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Review Article

Exponential Polynomials, Stirling Numbers, and Evaluation of Some Gamma Integrals

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This article is a short elementary review of the exponential polynomials (also called single-variable Bell polynomials) from the point of view of analysis. Some new properties are included, and several analysis-related applications are mentioned. At the end of the paper one application is described in details—certain Fourier integrals involving $\Gamma(a+it)$ and $\Gamma(a+it)\Gamma(b-it)$ are evaluated in terms of Stirling numbers.

1. Introduction

We review the exponential polynomials $\phi_n(x)$ and present a list of properties for easy reference. Exponential polynomials in analysis appear, for instance, in the rule for computing derivatives like $(d/dt)^n e^{ae^t}$ and the related Mellin derivatives:

$$\left(x\frac{d}{dx}\right)^n f(x), \qquad \left(\frac{d}{dx}x\right)^n f(x). \tag{1.1}$$

Namely, we have

$$\left(\frac{d}{dt}\right)^n e^{ae^t} = \phi_n(ae^t)e^{ae^t}, \tag{1.2}$$

or, after the substitution $x = e^t$,

$$\left(x\frac{d}{dx}\right)^n e^{ax} = \phi_n(ax)e^{ax}. \tag{1.3}$$

We also include in this review two properties relating exponential polynomials to Bernoulli numbers, B_k. One is the semiorthogonality

$$\int_{-\infty}^{0} \phi_n(x)\phi_m(x)e^{2x}\frac{dx}{x} = (-1)^n \frac{2^{n+m}-1}{n+m} B_{n+m}, \tag{1.4}$$

where the right-hand side is zero if $n+m$ is odd. The other property is (2.25).

At the end we give one application. Using exponential polynomials we evaluate the integrals

$$\begin{gathered} \int_{\mathbb{R}} e^{-it\lambda} t^n \Gamma(a+it)dt, \\ \int_{\mathbb{R}} e^{-it\lambda} t^n \Gamma(a+it)\Gamma(b-it)dt \end{gathered} \tag{1.5}$$

for $n = 0, 1, \ldots$, in terms of Stirling numbers.

2. Exponential Polynomials

The evaluation of the series

$$S_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad n = 0, 1, 2, \ldots \tag{2.1}$$

has a long and interesting history. Clearly, $S_0 = e$, with the agreement that $0^0 = 1$. Several reference books (e.g., [1]) provide the following numbers:

$$\begin{gathered} S_1 = e, \qquad S_2 = 2e, \qquad S_3 = 5e, \qquad S_4 = 15e, \qquad S_5 = 52e, \\ S_6 = 203e, \qquad S_7 = 877e, \qquad S_8 = 4140e. \end{gathered} \tag{2.2}$$

As noted by Gould in [2, page 93], the problem of evaluating S_n appeared in the Russian journal *Matematicheskii Sbornik*, 3 (1868), page 62, with solution ibid, 4 (1868-9), page 39. Evaluations are presented also in two papers by Dobinski and Ligowski. In 1877 Dobinski [3] evaluated the first eight series $S_1, \ldots, S_8$ by regrouping

$$\begin{aligned} S_1 &= \sum_{1}^{\infty} \frac{k}{k!} = 1 + \frac{2}{2!} + \frac{3}{3!} + \cdots = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots = e, \\ S_2 &= \sum_{1}^{\infty} \frac{k^2}{k!} = 1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \cdots = 1 + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \cdots \\ &= \left\{ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \right\} + \left\{ \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \cdots \right\} = e + S_1 = 2e, \end{aligned} \tag{2.3}$$

and continuing like that to S_8. For large n this method is not convenient. However, later that year Ligowski [4] suggested a better method, providing a generating function for the numbers S_n:

$$e^{e^z} = \sum_{k=0}^{\infty} \frac{e^{kz}}{k!} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{k^n}{k!} \frac{z^n}{n!} = \sum_{n=0}^{\infty} S_n \frac{z^n}{n!}. \tag{2.4}$$

Further, an effective iteration formula was found

$$S_n = \sum_{j=0}^{n-1} \binom{n-1}{j} S_j \tag{2.5}$$

by which every S_n can be evaluated starting from S_1.

These results were preceded, however, by the work [5] of Grunert (1797–1872), professor at Greifswalde. Among other things, Grunert obtained formula (2.9) from which the evaluation of (2.1) follows immediately.

The structure of the series S_n hints at the exponential function. Differentiating the expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{2.6}$$

and multiplying both sides by x we get

$$xe^x = \sum_{k=0}^{\infty} \frac{kx^k}{k!}, \tag{2.7}$$

which, for $x = 1$, gives $S_1 = e$. Repeating the procedure, we find $S_2 = 2e$ from

$$x(xe^x)' = \left(x + x^2\right)e^x = \sum_{k=0}^{\infty} \frac{k^2 x^k}{k!}, \tag{2.8}$$

and continuing like that, for every $n = 0, 1, 2, \ldots,$ we find the relation

$$\left(x\frac{d}{dx}\right)^n e^x = \phi_n(x)e^x = \sum_{k=0}^{\infty} \frac{k^n x^k}{k!}, \tag{2.9}$$

where ϕ_n are polynomials of degree n. Thus,

$$S_n = \phi_n(1)e, \quad \forall n \geq 0. \tag{2.10}$$

The polynomials ϕ_n deserve a closer look. From the defining relation (2.9) we obtain

$$x(\phi_n e^x)' = x(\phi_n' + \phi_n)e^x = \phi_{n+1}e^x, \tag{2.11}$$

that is,

$$\phi_{n+1} = x\left(\phi_n' + \phi_n\right), \tag{2.12}$$

which helps to find ϕ_n explicitly starting from ϕ_0:

$$\begin{aligned}
\phi_0(x) &= 1, \\
\phi_1(x) &= x, \\
\phi_2(x) &= x^2 + x, \\
\phi_3(x) &= x^3 + 3x^2 + x, \\
\phi_4(x) &= x^4 + 6x^3 + 7x^2 + x, \\
\phi_5(x) &= x^5 + 10x^4 + 25x^3 + 15x^2 + x,
\end{aligned} \tag{2.13}$$

and so on. Another interesting relation, easily proved by induction, is

$$\phi_{n+1}(x) = x\sum_{k=0}^{n}\binom{n}{k}\phi_k(x). \tag{2.14}$$

From (2.12) and (2.14) one finds immediately

$$\phi_n'(x) = \sum_{k=0}^{n-1}\binom{n}{k}\phi_k(x). \tag{2.15}$$

Obviously, $x = 0$ is a zero for all ϕ_n, $n > 0$. It can be seen that all the zeros of ϕ_n are real, simple, and nonpositive. The nice and short induction argument belongs to Harper [6].

The assertion is true for $n = 1$. Suppose that for some n the polynomial ϕ_n has n distinct real nonpositive zeros (including $x = 0$). Then the same is true for the function

$$f_n(x) = \phi_n(x)e^x. \tag{2.16}$$

Moreover, f_n is zero at $-\infty$ and by Rolle's theorem its derivative

$$\frac{d}{dx}f_n = \frac{d}{dx}\left(\phi_n(x)e^x\right) \tag{2.17}$$

has n distinct real negative zeros. It follows that the function

$$\phi_{n+1}(x)e^x = x\frac{d}{dx}\left(\phi_n(x)e^x\right) \tag{2.18}$$

has $n+1$ distinct real nonpositive zeros (adding here $x = 0$).

The polynomials ϕ_n can be defined also by the exponential generating function (extending Ligowski's formula)

$$e^{x(e^z-1)} = \sum_{n=0}^{\infty} \phi_n(x) \frac{z^n}{n!}. \tag{2.19}$$

It is not obvious, however, that the polynomials defined by (2.9) and (2.19) are the same, so we need the following simple statement.

Proposition 2.1. *The polynomials $\phi_n(x)$ defined by* (2.9) *are exactly the partial derivatives $(\partial/\partial z)^n e^{x(e^z-1)}$ evaluated at $z = 0$.*

Equation (2.19) follows from (2.9) after expanding the exponential e^{xe^z} in double series and changing the order of summation. A different proof will be given later.

Setting $z = 2k\pi i$, $k = \pm 1, \pm 2, \ldots$, in the generating function (2.19) one finds

$$e^{2k\pi i} = 1, \qquad e^{x(e^z-1)} = e^0 = 1, \tag{2.20}$$

which shows that the exponential polynomials are linearly dependent:

$$1 = \sum_{n=0}^{\infty} \phi_n(x) \frac{(2k\pi i)^n}{n!} \quad \text{or} \quad 0 = \sum_{n=1}^{\infty} \phi_n(x) \frac{(2k\pi i)^n}{n!}, \quad k = \pm 1, \pm 2, \ldots. \tag{2.21}$$

In particular, ϕ_n are not orthogonal for any scalar product on polynomials. (However, they have the semiorthogonality property mentioned in Section 1 and proved in Section 4.)

Comparing coefficient for z in the equation

$$e^{(x+y)e^z} = e^{xe^z} e^{ye^z} \tag{2.22}$$

yields the binomial identity

$$\phi_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \phi_k(x) \phi_{n-k}(y). \tag{2.23}$$

With $y = -x$ this implies the interesting "orthogonality" relation for $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k} \phi_k(x) \phi_{n-k}(-x) = 0. \tag{2.24}$$

Next, let B_n, $n = 0, 1, \ldots$, be the Bernoulli numbers. Then for $p = 0, 1, \ldots$, we have

$$\int_0^x \phi_p(t)dt = \frac{1}{p+1}\sum_{k=1}^{p+1}\binom{p+1}{k}B_{p+1-k}\phi_k(x). \tag{2.25}$$

For proof see Example 4 in [7, page 51], or [9].

Some Historical Notes

As already mentioned, formula (2.9) appears in the work of Grunert [5, page 260], where he gives also the representation (3.4) and computes explicitly the first six exponential polynomials. The polynomials ϕ_n were studied more systematically (and independently) by S. Ramanujan in his unpublished notebooks. Ramanujan's work is presented and discussed by Berndt in [7, Part 1, Chapter 3]. Ramanujan, for example, obtained (2.19) from (2.9) and also proved (2.14), (2.15), and (2.25). Later, these polynomials were studied by Bell [10] and Touchard [11, 12]. Both Bell and Touchard called them "exponential" polynomials, because of their relation to the exponential function, for example, (1.2), (1.3), (2.9), and (2.19). This name was used also by Rota [13]. As a matter of fact, Bell introduced in [10] a more general class of polynomials of many variables, $Y_{n,k}$, including ϕ_n as a particular case. For this reason ϕ_n are known also as the single-variable Bell polynomials [14–17]. These polynomials are also a special case of the actuarial polynomials introduced by Toscano [18] which, on their part, belong to the more general class of Sheffer polynomials [19]. The exponential polynomials appear in a number of papers and in different applications—see [9, 13, 20–24] and the references therein. In [25] they appear on page 524 as the horizontal generating functions of the Stirling numbers of the second kind (see (3.4)).

The numbers

$$b_n = \phi_n(1) = \frac{1}{e}S_n \tag{2.26}$$

are sometimes called exponential numbers, but a more established name is Bell numbers. They have interesting combinatorial and analytical applications [15, 16, 18, 26–32]. An extensive list of 202 references for Bell numbers is given in [33].

We note that (2.9) can be used to extend ϕ_n to ϕ_z for any complex number z by the formula

$$\phi_z(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k^z x^k}{k!} \tag{2.27}$$

(Butzer et al. [34, 35]). The function appearing here is an interesting entire function in both variables, x and z. Another possibility is to study the polyexponential function

$$e_s(x,\lambda) = \sum_{n=0}^{\infty}\frac{x^n}{n!(n+\lambda)^s}, \tag{2.28}$$

where $\operatorname{Re}\lambda > 0$. When s is a negative integer, the polyexponential can be written as a finite linear combination of exponential polynomials (see [9]).

3. Stirling Numbers and Mellin Derivatives

The iteration formula (2.12) shows that all polynomials ϕ_n have positive integer coefficients. These coefficients are the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ (or $S(n,k)$)—see [25, 28, 36–39]. Given a set of n elements, $\left\{ {n \atop k} \right\}$ represents the number of ways by which this set can be partitioned into k nonempty subsets ($0 \le k \le n$). Obviously, $\left\{ {n \atop 1} \right\} = 1$, $\left\{ {n \atop n} \right\} = 1$ and a short computation gives $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$. For symmetry one sets $\left\{ {0 \atop 0} \right\} = 1$, $\left\{ {n \atop 0} \right\} = 0$. The definition of $\left\{ {n \atop k} \right\}$ implies the property

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \tag{3.1}$$

(see [38, page 259]) which helps to compute all $\left\{ {n \atop k} \right\}$ by iteration. For instance,

$$\left\{ {n \atop 3} \right\} = \frac{3^{n-1} - 2^n + 1}{2}. \tag{3.2}$$

A general formula for the Stirling numbers of the second kind is

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n. \tag{3.3}$$

Proposition 3.1. *For every $n = 0, 1, 2, \ldots$*

$$\phi_n(x) = \left\{ {n \atop 0} \right\} + \left\{ {n \atop 1} \right\} x + \left\{ {n \atop 2} \right\} x^2 + \cdots + \left\{ {n \atop n} \right\} x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k. \tag{3.4}$$

The proof is by induction and is left to the reader. Setting here $x = 1$ we come to the well-known representation for the numbers S_n

$$S_n = e\left(\left\{ {n \atop 0} \right\} + \left\{ {n \atop 1} \right\} + \left\{ {n \atop 2} \right\} + \cdots + \left\{ {n \atop n} \right\} \right). \tag{3.5}$$

It is interesting that formula (3.4) is very old—it was obtained by Grunert [5, page 260] together with the representation (3.3) for the coefficients which are called now Stirling numbers of the second kind. In fact, coefficients of the form

$$\left\{ {n \atop k} \right\} k! \tag{3.6}$$

appear in the computations of Euler—see [37].

Next we turn to some special differentiation formulas. Let $D = d/dx$.

Mellin Derivatives

It is easy to see that the first equality in (2.9) extends to (1.3), where a is an arbitrary complex number, that is,

$$(xD)^n e^{ax} = \phi_n(ax)e^{ax} \tag{3.7}$$

by the substitution $x \to ax$. Even further, this extends to

$$(xD)^n e^{ax^p} = p^n \phi_n(ax^p)e^{ax^p} \tag{3.8}$$

for any a, p and $n = 0, 1, \dots$ (simple induction and (2.12)). Again by induction, it is easy to prove that

$$(xD)^n f(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k D^k f(x) \tag{3.9}$$

for any n-times differentiable function f. This formula was obtained by Grunert [5, pages 257-258] (see also [2, page 89], where a proof by induction is given).

As we know the action of xD on exponentials, formula (3.9) can be "discovered" by using Fourier transform. Let

$$F[f](t) = \int_{\mathbb{R}} e^{-ixt} f(x)dx \tag{3.10}$$

be the Fourier transform of some function f. Then

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt} F[f](t)dx, \\ (xD)^n f(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt}\phi_n(ixt)F[f](t)dx \\ &= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k F^{-1}\Big[(it)^k F[f]\Big](x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k D^k f(x). \end{aligned} \tag{3.11}$$

Next we turn to formula (1.2) and explain its relation to (1.3). If we set $x = e^t$, then for any differentiable function f

$$\frac{d}{dt} f = \left(\frac{d}{dx} f\right)\frac{dx}{dt} = \left(\frac{d}{dx} f\right)e^t = (xD)f, \tag{3.12}$$

and we see that (1.2) and (1.3) are equivalent:

$$\left(\frac{d}{dt}\right)^n e^{ae^t} = (xD)^n e^{ax} = \phi_n(ax)e^{ax} = \phi_n(ae^t)e^{ae^t}. \tag{3.13}$$

Proof of Proposition 2.1. We apply (1.2) to the function $f_x(z) = e^{x(e^z-1)} = e^{xe^z}e^{-x}$:

$$\left(\frac{d}{dz}\right)^n f_x(z) = \phi_n(xe^z)f_x(z). \tag{3.14}$$

From here, with $z = 0$

$$\left(\frac{d}{dz}\right)^n f_x(z)\Big|_{z=0} = \phi_n(x) \tag{3.15}$$

as needed. □

Now we list some simple operational formulas. Starting from the obvious relation

$$(xD)^n x^k = k^n x^k, \quad n = 0, 1, \ldots, \ k \in \mathbb{R} \tag{3.16}$$

for any function of the form

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \tag{3.17}$$

we define the differential operator

$$f(xD) = \sum_{n=0}^{\infty} a_n (xD)^n \tag{3.18}$$

with action on functions $g(x)$:

$$f(xD)g(x) = \sum_{n=0}^{\infty} a_n (xD)^n g(x). \tag{3.19}$$

When $g(x) = x^k$, (3.16) and (3.19) show that

$$f(xD)x^k = \sum_{n=0}^{\infty} a_n k^n x^k = f(k)x^k. \tag{3.20}$$

If now

$$g(x) = \sum_{k=0}^{\infty} c_k x^k \tag{3.21}$$

is a function analytical in a neighborhood of zero, the action of $f(xD)$ on this function is given by

$$f(xD)g(x) = \sum_{k=0}^{\infty} c_k f(k) x^k, \tag{3.22}$$

provided that the series on the right side converges. When f is a polynomial, formula (3.22) helps to evaluate series like

$$\sum_{k=0}^{\infty} c_k f(k) x^k \tag{3.23}$$

in a closed form. This idea was exploited by Schwatt [40] and more recently by the present author in [20]. For instance, when $g(x) = e^x$, (3.22) becomes

$$\sum_{k=0}^{\infty} f(k) \frac{x^k}{k!} = e^x \sum_{n=0}^{\infty} a_n \phi_n(x). \tag{3.24}$$

As shown in [20] this series transformation can be used for asymptotic series expansions of certain functions.

Leibniz Rule

The higher-order Mellin derivative $(xD)^n$ satisfies the Leibniz rule

$$(xD)^n (fg) = \sum_{k=0}^{n} \binom{n}{k} \left[(xD)^{n-k} f \right] \left[(xD)^k g \right]. \tag{3.25}$$

The proof is easy, by induction, and is left to the reader. We shall use this rule to prove the following proposition.

Proposition 3.2. *For all $n, m = 0, 1, 2, \ldots$*

$$\phi_{n+m}(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \left\{ {m \atop j} \right\} j^{n-k} x^j \phi_k(x). \tag{3.26}$$

Proof. One has

$$\phi_{n+m}(x) = (xD)^{n+m} e^x = (xD)^n (xD)^m e^x = (xD)^n \left(\phi_m(x) e^x \right), \tag{3.27}$$

which by the Leibniz rule (3.25) equals

$$\sum_{k=0}^{n} \binom{n}{k} \left[(xD)^{n-k} \phi_m(x) \right] \left[(xD)^k e^x \right]. \tag{3.28}$$

Using (3.3) and (3.16) we write

$$(xD)^{n-k}\phi_m(x) = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} j^{n-k} x^j, \tag{3.29}$$

and since also

$$(xD)^k e^x = \phi_k(x) e^x, \tag{3.30}$$

we obtain (3.26) from (3.27). The proof is completed. □

Setting $x = 1$ in (3.25) yields an identity for the Bell numbers:

$$b_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \left\{ {m \atop j} \right\} j^{n-k} b_k. \tag{3.31}$$

This identity was recently published by Spivey [32], who gave a combinatorial proof. After that Gould and Quaintance [16] obtained the generalization (3.26) together with two equivalent versions. The proof in [16] is different from the one above.

Using the Leibniz rule for xD we can prove also the following extension of property (2.24).

Proposition 3.3. *For any two integers $n, m \geq 0$*

$$(xD)^n \phi_m(x) = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} k^n x^k = \sum_{k=0}^{n} \binom{n}{k} \phi_{m+k}(x) \phi_{n-k}(-x). \tag{3.32}$$

The proof is simple. Just compute

$$\begin{aligned} (xD)^n \phi_m(x) &= (xD)^n \left[\left(e^{-x}\right) \left(\phi_m(x) e^x\right) \right] \\ &= \sum_{k=0}^{n} \binom{n}{k} \left[(xD)^{n-k} e^{-x} \right] \left[(xD)^k \left(\phi_m(x) e^x\right) \right] \end{aligned} \tag{3.33}$$

and (3.32) follows from (1.3).

For completeness we mention also the following three properties involving the operator Dx. Proofs and details are left to the reader:

$$(Dx)^n e^{ax} = \frac{\phi_{n+1}(ax)}{ax} e^{ax},$$
$$(Dx)^n f(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} x^k D^k f(x), \tag{3.34}$$
$$f(Dx) g(x) = \sum_{k=0}^{\infty} c_k f(k+1) x^k,$$

analogous to (1.3), (3.9), and (3.22) correspondingly.

For a comprehensive study of the Mellin derivative we refer to [41–43].

More Stirling Numbers

The polynomials ϕ_n, $n = 0, 1, \ldots$, form a basis in the linear space of all polynomials. Formula (3.4) shows how this basis is expressed in terms of the standard basis $1, x, x^2, \ldots, x^n, \ldots$. We can solve for x^k in (3.4) and express the standard basis in terms of the exponential polynomials

$$\begin{aligned} 1 &= \phi_0, \\ x &= \phi_1, \\ x^2 &= -\phi_1 + \phi_2, \\ x^3 &= 2\phi_1 - 3\phi_2 + \phi_3, \\ x^4 &= -6\phi_1 + 11\ \phi_2 - 6\phi_3 + \phi_4, \end{aligned} \tag{3.35}$$

and so forth. The coefficients here are also special numbers. If we write

$$x^n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \phi_k, \tag{3.36}$$

then $\begin{bmatrix} n \\ k \end{bmatrix}$ are the (absolute) Stirling numbers of first kind, as defined in [38]. (The numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ are nonnegative. The symbol $s(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$ is used for Stirling numbers of the first kind with changing sign—see [28, 33, 39] for more details.) $\begin{bmatrix} n \\ k \end{bmatrix}$ is the number of ways to arrange n objects into k cycles. According to this interpretation,

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad n \geq 1. \tag{3.37}$$

4. Semiorthogonality of ϕ_n

Proposition 4.1. *For every $n, m = 1, 2, \ldots$, one has*

$$\int_0^\infty \phi_n(-x)\phi_m(-x)e^{-2x}\frac{dx}{x} = (-1)^{n-1}\frac{2^{n+m}-1}{n+m}B_{n+m}. \tag{4.1}$$

Here B_k are the Bernoulli numbers. Note that the right-hand side is zero when $k+m$ is odd, as all Bernoulli numbers with odd indices > 1 are zeros.

Using the representation (3.4) in (4.1) and integrating termwise one obtains an equivalent form of (4.1):

$$\sum_{k=0}^{n}\sum_{j=0}^{m}(-1)^{k+j}\left\{\begin{matrix} n \\ k \end{matrix}\right\}\left\{\begin{matrix} m \\ j \end{matrix}\right\}\frac{(k+j-1)!}{2^{k+j}} = (-1)^{n-1}\frac{2^{n+m}-1}{n+m}B_{n+m}. \tag{4.2}$$

This (double sum) identity extends the known identity [38, page 317, Problem 6.76]

$$\sum_{j=0}^{m}(-1)^{j+1}\left\{\begin{matrix} m \\ j \end{matrix}\right\}\frac{j!}{2^{j+1}} = \frac{2^{m+1}-1}{m+1}B_{m+1}. \tag{4.3}$$

Namely, (4.3) results from (4.2) for $n = 1$. The presence of $(-1)^{n-1}$ at the right-hand side in (4.1) is not a "break of symmetry," because when $n+m$ is even, then n and m are both even or both odd.

Proof of the proposition. Starting from

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx, \tag{4.4}$$

we set $x = e^\lambda$, $z = a + it$, to obtain the representation

$$\Gamma(a+it) = \int_{-\infty}^{+\infty} e^{i\lambda t}e^{a\lambda}e^{-e^\lambda}d\lambda, \tag{4.5}$$

which is a Fourier transform integral. The inverse transform is

$$e^{a\lambda}e^{-e^\lambda} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t}\Gamma(a+it)dt. \tag{4.6}$$

When $a = 1$, this is

$$-e^\lambda e^{-e^\lambda} = \frac{d}{d\lambda}e^{-e^\lambda} = \frac{-1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t}\Gamma(1+it)dt. \tag{4.7}$$

Differentiating (4.7) $n-1$ times for λ we find

$$\left(\frac{d}{d\lambda}\right)^n e^{-e^\lambda} = \phi_n\left(-e^\lambda\right)e^{-e^\lambda} = \frac{-1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t}(-it)^{n-1}\Gamma(1+it)dt, \tag{4.8}$$

and Parceval's formula yields the equation

$$\int_{\mathbb{R}} \phi_n\left(-e^\lambda\right)\phi_m\left(-e^\lambda\right)e^{-2e^\lambda}d\lambda = \frac{1}{2\pi}\int_{\mathbb{R}}(-it)^{n-1}(it)^{m-1}|\Gamma(1+it)|^2 dt, \tag{4.9}$$

or, with $x = e^\lambda$

$$\int_0^\infty \phi_n(-x)\phi_m(-x)e^{-2x}\frac{dx}{x} = \frac{(-1)^n i^{n+m}}{2\pi}\int_{\mathbb{R}} t^{n+m-2}\frac{\pi t}{\sinh(\pi t)}it. \tag{4.10}$$

The right-hand side is 0 when $n+m$ is odd. When $n+m$ is even, we use the integral [1, page 351]

$$\int_0^\infty \frac{t^{2p-1}}{\sinh(\pi t)}dt = \frac{2^{2p}-1}{2p}(-1)^{p=1}B_{2p} \tag{4.11}$$

to finish the proof. □

Property (4.1) resembles the semiorthogonal property of the Bernoulli polynomials

$$\int_0^1 B_n(x)B_m(x)dx = (-1)^{n-1}\frac{n!m!}{(n+m)!}B_{n+m}, \tag{4.12}$$

see, for instance, [25, page 530].

5. Gamma Integrals

We use the technique in the previous section to compute certain Fourier integrals and evaluate the moments of $\Gamma(a+it)$ and $\Gamma(a+it)\Gamma(b-it)$.

Proposition 5.1. *For every $n = 0, 1, \ldots$ and $a, b > 0$ one has*

$$\begin{aligned}\int_{\mathbb{R}} & e^{-i\mu t} t^n \Gamma(a+it)\Gamma(b-it)\,dt \\ &= i^n 2\pi e^{-b\mu} \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \left\{ {k \atop m} \right\} (-1)^m a^{n-k} \frac{\Gamma(a+b+m)}{(1+e^{-\mu})^{a+b+m}},\end{aligned} \tag{5.1}$$

$$\begin{aligned}\int_{\mathbb{R}} & e^{-i\lambda t} t^n \Gamma(a+it)\,dt \\ &= i^n 2\pi e^{a\lambda} e^{-e^{\lambda}} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \sum_{m=0}^{k} \left\{ {k \atop m} \right\} (-1)^m e^{\lambda m}.\end{aligned} \tag{5.2}$$

In particular, when $\lambda = \mu = 0$, one obtains the moments

$$\begin{aligned}G_n(a,b) &\equiv \int_{\mathbb{R}} t^n \Gamma(a+it)\Gamma(b-it)\,dt \\ &= i^n \pi \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \left\{ {k \atop m} \right\} (-1)^m a^{n-k} \frac{\Gamma(a+b+m)}{2^{a+b+m-1}},\end{aligned} \tag{5.3}$$

$$G_n(a) \equiv \int_{\mathbb{R}} t^n \Gamma(a+it)\,dt = \frac{2\pi i^n}{e} \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \left\{ {k \atop m} \right\} (-1)^m a^{n-k}. \tag{5.4}$$

When $n = 0$ in (5.1) *one has the known integral*

$$\int_{\mathbb{R}} e^{-i\mu t} \Gamma(a+it)\Gamma(b-it)\,dt = 2\pi \Gamma(a+b) e^{-b\mu} \left(1+e^{-\mu}\right)^{-a-b}, \tag{5.5}$$

which can be found in the form of an inverse Mellin transform in [44].

Proof. Using again (4.6)

$$e^{a\lambda} e^{-e^{\lambda}} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda t} \Gamma(a+it)\,dt, \tag{5.6}$$

we differentiate both side n times

$$\left(\frac{d}{d\lambda}\right)^n \left[e^{a\lambda} e^{-e^{\lambda}}\right] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda t} (-it)^n \Gamma(a+it)\,dt, \tag{5.7}$$

and then, according to the Leibniz rule and (1.2) the left-hand side becomes

$$\left(\frac{d}{d\lambda}\right)^n \left[e^{a\lambda} e^{-e^{\lambda}}\right] = e^{a\lambda} e^{-e^{\lambda}} \sum_{k=0}^{n} \binom{n}{k} \phi_k\left(-e^{\lambda}\right) a^{n-k}. \tag{5.8}$$

Therefore,

$$e^{a\lambda}e^{-e^{\lambda}}\sum_{k=0}^{n}\binom{n}{k}\phi_k\left(-e^{\lambda}\right)a^{n-k}=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-i\lambda t}(-it)^n\Gamma(a+it)dt, \tag{5.9}$$

and (5.2) follows from here.

Replacing λ by $\lambda-\mu$ we write (5.6) in the form

$$e^{b\lambda}e^{-b\mu}e^{-e^{\lambda}e^{-\mu}}=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-i\lambda t}e^{i\mu t}\Gamma(b+it)dt, \tag{5.10}$$

and then Parceval's formula for Fourier integrals applied to (5.9) and (5.10) yields

$$\begin{aligned}e^{-b\mu}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}\int_{\mathbb{R}}e^{(a+b)\lambda}e^{-e^{\lambda}(1+e^{-\mu})}\phi_k\left(-e^{\lambda}\right)d\lambda\\ =\frac{(-i)^n}{2\pi}\int_{\mathbb{R}}e^{-i\mu t}t^n\Gamma(a+it)\Gamma(b-it)dt.\end{aligned} \tag{5.11}$$

Returning to the variable $x=e^{\lambda}$ we write this in the form

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{R}}e^{-i\mu t}t^n\Gamma(a+it)\Gamma(b-it)dt\\ &\quad=i^ne^{-b\mu}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}\int_0^{\infty}\phi_k(-x)x^{a+b-1}e^{-x(1+e^{-\mu})}dx\\ &\quad=i^ne^{-b\mu}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\left\{{k \atop j}\right\}a^{n-k}(-1)^j\int_0^{\infty}x^{a+b+j-1}e^{-x(1+e^{-\mu})}dx\\ &\quad=i^ne^{-b\mu}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\left\{{k \atop j}\right\}a^{n-k}(-1)^j\frac{\Gamma(a+b+j)}{(1+e^{-\mu})^{a+b+j}},\end{aligned} \tag{5.12}$$

which is (5.1). The proof is complete. □

Next, we observe that for any polynomial

$$p(t)=\sum_{n=0}^{m}a_nt^n \tag{5.13}$$

one can use (5.4) to write the following evaluation:

$$\int_{\mathbb{R}}p(t)\Gamma(a+it)dt=\sum_{n=0}^{m}a_nG_n(a). \tag{5.14}$$

In particular, when $a = 1$ we have

$$G_n(1) = 2\pi i^n e^{-1} \phi_{n+1}(-1), \tag{5.15}$$

and therefore,

$$\int_{\mathbb{R}} p(t)\Gamma(1+it)dt = \frac{2\pi}{e} \sum_{n=0}^{m} a_n i^n \phi_{n+1}(-1). \tag{5.16}$$

More applications can be found in the recent papers [9, 20, 21].

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