } L^2(\Omega), \\ \varphi_\gamma &\rightharpoonup 0 \quad \text{in } L^2(\omega). \end{aligned} \tag{2.13}$$

Hence, $\varphi^* \in H^1(\Omega \setminus \bar{\omega})$, $\|\varphi^*\|_{L^2(\Omega \setminus \bar{\omega})} = 1$, $\varphi^* \equiv 0$ in ω , and one may infer that $\varphi^* = 0$ on $\partial\omega$. Thus by (2.11) we get that

$$\lim_{\gamma \rightarrow \infty} \lambda_{1,\gamma}^p \geq \lambda_1^{\omega,p}. \tag{2.14}$$

By (2.10) and (2.14) we get the conclusion of Lemma 2.2. □

LEMMA 2.3. *Let (h, p) be a nonnegative solution to (1.5), corresponding to the control $u \in L_{\text{loc}}^\infty(\bar{\omega} \times [0, \infty))$. If*

$$\lim_{t \rightarrow \infty} p(t) = 0 \quad \text{in } L^\infty(\Omega), \tag{2.15}$$

then

$$\lim_{t \rightarrow \infty} (h(t) - \tilde{h}(t)) = 0 \quad \text{in } L^\infty(\Omega), \tag{2.16}$$

where \tilde{h} is the unique nontrivial nonnegative solution to (1.11).

Proof. Since

$$\lim_{t \rightarrow \infty} p(t) = 0 \quad \text{in } L^\infty(\Omega), \tag{2.17}$$

it follows that, for every small enough $\delta > 0$, there exists $t_\delta > 0$ such that

$$0 \leq p(t, x) \leq \delta \quad \text{a.e. } x \in \Omega, \quad \forall t \geq t_\delta. \tag{2.18}$$

By (H3) we get that

$$0 \leq f_1(t, h(x, t), p(x, t))p \leq C\delta, \quad \text{a.e. } x \in \Omega, \quad \forall t \geq t_\delta. \tag{2.19}$$

Let us denote now by h_1 and h_2 the solutions to the following problems, respectively:

$$\begin{aligned}
 (h_1)_t - d_1 \Delta h_1 &= r(t)h_1 - k(t)h_1^2 - C\delta h_1, & x \in \Omega, t > t_\delta, \\
 \frac{\partial h_1}{\partial \nu} &= 0, & x \in \partial\Omega, t > t_\delta, \\
 h_1(x, t_\delta) &= \rho_1, & x \in \Omega, \\
 (h_2)_t - d_1 \Delta h_2 &= r(t)h_2 - k(t)h_2^2, & x \in \Omega, t > t_\delta, \\
 \frac{\partial h_2}{\partial \nu} &= 0, & x \in \partial\Omega, t > t_\delta, \\
 h_2(x, t_\delta) &= \rho_2, & x \in \Omega,
 \end{aligned} \tag{2.20}$$

where $\rho_1 > 0$ is a small enough constant and ρ_2 is a large enough constant, such that

$$0 < \rho_1 < h(x, t_\delta) < \rho_2 \quad \text{a.e. } x \in \Omega \tag{2.21}$$

(existence of such ρ_1 is a consequence of a result in [5]).

Then, by the comparison principle for the parabolic equations, we obtain

$$h_1(x, t) \leq h(x, t) \leq h_2(x, t), \quad \text{a.e. } x \in \Omega, \forall t \geq t_\delta. \tag{2.22}$$

As in the proof of Lemma 1.3 we can prove that h_2 satisfies

$$\begin{aligned}
 \lim_{t \rightarrow \infty} |h_2(t) - \tilde{h}(t)| &= 0, \\
 \lim_{t \rightarrow \infty} |h_1(t) - \tilde{h}_\delta(t)| &= 0,
 \end{aligned} \tag{2.23}$$

where \tilde{h}_δ is the unique nontrivial nonnegative solution to

$$\begin{aligned}
 \tilde{h}_t - d_1 \Delta \tilde{h} &= r(t)\tilde{h} - k(t)\tilde{h}^2 - C\delta \tilde{h}, & x \in \Omega, t > 0, \\
 \frac{\partial \tilde{h}}{\partial \nu} &= 0, & x \in \partial\Omega, t > 0, \\
 \tilde{h}(x, t) &= \tilde{h}(x, t + T), & x \in \Omega, t \geq 0.
 \end{aligned} \tag{2.24}$$

Since $\delta \mapsto \tilde{h}_\delta$ is a decreasing function, then we may pass to the limit in (2.24) and get that

$$\lim_{t \rightarrow \infty} |\tilde{h}_\delta(t) - \tilde{h}(t)| = 0. \tag{2.25}$$

By (2.22)–(2.24) we get the conclusion. \square

Proof of Theorem 2.1. Assume that $p_0(x) > 0$ a.e. $x \in \Omega$ and let (h, p) be a nonnegative solution to (1.5) corresponding to the p -stabilizing control $u \in L_{\text{loc}}^\infty(\bar{\omega} \times [0, \infty))$. Since

$$\lim_{t \rightarrow \infty} \|p(t)\|_{L^\infty(\Omega)} = 0, \tag{2.26}$$

it follows by Lemma 2.3 that

$$\lim_{t \rightarrow \infty} \|h(t) - \tilde{h}(t)\|_{L^\infty(\Omega)} = 0, \quad (2.27)$$

which implies, due to the continuity of the function f_2 , that, for any $\varepsilon > 0$, there exists $t_\varepsilon \geq 0$ such that

$$\|h(t)f_2(t, h(t), p(t)) - \tilde{h}(t)f_2(t, \tilde{h}(t), 0)\|_{L^\infty(\Omega)} < \varepsilon, \quad (2.28)$$

for any $t \geq t_\varepsilon$.

Let $\varepsilon > 0$ be arbitrary but fixed. Denoting now by p_1 the solution to the following problem:

$$\begin{aligned} (p_1)_t - d_2 \Delta p_1 &= -a(t)p_1 + f_2(t, \tilde{h}(t), 0)\tilde{h}(t)p_1 - \varepsilon p_1, & x \in \Omega \setminus \bar{\omega}, t > t_\varepsilon, \\ p_1 &= 0, & x \in \partial\omega, t > t_\varepsilon, \\ \frac{\partial p_1}{\partial \nu} &= 0, & x \in \partial\Omega, t > t_\varepsilon, \\ p_1(x, t_\varepsilon) &= p(x, t_\varepsilon), & x \in \Omega \setminus \bar{\omega}, \end{aligned} \quad (2.29)$$

we obtain via the comparison principle for parabolic equations and using (2.28) that

$$0 \leq p_1(x, t) \leq p(x, t), \quad \text{a.e. } x \in \Omega \setminus \bar{\omega}, \forall t \geq t_\varepsilon. \quad (2.30)$$

Let φ_1 be an eigenfunction corresponding to $\lambda_1^{\omega, p}$ and satisfying $\|\varphi_1\|_{L^2(\Omega \setminus \bar{\omega})} = 1$, $\varphi_1(x) \geq 0$ a.e. $x \in \Omega \setminus \bar{\omega}$ and denote by $\langle \cdot, \cdot \rangle$ the usual inner product in $L^2(\Omega \setminus \bar{\omega})$. Then

$$\langle p_1(t), \varphi_1 \rangle' + (\lambda_1^{\omega, p} - l(t) + \varepsilon) \langle p_1(t), \varphi_1 \rangle = 0, \quad \forall t \geq t_\varepsilon. \quad (2.31)$$

We infer that

$$\langle p_1(t), \varphi_1 \rangle = \exp \left\{ -\lambda_1^{\omega, p}(t - t_\varepsilon) + \int_{t_\varepsilon}^t (l(s) - \varepsilon) ds \right\} \langle p(t_\varepsilon), \varphi_1 \rangle, \quad \forall t \geq t_\varepsilon. \quad (2.32)$$

The p -zero stabilizability and (2.30) imply that

$$\lim_{t \rightarrow \infty} p_1(t) = 0 \quad \text{in } L^\infty(\Omega \setminus \bar{\omega}). \quad (2.33)$$

Since $p(x, t_\varepsilon) > 0$ a.e. $x \in \Omega$ (see [5]), we conclude that

$$-\lambda_1^{\omega, p} T + \int_0^T l(t) dt - \varepsilon T < 0. \quad (2.34)$$

Making $\varepsilon \rightarrow 0$ we get the conclusion. □

Conversely, assume that $\lambda_1^{\omega,p} > \alpha^*$. Then, by Lemma 2.2, we have that for $\varepsilon > 0$ small enough and for $\gamma \geq 0$ large enough

$$\lambda_{1,\gamma}^p - \varepsilon > \alpha^*. \quad (2.35)$$

Set now $u := -\gamma p$ and let (h, p) be the corresponding solution to (1.5). Using (1.9) and Lemma 1.3, we get that for every $\varepsilon > 0$, there exists $T_\varepsilon \geq 0$, such that

$$h(t, x) f_2(t, h(t, x), p(t, x)) < \tilde{h}(t) f_2(t, \tilde{h}(t), 0) + \varepsilon, \quad \text{a.e. } x \in \Omega, \quad \forall t \geq T_\varepsilon. \quad (2.36)$$

Denote by p_2 the solution to the following problem:

$$\begin{aligned} (p_2)_t - d_2 \Delta p_2 &= -a(t) p_2 + f_2(t, \tilde{h}(t), 0) \tilde{h}(t) p_2 + \varepsilon p_2 - m(x) \gamma p_2, \quad x \in \Omega, \quad t > T_\varepsilon, \\ \frac{\partial p_2}{\partial \nu} &= 0, \quad x \in \partial \Omega, \quad t > T_\varepsilon, \\ p_2(x, T_\varepsilon) &= \varphi_{1,\gamma}(x), \quad x \in \Omega, \end{aligned} \quad (2.37)$$

where $\varphi_{1,\gamma}$ is an eigenfunction of (2.4) corresponding to $\lambda := \lambda_{1,\gamma}^p$ and satisfying $\varphi_{1,\gamma}(x) \geq p(x, T_\varepsilon)$ a.e. $x \in \Omega$.

Applying the comparison result for parabolic equations, we conclude that

$$0 \leq p(x, t) \leq p_2(x, t), \quad \text{a.e. } x \in \Omega, \quad \forall t \geq T_\varepsilon. \quad (2.38)$$

This yields

$$p_2(x, t) \leq \varphi_{1,\gamma}(x) \exp \left\{ -\lambda_{1,\gamma}^p (t - T_\varepsilon) + \int_{T_\varepsilon}^t (l(s) + \varepsilon) ds \right\}, \quad \text{a.e. } x \in \Omega, \quad \forall t \geq T_\varepsilon. \quad (2.39)$$

Since $\lambda_{1,\gamma}^p > (1/T) \int_0^T l(s) ds + \varepsilon$, it follows that

$$p_2(t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega), \quad (2.40)$$

which implies that

$$p(t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega), \quad (2.41)$$

as $t \rightarrow +\infty$, at the same rate as $\exp \{ (-\lambda_{1,\gamma}^p + \alpha^* + \varepsilon) t \}$.

Remark 2.4. Since

$$\lim_{\gamma \rightarrow +\infty, \varepsilon \rightarrow 0+} (\lambda_{1,\gamma}^p - \varepsilon) = \lambda_1^{\omega,p}, \quad (2.42)$$

we see how important it would be to maximize $\lambda_1^{\omega,p}$ with respect to the location and geometry of ω and Ω .

3. The h -zero stabilization of the predator population

In this section, we are looking for a stabilizing control v acting indirectly (acting on the prey population). Let us consider (h, p) a solution to (1.7) corresponding to the feedback control $v := -\gamma h$. The system becomes

$$\begin{aligned} h_t - d_1 \Delta h &= r(t)h - k(t)h^2 - f_1(t, h, p)hp - m(x)\gamma h, & x \in \Omega, t > 0, \\ p_t - d_2 \Delta p &= -a(t)p + f_2(t, h, p)hp, & x \in \Omega, t > 0, \\ \frac{\partial h}{\partial \nu} &= \frac{\partial p}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ h(x, 0) &= h_0(x), \quad p(x, 0) = p_0(x), & x \in \Omega. \end{aligned} \tag{3.1}$$

For any $\gamma \geq 0$ we consider the following eigenvalue problem:

$$\begin{aligned} -d_1 \Delta \Psi + m(x)\gamma \Psi &= \lambda \Psi \quad \text{in } \Omega, \\ \frac{\partial \Psi}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

and denote by $\lambda_{1,\gamma}^h$ its principal eigenvalue. Next, we denote by $\lambda_1^{\omega,h}$ the principal eigenvalue to

$$\begin{aligned} -d_1 \Delta \Psi &= \lambda \Psi, \quad x \in \Omega \setminus \bar{\omega}, \\ \Psi &= 0, \quad x \in \partial\omega, \\ \frac{\partial \Psi}{\partial \nu} &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{3.3}$$

It is a consequence of Rayleigh's principle that the mapping $\gamma \mapsto \lambda_{1,\gamma}^h$ is increasing and continuous, and

$$\lambda_{1,\gamma}^h \longrightarrow \lambda_1^{\omega,h} \quad \text{as } \gamma \longrightarrow \infty. \tag{3.4}$$

Let

$$\tilde{\alpha}^* = \frac{1}{T} \int_0^T r(s) ds. \tag{3.5}$$

In the same manner as in Section 2 it follows the next result.

THEOREM 3.1. *If for a $\gamma \geq 0$ one has that $\lambda_{1,\gamma}^h > \tilde{\alpha}^*$, then the predator population is h -zero stabilizable and the feedback control $v := -\gamma h$ realizes (1.6), where (h, p) is the solution to (1.7) corresponding to $v := -\gamma h$. Moreover,*

$$\lim_{t \rightarrow +\infty} h(t) = 0 \quad \text{in } L^\infty(\Omega). \tag{3.6}$$

Remark 3.2. Assume that the hypotheses in Theorem 3.1 hold. Since $h(t) \rightarrow 0$ in $L^\infty(\Omega)$, as $t \rightarrow +\infty$, then it follows (as in Section 2) that $p(t) \rightarrow 0$ in $L^\infty(\Omega)$, as $t \rightarrow +\infty$, at the rate of

$$\exp \left\{ - \left(\frac{1}{T} \int_0^T a(s) ds + \varepsilon \right) t \right\} \quad (3.7)$$

(for $\varepsilon > 0$ small enough).

If, in addition, $(1/T) \int_0^T a(s) ds > \lambda_1^{\omega, p}$, then the second strategy (when we act on prey) leads to a faster convergence to zero of p , so it is better.

Remark 3.3. If $\lambda_1^{\omega, h} > \tilde{\alpha}^*$, then there exists $\gamma \geq 0$ such that $\lambda_{1, \gamma}^h > \tilde{\alpha}^*$. The solution (h, p) to (3.1) satisfies

$$h(t) \rightarrow 0 \quad \text{in } L^\infty(\Omega), \quad (3.8)$$

as $t \rightarrow +\infty$. Therefore,

$$p(t) \rightarrow 0 \quad \text{in } L^\infty(\Omega), \quad (3.9)$$

as $t \rightarrow +\infty$.

Remark 3.4. In general, the habitat of preys is larger than Ω . The strategy to eradicate the predators via indirect control is the following one: we isolate the domain Ω (we do not permit migration through the boundary of it), then we eradicate firstly the preys in Ω and consequently the predators will extinct. Next, the preys are allowed to repopulate the domain Ω .

4. Final comments

The results in Sections 2 (and 3) show how important is to find the position and the geometry of ω and Ω in order to get a great value for $\lambda_1^{\omega, p}$ (and $\lambda_1^{\omega, h}$).

This yields

$$\lambda_1^{\omega, p} = d_2 \lambda_1(\omega, \Omega), \quad \lambda_1^{\omega, h} = d_1 \lambda_1(\omega, \Omega), \quad (4.1)$$

where $\lambda_1(\omega, \Omega)$ is the principal eigenvalue to

$$\begin{aligned} -\Delta \varphi(x) &= \lambda \varphi(x), & x &\in \Omega \setminus \bar{\omega}, \\ \varphi(x) &= 0, & x &\in \partial \omega, \\ \frac{\partial \varphi}{\partial \nu} &= 0, & x &\in \partial \Omega. \end{aligned} \quad (4.2)$$

The following result has been proved in [8] using rearrangement techniques and can be used to obtain upper and lower bounds for $\lambda_1(\omega, \Omega)$.

THEOREM 4.1. Assume that φ^* is an eigenfunction of (4.2), corresponding to $\lambda := \lambda_1(\omega, \Omega)$, that satisfies in addition

$$\begin{aligned} 0 < \varphi^*(x) < M, \quad \forall x \in \Omega \setminus \bar{\omega}, \\ \varphi^*(x) = M, \quad \forall x \in \partial\Omega, \end{aligned} \tag{4.3}$$

where $M > 0$ is a constant. Then

$$\lambda_1(\omega, \Omega) > \lambda_1(\omega, \tilde{\Omega}), \tag{4.4}$$

for any domain $\tilde{\Omega} \subset \mathbb{R}^N$ with smooth boundary and such that $\omega \subset \subset \tilde{\Omega}$, $\text{meas}(\tilde{\Omega}) = \text{meas}(\Omega)$, and $\tilde{\Omega} \neq \Omega$.

Remark 4.2. If ω and Ω are balls with the same center, there exists such φ^* .

Remark 4.3. If there exists φ^* an eigenfunction of (4.2) corresponding to $\lambda := \lambda_1(\omega, \Omega)$ and satisfying (4.3), then

$$\begin{aligned} \lambda_1(\omega, \Omega) &= \max \{ \lambda_1(\omega, \tilde{\Omega}); \tilde{\Omega} \subset \mathbb{R}^N \text{ is a domain with smooth} \\ &\quad \text{boundary and satisfying } \omega \subset \subset \tilde{\Omega}, \text{meas}(\tilde{\Omega}) = \text{meas}(\Omega) \} \\ &= \max \{ \lambda_1(\tilde{\omega}, \Omega); \tilde{\omega} \subset \subset \Omega \text{ is an isometric transform of } \omega \}. \end{aligned} \tag{4.5}$$

Remark 4.4. If ω is a ball, $\omega \subset \subset \Omega$, then we may conclude by Theorem 4.1 that

$$\lambda_1(\omega, \Omega) \leq \lambda_1(\omega, B), \tag{4.6}$$

where B is a ball with the same measure as Ω and with the same center as ω . Moreover, we have equality only for $\Omega \equiv B$ and we conclude that the maximal value for $\lambda_1(\omega, \Omega)$, subject to all domains $\Omega \subset \mathbb{R}^N$ with smooth boundary and satisfying $\omega \subset \subset \Omega$ and having a prescribed measure, is attained for the ball B of the same measure and with the same center as ω .

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