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Research Article

On Existence of Solution for a Class of Semilinear Elliptic Equations with Nonlinearities That Lies between Different Powers

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We prove that the semilinear elliptic equation $-\Delta u = f(u)$, in Ω, $u = 0$, on $\partial\Omega$ has a positive solution when the nonlinearity f belongs to a class which satisfies $\mu t^q \leq f(t) \leq Ct^p$ at infinity and behaves like t^q near the origin, where $1 < q < (N+2)/(N-2)$ if $N \geq 3$ and $1 < q < +\infty$ if $N = 1, 2$. In our approach, we do not need the Ambrosetti-Rabinowitz condition, and the nonlinearity does not satisfy any hypotheses such those required by the blowup method. Furthermore, we do not impose any restriction on the growth of p.

1. Introduction

Since the 70s, several authors have been studying existence of solutions for the semilinear elliptic Dirichlet problem

$$\begin{aligned} -\Delta u = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$. Most part of these results were done under some hypotheses on the nonlinearity f in order to make variational methods work. The most used hypothesis is the Ambrosetti-Rabinowitz condition [1] which makes the Euler-Lagrange functional associated to subcritical problem (P) satisfies the Palais-Smale condition. Also, the blowup method, due to Gidas and Spruck [2], is used to get existence of solutions and, to work, it needs an asymptotical behaviour like t^p for the subcritical nonlinearity f (this is, for the case $1 < p < (N+2)/(N-2)$, when $N \geq 3$ or $1 < p < \infty$, when $N = 2$). Most recently, two existence results related to the subject have been published: de Figueiredo and Yang [3] considered problem (P), where variational techniques were applied together with Morse's index, in Azizieh and Clément [4] using also variational techniques and the blowup method; problem (P) was studied for the m-Laplacean operator, with $m > 1$. In both papers [3, 4], the existence of solutions depends on the following condition:

$$\mu t^p \leq f(t) \leq C t^p \quad \text{for large } t. \tag{1.1}$$

Our results complete those above in the sense that this paper treats this subject for a class of nonlinearities that lies between two different powers t^q and t^p, $1 < q < p$, with no restrictions on p. A multiplicity of solutions result was done by Li and Liu [5] with critical growth restriction on p. Here, we will suppose that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Hölder continuous function with the following decomposition $f(s) = z(s) + \lambda g(s)$. The hypotheses on the functions z and g are the following.

(H_1) $\lim_{s\to 0}(z(s)/s) = 0$.

(H_2) There exists $\theta > 2$ such that

$$0 < \theta Z(s) \leq z(s)s \quad \text{for } s \geq 0, \tag{1.2}$$

where $Z(s) = \int_0^s z(t)dt$.

(H_3) If $N \geq 3$, there exists $q \in (1, (N+2)/(N-2))$, and $q \in (1, +\infty)$ if $N = 1, 2$ such that

$$\left| z(s) \right| \leq |s|^q \quad \forall s \in \mathbb{R}. \tag{1.3}$$

(H_4) There is a positive real sequence $\{M_n\}$ satisfying $\lim_{n\to+\infty} M_n = +\infty$, $g(M_n) > 0$, and

$$\frac{g(M_n)}{M_n^q} \geq \frac{g(s)}{s^q} \quad \forall s \in [0, M_n], \forall n \in \mathbb{N}. \tag{1.4}$$

For convenience, we rewrite problem (P) in the following way:

$$\begin{aligned} -\Delta u &= z(u) + \lambda g(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{P}$$

Observe that conditions (H_1), (H_3), and (H_4) imply

$$\limsup_{s\to 0^+} \frac{f(s)}{s} < \lambda_1, \tag{1.5}$$

where λ_1 is the first eigenvalue of $(-\Delta; H_o^1(\Omega))$. Moreover, since $f(s) \geq z(s)$, for all $s > 0$, a necessary condition for the existence of solution for problem (P) is that the nonlinearity f crosses the line $\lambda_1 s$.

As an example, we may consider

$$z(s) = s^q, \quad g(s) = s^p(1 + \cos s), \quad \theta = q + 1, \quad M_n = 2n\pi. \tag{1.6}$$

The function $f(s) = \lambda s^p(1 + \cos s) + s^q$ satisfies the above conditions but does not satisfy Ambrosetti-Rabinowitz conditions. Moreover, f also does not belong to any class of solutions contained in the references [1–3] or [4].

Our main result is the following.

Theorem 1.1. *There exists $\lambda_* > 0$ such that problem (P) has a positive solution for all $\lambda \in (0, \lambda_*]$.*

In our proof, we adapt an idea explored by Chabrowski and Yang in [6]. Comparing our approach to others cited in the literature, it is important to stress that ours does not use Morse Index method and consequently we do not need derivatives of the nonlinearity as in [3], and we do not have any restriction $1 < p \leq N/(N-2)$ as required in [4], for the case $m = 2$.

2. Proof of Theorem 1.1

Hereafter, let us denote by $\|v\|$ the usual norm of $H_o^1(\Omega)$, that is,

$$\|v\| = \left(\int_\Omega |\nabla v|^2 dx\right)^{1/2}, \tag{2.1}$$

and by c_q the minimax level obtained by the mountain-pass theorem of Ambrosetti and Rabinowitz applied to the functional

$$J_q(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega Z(v_+)dx, \tag{2.2}$$

where $v_+(x) = \max\{0, v(x)\}$.

The proposition below establishes an estimate involving the L^∞-norm of a solution related to a subcritical problem. This estimate is an important point in our approach, and its proof is an immediate consequence of bootstrap arguments. The constant λ_* that appears in Theorem 1.1 depends only on the constant k below and on the sequence (M_n) given by (H_4), as we can see in this section.

Proposition 2.1. *Let v be a solution of the problem*

$$\begin{aligned} -\Delta v &= h(v) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a continuous functions verifying $|h(s)| \leq 2|s|^q$, for all $s > 0$. Then, for all $C > 0$, there exists a constant $k = k(\Omega, q, C) > 0$ such that if

$$\|v\|^2 \leq C, \quad \|v\|_\infty \leq k. \tag{2.4}$$

Applying Proposition 2.1, for the constant $C = 2c_q(1/2 - 1/(q+1))^{-1}$ there exists a constant $k = k(\Omega, q)$ such that (2.4) holds.

Let us fix $n \in \mathbb{N}$ such that $M_n > k = k(\Omega, q)$, and let $\lambda_* > 0$ satisfy

$$\lambda_* \frac{g(M_n)}{M_n^q} \leq 1, \qquad \lambda_*\left(\frac{1}{\theta} + \frac{1}{q+1}\right) g(M_n) M_n |\Omega| \leq c_q, \tag{2.5}$$

where $|\Omega|$ denotes the Lebesgue's measure of the set Ω.

For that $n \in \mathbb{N}$ previously fixed , let us consider the function $h_n : \mathbb{R} \to \mathbb{R}$ given by

$$h_n(s) = \begin{cases} 0, & \text{if } s \le 0, \\ g(s), & \text{if } 0 \le s \le M_n, \\ \dfrac{g(M_n)}{M_n^q} s^q, & \text{if } s \ge M_n. \end{cases} \tag{2.6}$$

From condition (H_4),

$$|h_n(s)| \le \frac{g(M_n)}{M_n^q}|s|^q, \quad \forall s, \tag{2.7}$$

and then $f_n(s) = z(s) + \lambda h_n(s)$ satisfies the inequality $f_n(s) \le 2|s|^q$, since $\lambda \le \lambda_*$.

It is easy to check that the truncated function f_n satisfies Ambrosetti-Rabinowitz condition in the interval $[M_n, +\infty)$, that is,

$$\theta F_n(s) - s f_n(s) \le 0, \quad \forall s \ge M_n, \tag{2.8}$$

where $F_n(s) = \int_o^s f_n(t)dt$. In fact, denoting $H_n(s) = \int_o^s h_n(t)dt$ we have

$$H_n(s) = \int_o^{M_n} g(t)dt + \int_{M_n}^s \frac{g(M_n)}{M_n^q} t^q dt, \tag{2.9}$$

for $s > M_n$, which implies

$$H_n(s) = \int_o^{M_n} g(t)dt + \frac{1}{q+1}\frac{g(M_n)}{M_n^q}\left[s^{q+1} - M_n^{q+1}\right]. \tag{2.10}$$

Integrating inequality (H_4), from 0 to M_n, we have

$$\int_o^{M_n} g(t)dt \le \frac{1}{q+1}\frac{g(M_n)}{M_n^q} M_n^{q+1}. \tag{2.11}$$

Consequently, from (2.10) and (2.11), the inequality

$$H_n(s) \le \frac{1}{q+1}\frac{g(M_n)}{M_n^q} s^{q+1} \tag{2.12}$$

holds. Thus,

$$(q+1)H_n(s) - s h_n(s) \le 0 \quad \forall s > M_n, \tag{2.13}$$

which, together with (H_2), implies

$$\theta F_n(s) - s f_n(s) \le 0 \quad \forall s \ge M_n, \tag{2.14}$$

as we wanted to check (observe that (H_2) and (H_3) imply that $\theta \le q + 1$).

Now, consider the following truncated problem:

$$\begin{aligned} -\Delta u &= z(u) + \lambda h_n(u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{P_n}$$

Let us recall that $\lambda \leq \lambda_*$ and $M_n > k$. It is well known that problem (P_n) has a positive solutions $u \in W_o^{1,p}(\Omega)$. This occurs since the energy functional related to (P_n) given by

$$J_n(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega Z(v_+)dx - \lambda \int_\Omega H_n(v)dx \tag{2.15}$$

or

$$J_n(v) = J_q(v) - \lambda \int_\Omega H_n(v)dx \tag{2.16}$$

satisfies the hypotheses of the mountain pass theorem [1]. Moreover, from the inequality $J_n(v) \leq J_q(v)$, for all $v \in H_o^1(\Omega)$, the minimax level b_n of functional J_n satisfies $b_n \leq c_q$.

Thus, from equality $b_n = J_n(u) = J_n(u) - (1/\theta)J_n'(u)u$, we have

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 + \int_\Omega \left(\frac{1}{\theta}uf_n(u) - F_n(u)\right)dx = b_n \tag{2.17}$$

and from (2.8) we have

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 \leq \lambda \int_{\{u \leq M_n\}} \left|\frac{1}{\theta}uh_n(u) - H_n(u)\right|dx + b_n. \tag{2.18}$$

From inequality (2.7), we have $|(1/\theta)uh_n(u) - H_n(u)| \leq (1/\theta + 1/(q+1))g(M_n)M_n$ in $\{x \in \Omega : u(x) \leq M_n\}$, hence

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 \leq \lambda\left(\frac{1}{\theta} + \frac{1}{q+1}\right)g(M_n)M_n|\Omega| + c_q. \tag{2.19}$$

From the choice of λ_*, we get

$$\|u\|^2 \leq 2c_q\left(\frac{1}{2} - \frac{1}{q+1}\right)^{-1}. \tag{2.20}$$

Using the constant C obtained in Proposition 2.1 and from the last inequality, we have that $u(x) \leq k$, for all $x \in \Omega$. But $M_n > k$ implies that $u(x) \leq M_n$ and thus u is a solution for (P). The proof is done.

Remark 2.2. Hypothesis (H_4) can be replaced by

$(H_4)'$ There is a positive real $M > k = k(\Omega, q)$ satisfying

$$\frac{g(M)}{M^q} \geq \frac{g(s)}{s^q} \quad \forall s \in [0, M]. \tag{2.21}$$

3. Final comments

The methods applied in this paper with few modifications can be used to establish existence of positive solutions for the problem

$$-\Delta_p u = z(u) + \lambda g(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

with $N > p > 1$ and z, g verifying

(H_5) $\lim_{s\to 0}(z(s)/|s|^{p-2}s) = 0$.

(H_6) There exists $\theta > p$ such that

$$0 < \theta Z(s) \le z(s)s, \quad \text{for } s \ge 0, \tag{3.2}$$

where $Z(s) = \int_0^s z(t)dt$.

(H_7) There exists $q \in (p-1, (N(p-1)+p)/(N-p))$ such that

$$|z(s)| \le |s|^q \quad \forall s \in \mathbb{R}. \tag{3.3}$$

(H_8) There is a positive real sequence $\{M_n\}$ satisfying $\lim_{n\to+\infty} M_n = +\infty$, $g(M_n) > 0$, and

$$\frac{g(M_n)}{M_n^q} \ge \frac{g(s)}{s^q} \quad \forall s \in [0, M_n], \forall n \in \mathbb{N}. \tag{3.4}$$

The ideas developed in this paper may be used when we deal with the p-Laplacian operator. The main difference is concerned with the proof of Proposition 2.1, where we should replace bootstrap arguments by Moser's iteration methods, and then repeat the same approach explored by Chabrowski and Yang [6].

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