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Research Article

Existence Results for Generalized Vector Equilibrium Problems with Multivalued Mappings via KKM Theory

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We first define upper sign continuity for a set-valued mapping and then we consider two types of generalized vector equilibrium problems in topological vector spaces and provide sufficient conditions under which the solution sets are nonempty and compact. Finally, we give an application of our main results. The paper generalizes and improves results obtained by Fang and Huang in (2005).

1. Introduction and preliminaries

Throughout this paper, unless otherwise specified, we always let X and Y be real Hausdorff topological vector spaces, $K \subseteq X$ a nonempty convex set, $C : K \to 2^Y$ with pointed closed cone convex values (we recall that a subset A of Y is convex cone and pointed whenever $A + A \subseteq Y$, $tA \subseteq A$, for $t \geq 0$, and $A \cap -A = \{0\}$, resp.), where 2^Y denotes all the subsets of Y. Denote by $L(X,Y)$ the set of all continuous linear mappings from X into Y. For any given $l \in L(X,Y)$, $x \in X$, let $\langle l, x\rangle$ denote the value of l at x. Let $T : K \to L(X,Y)$ and $g : K \to K$ be two mappings. Finally, let $F : K \times K \to 2^Y$ be a set-valued mapping. We need the following definitions and results in the sequel.

Definition 1.1. Let $F : K \times K \to 2^Y$ be a set-valued mapping. One says that F is

(i) strongly C-pseudomonotone if, for any given x and $y \in K$,

$$\not\subseteq -\operatorname{int} C(x) \Longrightarrow F(y,x) \subseteq -C(y). \tag{1.1}$$

(ii) C-pseudomonotone if, for any given x and $y \in K$,

$$F(x,y) \not\subseteq -C(x) \setminus \{0\} \Longrightarrow F(y,x) \subseteq -C(y). \tag{1.2}$$

Remark 1.2. (1) Strongly C-pseudomonotonicity implies C-pseudomonotonicity.

(2) Let $T : K \to 2^{L(X,Y)}$ be a set-valued mapping and let $g : K \to K$ be a mapping. If we define $F(x,y) = \langle Tx, y - g(x)\rangle$, for each $(x,y) \in K \times K$, then strongly C-pseudomonotonicity and C-pseudomonotonicity reduce to the strongly C-pseudomonotonicity and C-pseudomonotonicity, of T with respect to g, respectively, introduced in [1].

Definition 1.3. Let X and Y be two topological spaces. A set-valued mapping $G : X \to 2^Y$ is called

(i) *upper semicontinuous* (u.s.c.) at $x \in X$ if for each open set V containing $G(x)$, there is an open set U containing x such that for each $t \in U$, $G(t) \subseteq V$; G is said to be u.s.c. on X if it is u.s.c. at all $x \in X$;

(ii) *lower semicontinuous* (l.s.c.) at $x \in X$ if for each open set V with $G(x) \cap V \neq \varnothing$, there is an open set U containing x such that for each $t \in U$, $G(t) \cap V \neq \varnothing$; G is said to be l.s.c. on X if it is l.s.c. at all $x \in X$;

(iii) *closed* if the graph of G, that is, the set $\{(x,y) : x \in X,\ y \in G(x)\}$, is a closed set in $X \times Y$;

(iv) *compact* if the closure of range G, that is, $\operatorname{cl} G(X)$, is compact, where $G(X) = \bigcup_{x \in X} G(x)$.

Remark 1.4. One can see that (ii) is equivalent to the following statement:

G is l.s.c. at $x \in X$ if for each closed set $C \subseteq Y$, any net $\{x_\alpha\} \subseteq K$, x_α converges to x, and $G(x_\alpha) \subseteq C$, for all α imply that $G(x) \subseteq C$.

Lemma 1.5 (see [2]). *Let X and Y be two topological spaces. Suppose that $G : X \to 2^Y$ is a set-valued mapping. Then the following statements are true.*

(a) *If G is closed and compact, then G is u.s.c.*

(b) *Let, for any $x \in X$, $G(x)$ be compact. If G is u.s.c. on X then for any net $\{x_\alpha\} \subset X$ such that $x_\alpha \to x$ and for every $y_\alpha \in G(x_\alpha)$, there exist $y \in G(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$.*

For the converse of (b) in Lemma 1.5, we refer the reader to [3].

Definition 1.6. Let X be a topological vector space and Y a topological space. A set-valued mapping $G : X \to 2^Y$ is called upper hemicontinuous if the restriction of G on straight lines is upper semicontinuous.

Definition 1.7. One says that the mapping $G : K \times K \to 2^Y$ is C-upper sign continuous if, for all $x, y \in K$, the following implication holds:

$$G((1-t)x+ty,y)\cap C((1-t)x+ty)\neq\varnothing,\quad \forall t\in]0,1[\Longrightarrow G(x,y)\cap C(x)\neq\varnothing. \tag{1.3}$$

Remark 1.8. Let $f : K\times K \to \mathbb{R}$ be a mapping. If we define $G(x,y) = \{f(x,y)\}$, for all $x, y \in K$ and $C(x) = [0,\infty)$, then Definition 1.7 reduces to the upper sign continuous introduced by Bianchi and Pini in [4]. The upper sign continuity notion was first introduced by Hadjisavvas [5] for a single-valued mapping in the framework of variational inequality problems.

2. Main results

In this section, we consider the following generalized vector equilibrium problems (for short, GVEPs) in the topological vector space setting:

(GVEP_1) find $x \in K$ such that $F(x,y) \nsubseteq -\operatorname{int} C(x)$, $\forall y \in K$, and

(GVEP_2) find $x \in K$ such that $F(x,y) \nsubseteq -C(x)\setminus\{0\}$, $\forall y \in K$.

Clearly, a solution of GVEP_2 is also a solution of problem GVEP_1. We need the following lemma in the sequel.

Lemma 2.1. *Suppose that*

(i) *F is C-pseudomonotone;*

(ii) *$F(x,x)\cap C(x)\neq\varnothing$ for each $x\in K$;*

(iii) *F is C-upper sign continuous;*

(iv) *for each fixed $x \in K$, the mapping $z \to F(x,z)$ is convex, that is, $F(x,(1-t)y+tz)\subseteq (1-t)F(x,y)+tF(x,z)-C(x)$, $\forall y\in K$, $\forall t\in]0,1[$.*

Then for any given $y \in K$, the following are equivalent:

(I) *$F(y,z)\nsubseteq -C(y)\setminus\{0\}$, $\forall z\in K$;*

(II) *$F(z,y)\subseteq -C(z)$, $\forall z\in K$.*

Proof. (I) $\Rightarrow$ (II) is obvious from the definition of C-pseudomonotonicity of F. Suppose that (II) holds. For each $z\in K$, put $z_t = y+t(z-y)$, where $t\in]0,1[$ and $y\in K$ as above. By (II), we have

$$F(z_t,y)\subseteq -C(z_t),\quad \forall t\in]0,1[. \tag{2.1}$$

We claim that $F(z_t,z)\cap C(z_t)\neq\varnothing$. Suppose $F(z_t,z)\cap C(z_t)=\varnothing$, for some $t\in]0,1[$. Then

$$F(z_t,z)\subseteq Y\setminus C(z_t),\quad \text{for this } t\in]0,1[, \tag{2.2}$$

and so $F(z_t,z_t)\subseteq (1-t)F(z_t,y)+tF(z_t,z)-C(z_t)\subseteq -C(z_t)+(Y\setminus C(z_t))-C(z_t) = (Y\setminus C(z_t))-C(z_t)\subseteq Y\setminus C(z_t)$, which contradicts (ii) (note the first inclusion follows from (iv), the second inclusion follows from (2.1) and (2.2), and the third follows from the relation $Y \setminus C(z_t) - C(z_t) \subseteq Y \setminus C(z_t)$). Therefore, for all $t \in]0,1[$, the set $F(z_t, z) \cap C(z_t)$ is nonempty. Thus, by (iii) there is a $u \in F(y,z) \cap C(y)$. Hence, since $C(y) \cap -C(y) \setminus \{0\} = \emptyset$, we get $u \notin (-C(y) \setminus \{0\})$. Consequently, $F(y,z) \nsubseteq -C(y) \setminus \{0\}$. This completes the proof. □

Remark 2.2. If the set-valued mapping $C : K \to 2^Y$ has closed graph and for each fixed $z \in K$ the mapping $x \to F(x,z)$ is upper hemicontinuous with nonempty compact values, then condition (iii) in Lemma 2.1 holds. To see this, let x and y be arbitrary elements of K and $u_t \in F(z_t, y) \cap C(z_t) \neq \emptyset$, where $z_t = (1-t)x + ty$, $t \in]0,1[$. By Lemma 1.5(b), there exists a subnet of (u_t) (without loss of generality (u_t)) and $u \in F(x,y)$ such that $u_t \to u$, where $t \to 0$. Now, since $C : K \to 2^Y$ has closed graph (note $u_t \to u$ and $z_t \to x$ as $t \to 0$) and $u_t \in C(z_t)$, we have $u \in C(x)$. Hence, $u \in F(x,y) \cap C(x)$ and so $F(x,y) \cap C(x) \neq \emptyset$. This shows that F is C-upper sign continuous. Therefore, Lemma 2.1 improves Lemma 2.3 in [1].

By a similar argument as in Lemma 2.1 and using Remark 2.2, we can deduce the following result.

Lemma 2.3. *Suppose that*

(i) *for each fixed $z \in K$, the mapping $x \to F(x,z)$ is upper semicontinuous with compact values;*

(ii) *F is strongly C-pseudomonotone;*

(iii) *$F(x,x) \nsubseteq -\operatorname{int} C(x)$, for each $x \in K$;*

(iv) *the mapping $x \to W(x) = Y \setminus (-\operatorname{int} C(x))$, for each $x \in K$, has closed graph;*

(v) *for each fixed $x \in K$, the mapping $z \to F(x,z)$ is convex.*

Then for any given $y \in K$, the following are equivalent:

(I) $F(y,z) \nsubseteq -\operatorname{int} C(y)$, $\forall z \in K$;

(II) $F(z,y) \subseteq -C(z)$, $\forall z \in K$.

Remark 2.4. Let $T : K \to 2^{L(X,Y)}$ be a set-valued mapping. If we define $F(x,y) = \langle Tx, y - x \rangle$, where $x, y \in K$, then Lemma 2.3 reduces to Lemma 3 of Yin and Xu [6].

Lemma 2.5. *Under the assumptions of Lemma 2.1, the solution set of (GVEP$_2$) is convex.*

Proof. Let x_1 and x_2 be solutions of (GVEP$_2$). By Lemma 2.1, we have

$$F(z, x_i) \subseteq -C(z), \quad \forall z \in K, \; i = 1,2. \tag{2.3}$$

From this and condition (iv) of Lemma 2.1, for all $t \in]0,1[$, we deduce that

$$F(z, (1-t)x_1 + tx_2) \subseteq (1-t)F(z, x_1) + tF(z, x_2) - C(z) \subseteq -C(z), \tag{2.4}$$

for all $z \in K$. Hence, from Lemma 2.1, we get

$$F((1-t)x_1 + tx_2, z) \nsubseteq -C((1-t)x_1 + tx_2) \setminus \{0\}, \quad \forall z \in K. \tag{2.5}$$

This means that $(1-t)x_1 + tx_2$ is a solution of (GVEP$_2$). The proof is complete. □

Similarly, we can prove the following lemma.

Lemma 2.6. *Under the assumptions of Lemma 2.3, the solution set of ($GVEP_1$) is convex.*

Remark 2.7. Lemma 2.5 extends Theorem 3 of Yin and Xu [6] and Lemma 2.5 of Fang and Huang [1].

Definition 2.8. Let K_0 be a nonempty subset of K. A set-valued mapping $\Gamma : K_0 \to 2^K$ is said to be a *KKM map* if $\text{co}\, A \subseteq \bigcup_{x\in A}\Gamma(x)$, for every finite subset A of K_0, where co denotes the convex hull.

Lemma 2.9 ((Fan-KKM lemma) [7]). *Let K be a nonempty subset of a topological vector space X and $\Gamma : K \to 2^X$ be a KKM mapping with closed values. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x\in B}\Gamma(x)$ is compact. Then*

$$\bigcap_{x\in K}\Gamma(x) \neq \varnothing. \tag{2.6}$$

Lemma 2.10 (see [8]). *Let K be a convex subset of a metrizable topological vector space X and $F : K \to 2^K$ be a compact upper semicontinuous set-valued mapping with nonempty closed convex values. Then F has a fixed point in K.*

Theorem 2.11. *Let all the assumptions of Lemma 2.1 hold and for each fixed $x \in K$, the mapping $y \to F(x,y)$ is lower semicontinuous, where $y \in K$. If there exist a nonempty compact subset B of K and a nonempty convex compact subset D of K such that, for each $x \in K \setminus B$ there exists $y \in D$ such that $F(y,x) \nsubseteq -C(y)$, then the solution set of problem ($GVEP_2$) is nonempty and compact in K.*

Proof. Define $\Gamma, \widehat{\Gamma} : K \to 2^K$ by

$$\begin{aligned} \Gamma(y) &= \{x \in K : F(x,y) \nsubseteq -C(x) \setminus \{0\}\}, \\ \widehat{\Gamma}(y) &= \{x \in K : F(y,x) \subseteq -C(y)\}. \end{aligned} \tag{2.7}$$

We claim that Γ is a KKM mapping. If not, there exist $y_1, y_2, \dots, y_n \in K$ and $t_i > 0$, $\sum_{i=1}^n t_i = 1$ such that $z = \sum_{i=1}^n t_i y_i \notin \Gamma(y_i)$, that is,

$$F(z, y_i) \subseteq -C(z) \setminus \{0\}, \quad i = 1,2,3,\dots,n, \tag{2.8}$$

and so, since $-C(z)$ is a closed convex pointed cone,

$$\sum_{i=1}^n t_i F(z, y_i) \subseteq -C(z) \setminus \{0\}. \tag{2.9}$$

It follows from condition (iv) of Lemma 2.1 that

$$F(z,z) \subseteq \sum_{i=1}^n t_i F(z, y_i) - C(z). \tag{2.10}$$

Now, by combining (2.9) and (2.10), we get

$$F(z,z) \subseteq \sum_{i=1}^{n} t_i F(z,y_i) - C(z) \subseteq -C(z) \setminus \{0\} - C(z) \subseteq -C(z) \setminus \{0\}, \tag{2.11}$$

which is a contradiction to condition (ii) of Lemma 2.1. Therefore, Γ is a KKM mapping and so $\widehat{\Gamma}$ is also a KKM mapping (note, $\Gamma(y) \subseteq \widehat{\Gamma}(y)$, for all $y \in K$). By Remark 1.4, the values of $\widehat{\Gamma}$ are closed in K (note, for each fixed $x \in K$, the mapping $y \to F(x,y)$ is lower semicontinuous) and by our assumption, we obtain that $\bigcap_{y\in D}\widehat{\Gamma}(y)$ is a closed subset of the compact set B and hence $\bigcap_{y\in D}\widehat{\Gamma}(y)$ is compact in K. Therefore, $\widehat{\Gamma}$ fulfils all the assumptions of Lemma 2.9 and so $\bigcap_{y\in K}\widehat{\Gamma}(y) \neq \emptyset$. This means that there exists $z \in K$ such that

$$F(y,z) \subseteq -C(z), \quad \forall y \in K. \tag{2.12}$$

Now, it follows from Lemma 2.1 that

$$F(z,y) \nsubseteq -C(y) \setminus \{0\}, \quad \forall y \in K, \tag{2.13}$$

and hence z is a solution of the problem (GVEP$_2$). This proves that the solution set of (GVEP$_2$) is nonempty. By Lemma 2.1, the solution set of (GVEP$_2$) equals $\bigcap_{y\in K}\widehat{\Gamma}(y)$ and so it is a compact set in K (note, in the above that the set $\bigcap_{y\in K}\widehat{\Gamma}(y)$ is a closed subset of the compact set B). The proof is complete. □

As an application of Theorem 2.11, we derive the existence result for a solution of the following problem which consists of finding a $u \in K$ such that

$$\langle A(u,u), v - g(u)\rangle \nsubseteq -C(u) \setminus \{0\}, \quad \forall v \in K, \tag{2.14}$$

where $A : K \times K \to 2^{L(X,Y)}$ and $g : K \to K$.

This problem was considered by Fang and Huang [1] in reflexive Banach spaces setting for a set-valued mapping which is demi-C-pseudomonotone.

Theorem 2.12. *Let X be metriziable topological vector space, K nonempty convex subset of X, $A : K \times K \to 2^{L(X,Y)}$, and $g : K \to K$ be two mappings. Assume that*

(i) *for each fixed $w \in K$, the mapping $(u,v) \to \langle A(w,u), v - g(u)\rangle$ is C-pseudomonotone and C-upper sign continuous;*

(ii) *$\langle A(w,u), u - g(u)\rangle \cap C(u) \neq \emptyset$, for each (w,u);*

(iii) *for each fixed $v \in K$, the mapping $(w,u) \to \langle A(w,u), u - g(v)\rangle$ is lower semicontinuous;*

(iv) *for each finite dimensional subspace M of X with $K_M = K \cap M \neq \emptyset$, there exist compact subset B_M and compact convex subset D_M of K_M such that $\forall (w,z) \in K_M \times (K_M \setminus B_M)$, $\exists u \in D_M$ such that $\langle A(w,u), z - g(u)\rangle \nsubseteq -C(u)$.*

Then there exists $u \in K$ such that

$$\langle A(u,u), v - g(u)\rangle \nsubseteq -C(u) \setminus \{0\}, \quad \forall v \in K. \tag{2.15}$$

Proof. Let $M \subset X$ be a finite dimensional subspace with $K_M = K \cap M \neq \varnothing$. For each fixed $w \in K$, consider the problem of finding a $u \in K_M$ such that

$$\langle A(w,u), v - g(u) \rangle \nsubseteq -C(u) \setminus \{0\}, \quad \forall v \in K_M. \tag{2.16}$$

By Theorem 2.11, the problem (2.16) has a nonempty compact solution set in K (note, in Theorem 2.11 take $F(u,v) = \langle A(w,u), v - g(u) \rangle,\ (u,v) \in K \times K$). For $w \in K_M$, we define a set-valued mapping $T : K_M \to 2^{K_M}$ by

$$T(w) = \{u \in K_M : \langle A(w,u), v - g(u) \rangle \nsubseteq -C(u) \setminus \{0\},\ \forall v \in K_M\}. \tag{2.17}$$

Then $T(w)$ is a nonempty closed subset of B_M, in fact, $T(w)$ is the solution set of (2.16) corresponding to w. By Lemma 2.1, we have

$$T(w) = \{u \in K_M : \langle A(w,v), u - g(v) \rangle \subseteq -C(v),\ \forall v \in K_M\}, \tag{2.18}$$

which is a convex set. By condition (iii) via Remark 1.4, T is closed. By (iv), we have $T(K_M) = \bigcup_{w \in K_M} T(w) \subseteq B_M$. Hence, Lemma 1.5(a) implies that T is upper semicontinuous. Hence, T satisfies all the assumptions of Lemma 2.1 and so T has a fixed point $w_0 \in K_M$, that is,

$$\langle A(w_0, v), w_0 - g(v) \rangle \subseteq -C(v), \quad \forall v \in K_M. \tag{2.19}$$

Set $\mathcal{M} = \{M \subset X : M$ is a finite dimensional subspace with $K_M \neq \varnothing\}$ and for $M \in \mathcal{M}$,

$$W_M = \{u \in K_M : \langle A(u,v), u - g(v) \rangle \subseteq -C(v),\ \forall v \in K_M\}. \tag{2.20}$$

By (2.19) and conditions (iii) and (iv), W_M is a nonempty and closed subset of the compact set B_M and hence W_M is compact in K. Let $\{M_i\}_{i=1}^n$ be a finite subset of $\mathcal{M}$. From the definition of W_M, we have $W_{\cup_i M_i} \subset \bigcap_{i=1}^n W_{M_i}$ and so $\{W_M : M \in \mathcal{M}\}$ has the finite intersection property, so, there is $u \in \bigcap_{M \in \mathcal{M}} W_M$ (note, if $\bigcap_{M \in \mathcal{M}} W_M = \varnothing$, then $W_{M_0} \subseteq \bigcup_{M \neq M_0} K \setminus W_M$, where M_0 is an arbitrary element of $\mathcal{M}$, so the family $\{K \setminus W_M\}_{M \neq M_0}$ is an open covering for the compact set W_{M_0} and so there exist $M_{i_1}, \ldots, M_{i_n}$ such that $W_{M_0} \subseteq \bigcup_{j=1}^n K \setminus W_{i_j}$, which implies that $\bigcap_{j=1}^n W_{i_j} \bigcap W_{M_0} = \varnothing$, which is a contradiction).

We claim that

$$\langle A(u,u), v - g(u) \rangle \nsubseteq -C(u) \setminus \{0\}, \quad \forall v \in K. \tag{2.21}$$

Indeed, for each $v \in K$, there is $M_v \in \mathcal{M}$ such that $v \in K_M$. Hence, by $u \in W_{M_v}$ (note, $u \in \bigcap_{M \in \mathcal{M}} W_M$) and the definition of W_{M_v}, we have

$$\langle A(u,v), u - g(v) \rangle \subseteq -C(v), \tag{2.22}$$

and so since v was an arbitrary element of K, then (2.21) is true, for all $v \in K$. This completes the proof of claim. From (2.21) and Lemma 2.1, we have

$$\langle A(u,u), v - g(u) \rangle \not\subseteq -C(u) \setminus \{0\}, \quad \forall v \in K, \tag{2.23}$$

and so the proof of the theorem is complete. □

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