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Research Article

Monotone Hybrid Projection Algorithms for an Infinitely Countable Family of Lipschitz Generalized Asymptotically Quasi-Nonexpansive Mappings

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We prove a weak convergence theorem of the modified Mann iteration process for a uniformly Lipschitzian and generalized asymptotically quasi-nonexpansive mapping in a uniformly convex Banach space. We also introduce two kinds of new monotone hybrid methods and obtain strong convergence theorems for an infinitely countable family of uniformly Lipschitzian and generalized asymptotically quasi-nonexpansive mappings in a Hilbert space. The results improve and extend the corresponding ones announced by Kim and Xu (2006) and Nakajo and Takahashi (2003).

1. Introduction

Let C be a nonempty, closed, and convex subset of a real Banach space E. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. Then T is said to be

(i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(ii) *asymptotically nonexpansive* if there exists a sequence $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and

$$\left\|T^n x - T^n y\right\| \leq k_n \left\|x - y\right\| \tag{1.1}$$

for all $x, y \in C$ and $n \geq 1$;

(iii) *asymptotically quasi-nonexpansive* if there exists a sequence $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \leq k_n \|x - p\| \tag{1.2}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

(iv) *generalized asymptotically nonexpansive* [1] if there exist nonnegative real sequences $\{k_n\}$ and $\{c_n\}$ with $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} c_n = 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| + c_n \tag{1.3}$$

for all $x, y \in C$ and $n \geq 1$;

(v) *generalized asymptotically quasi-nonexpansive* [1] if there exist nonnegative real sequences $\{k_n\}$ and $\{c_n\}$ with $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} c_n = 0$ such that

$$\|T^n x - p\| \leq k_n \|x - p\| + c_n \tag{1.4}$$

for all $x \in C, p \in F(T)$ and $n \geq 1$;

(vi) *asymptotically nonexpansive in the weak sense* [2] if

$$\limsup_{n\to\infty} \sup_{y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \leq 0 \tag{1.5}$$

for all $x \in C$.

(vii) *asymptotically nonexpansive in the intermediate sense* [3] if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \leq 0; \tag{1.6}$$

(viii) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \tag{1.7}$$

for all $x, y \in C$ and $n \geq 1$.

It is clear that a generalized asymptotically quasi-nonexpansive mapping is to unify various classes of mappings associated with the class of generalized asymptotically nonexpansive mapping, asymptotically nonexpansive mappings, and nonexpansive mappings. However, the converse of each of above statement may be not true. The example shows that a generalized asymptotically quasi-nonexpansive mapping is not an asymptotically quasi-nonexpansive mapping; see [1]. Note that if T is asymptotically nonexpansive in the weak sense, we have that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| + c_n \tag{1.8}$$

for all $x, y \in C$, where $c_n = \max\{0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|)\}$ so that $\lim_{n\to\infty} c_n = 0$. Hence, T is a generalized asymptotically nonexpansive mapping.

The mapping $T : C \to C$ is said to be *demiclosed* at 0 if for each sequence $\{x_n\}$ in C converging weakly to x and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$.

A Banach space E is said to satisfy *Opial's property*, see [4], if for each $x \in E$ and each sequence $\{x_n\}$ weakly convergent to x, the following condition holds for all $x \neq y$:

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|. \tag{1.9}$$

Let τ be a Hausdorff linear topology and let E satisfy the uniform τ-Opial property. In 1993, Bruck, Kuczumow, and Reich proved that $\{T^n x\}$ is τ-convergent if and only if $\{T^n x\}$ is τ-asymptotically regular, that is,

$$T^{n+1}x - T^n x \xrightarrow{\tau} 0. \tag{1.10}$$

Moreover, they also proved that the τ-limit of $\{T^n x\}$ is a fixed point of T.

In 1953, Mann [5] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \tag{1.11}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

However, we note that Mann's iteration process (1.11) has only weak convergence, in general; for instance, see [6–8].

In 2003, Nakajo and Takahashi [9] proposed the following modification of the Mann iteration for a single nonexpansive mapping T in a Hilbert space. They proved the following theorem.

Theorem 1.1. *Let C be a closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the following algorithm:*

$$\begin{gathered} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_0. \end{gathered} \tag{1.12}$$

Then $\{x_n\}$ defined by (1.12) *converges strongly to $P_{F(T)}x_0$.*

Recently, Kim and Xu [10] extended the result of Nakajo and Takahashi [9] from nonexpansive mappings to asymptotically nonexpansive mappings. They proved the following theorem.

Theorem 1.2. *Let C be a nonempty, bounded, closed, and convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty}\alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$
\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily}, \\
y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\
C_n = \left\{ z \in C : \left\| y_n - z \right\|^2 \leq \|x_n - z\|^2 + \theta_n \right\}, \\
Q_n = \{ z \in C : \langle x_0 - x_n, x_n - z \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{gathered}
\tag{1.13}
$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0$, as $n \to \infty$. Then $\{x_n\}$ defined by (1.13) converges strongly to $P_{F(T)}x_0$.

Since 2003, the strong convergence problems of the CQ method for fixed point iteration processes in a Hilbert space or a Banach space have been studied by many authors; see [9–20].

Let $\{T_i\}_{i=1}^{\infty}$ be an infinitely family of uniformly L_i-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings and let $F := \bigcap_{i=1}^{\infty} F(T_i)$. In this paper, motivated by Kim and Xu [10] and Nakajo and Takahashi [9], we introduce two kinds of new algorithms for finding a common fixed point of a countable family of uniformly Lipschitzian and generalized asymptotically quasi-nonexpansive mappings which are modifications of the normal Mann iterative scheme. Our iterative schemes are defined as follows.

Algorithm 1.3. *For an initial point $x_0 \in C$, compute the sequence $\{x_n\}$ by the iterative process:*

$$
\begin{gathered}
y_{i,n} = \alpha_{i,n} x_n + (1-\alpha_{i,n})T_i^n x_n, \\
C_{i,n} = \left\{ z \in C : \left\| y_{i,n} - z \right\|^2 \leq \|x_n - z\|^2 - \alpha_{i,n}(1-\alpha_{i,n})\left\| T_i^n x_n - x_n \right\|^2 + (1-\alpha_{i,n})\theta_{i,n} \right\}, \\
C_n = \bigcap_{i=1}^{\infty} C_{i,n}, \\
Q_0 = C, \\
Q_n = \{ z \in Q_{n-1} : \langle z - x_n, x_0 - x_n \rangle \leq 0 \}, \quad n \geq 1, \\
x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0,
\end{gathered}
\tag{1.14}
$$

where $\theta_{i,n} = (k_{i,n}^2 - 1)\nabla_n^2 + 2k_{i,n}c_{i,n}\nabla_n + c_{i,n}^2$, $\nabla_n = \sup_{n\in\mathbb{N}}\{\|x_n - z\| : z \in F\} < \infty$.

Algorithm 1.4. *For an initial point $x_0 \in C$, compute the sequence $\{x_n\}$ by the iterative process:*

$$
\begin{gathered}
C_{i,0} = C, \\
y_{i,n} = \alpha_{i,n} x_n + (1 - \alpha_{i,n}) T_i^n x_n, \\
C_{i,n+1} = \left\{ z \in C_{i,n} : \left\| y_{i,n} - z \right\|^2 \le \|x_n - z\|^2 - \alpha_{i,n}(1 - \alpha_{i,n}) \left\| T_i^n x_n - x_n \right\|^2 + (1 - \alpha_{i,n}) \theta_{i,n} \right\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \ge 0,
\end{gathered}
\tag{1.15}
$$

where $\theta_{i,n} = (k_{i,n}^2 - 1)\nabla_n^2 + 2k_{i,n} c_{i,n} \nabla_n + c_{i,n}^2$, $\nabla_n = \sup_{n \in \mathbb{N}} \{\|x_n - z\| : z \in F\} < \infty$.

2. Preliminaries

In this section, we present some useful lemmas which will be used in our main results.

Lemma 2.1 (see [21]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three sequences of nonnegative numbers such that $b_n \ge 1$ and*

$$
a_{n+1} \le b_n a_n + c_n, \quad \forall n \ge 1, \tag{2.1}
$$

if $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} (b_n - 1) < \infty$, then $\lim_{n \to \infty} a_n$ exists.

Lemma 2.2 (see [22]). *Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$
\left\| \lambda x + (1 - \lambda) y \right\|^p \le \lambda \|x\|^p + (1 - \lambda) \left\| y \right\|^p - \omega_p(\lambda) g\left(\left\| x - y \right\| \right) \tag{2.2}
$$

for all $x, y \in B_r(0) = \{x \in X : \|x\| \le r\}$ and $\lambda \in [0, 1]$ where $\omega_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

Lemma 2.3. *Let C be a nonempty subset of a Banach space E and let $T : C \to C$ be a uniformly L-Lipschitzian mapping. Let $\{x_n\}$ be a sequence in C such that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n \to \infty} \|x_n - T^n x_n\| = 0$. Then $\lim_{n \to \infty} \|x_n - T x_n\| = 0$.*

Proof. Since T is uniformly L-Lipschitzian, we have

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + \left\|T^{n+1}x_{n+1} - T^{n+1}x_n\right\| \\ &\quad + \left\|T^{n+1}x_n - Tx_n\right\| \\ &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + L\|x_n - x_{n+1}\| \\ &\quad + L\|T^n x_n - x_n\|.\end{aligned} \tag{2.3}$$

It follows that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. □

The following lemmas give some characterizations and useful properties of the metric projection P_C in a Hilbert space.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.4}$$

P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C.

Lemma 2.4 (see [12]). *Let C be a closed and convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if the following holds:*

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in C. \tag{2.5}$$

Lemma 2.5 (see [9]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $P_C : H \to C$ be the matric projection from H onto C. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \le \|x - y\|^2, \quad \forall x \in H,\ \forall y \in C. \tag{2.6}$$

Lemma 2.6 (see [12]). *Let H be a real Hilbert space. Then the following equations hold:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, *for all* $x, y \in H$;

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, *for all* $t \in [0,1]$ *and* $x, y \in H$.

Lemma 2.7 (see [10]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set*

$$\left\{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\right\} \tag{2.7}$$

is convex and closed.

Lemma 2.8. *Let C be a closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mapping with nonnegative real sequences $\{k_n\}$, $\{c_n\}$ such that $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} c_n = 0$. Then $F(T)$ is a closed and convex subset of C.*

Proof. Since T is continuous, $F(T)$ is closed. Next, we show that $F(T)$ is convex. Let $p_1, p_2 \in F(T)$ and $t \in (0,1)$. Put $p = tp_1 + (1-t)p_2$. By Lemma 2.6, we have

$$\begin{aligned}
\|p - T^n p\|^2 &= \|t(T^n p - p_1) + (1-t)(T^n p - p_2)\|^2 \\
&= t\|T^n p - p_1\|^2 + (1-t)\|T^n p - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\
&\leq t(k_n\|p - p_1\| + c_n)^2 + (1-t)(k_n\|p - p_2\| + c_n)^2 \\
&\quad - t(1-t)\|p_1 - p_2\|^2 \\
&= t(1-t)^2 k_n^2 \|p_1 - p_2\|^2 + (1-t)t^2 k_n^2 \|p_1 - p_2\|^2 \\
&\quad - t(1-t)\|p_1 - p_2\|^2 + 2tk_n c_n \|p - p_1\| + tc_n^2 \\
&\quad + 2(1-t)k_n c_n \|p - p_2\| + (1-t)c_n^2 \\
&= \left((1-t)t\left(k_n^2 - 1\right)\right)\|p_1 - p_2\|^2 + 2tk_n c_n \|p - p_1\| + tc_n^2 \\
&\quad + 2(1-t)k_n c_n \|p - p_2\| + (1-t)c_n^2.
\end{aligned} \tag{2.8}$$

This implies that $\lim_{n\to\infty}\|p - T^n p\| = 0$. Since T is continuous, we have $\lim_{n\to\infty}\|Tp - T^{n+1}p\| = 0$, so that $p = Tp$. Hence $p \in F(T)$. □

3. Main Results

First, we prove a weak convergence theorem for a single uniformly Lipschitzian and generalized asymptotically quasi-nonexpansive mapping in a uniformly convex Banach space.

Theorem 3.1. *Let E be a uniformly convex Banach space E which satisfies Opial's property. Let C be a nonempty, closed, and convex subset of E, and $T : C \to C$ a uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mapping with nonnegative real sequences $\{k_n\}$, $\{c_n\}$ such that $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} c_n = 0$. Assume that $I - T$ is demiclosed at 0, where I is the identity mapping of C and $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{x_n\}$ be the sequence in C generated by the modified Mann iteration process:*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Then $\{x_n\}$ converges weakly to a fixed point of $F(T)$.

Proof. Let $p \in F(T)$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|T^n x_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)k_n\|x_n - p\| + (1-\alpha_n)c_n \\ &\leq (1 + (k_n - 1)(1-\alpha_n))\|x_n - p\| + (1-\alpha_n)c_n. \end{aligned} \tag{3.2}$$

Since $\Sigma_{n=1}^{\infty}(k_n - 1) < \infty$, $\Sigma_{n=1}^{\infty} c_n < \infty$, then by Lemma 2.1 and (3.2), we obtain that

$$\lim_{n\to\infty} \|x_n - p\| \tag{3.3}$$

exists.

This implies that $\{T^n x_n - p\}$ is bounded. Put $r = \max\{\sup_{n\in N}\|T^n x_n - p\|, \sup_{n\in N}\|x_n - p\|\}$. By Lemma 2.2, there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(T^n x_n - p)\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T^n x_n - p\|^2 - \alpha_n(1-\alpha_n)g(\|T^n x_n - x_n\|) \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n^2\|x_n - p\|^2 \\ &\quad + 2(1-\alpha_n)k_n c_n\|x_n - p\| \\ &\quad + (1-\alpha_n)c_n^2 - \alpha_n(1-\alpha_n)g(\|T^n x_n - x_n\|). \end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned} \alpha_n(1-\alpha_n)g(\|T^n x_n - x_n\|) \leq{}& \left(k_n^2 - \alpha_n\left(k_n^2 - 1\right)\right)\|x_n - p\|^2 \\ &- \|x_{n+1} - p\|^2 \\ &+ 2(1-\alpha_n)k_n c_n\|x_n - p\| \\ &+ (1-\alpha_n)c_n^2. \end{aligned} \tag{3.5}$$

By our assumptions and (3.3), we get $\lim_{n\to\infty} g(\|T^n x_n - x_n\|) = 0$. Since g is continuous strictly increasing with $g(0) = 0$, we can conclude that $\lim_{n\to\infty}\|T^n x_n - x_n\| = 0$. Observe that $\|x_{n+1} - x_n\| \leq \alpha_n\|T^n x_n - x_n\| \to 0$. It follows from Lemma 2.3 that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $q_1 \in C$. From $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed at 0, we obtain that $Tq_1 = q_1$. That is, $q_1 \in F(T)$. Next, we show that $\{x_n\}$ converges weakly to q_1 and take another subsequence $\{x_{m_k}\}$ of $\{x_n\}$

converging weakly to some $q_2 \in C$. Again, as above, we can conclude that $q_2 \in F(T)$. Finally, we show that $q_1 = q_2$. Assume $q_1 \neq q_2$. Then by Opial's property of E, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - q_1\| &= \lim_{k\to\infty}\|x_{n_k} - q_1\| \\
&< \lim_{k\to\infty}\|x_{n_k} - q_2\| \\
&= \lim_{n\to\infty}\|x_n - q_2\| \\
&= \lim_{k\to\infty}\|x_{m_k} - q_2\| \\
&< \lim_{k\to\infty}\|x_{m_k} - q_1\| \\
&= \lim_{n\to\infty}\|x_n - q_1\|,
\end{aligned} \tag{3.6}$$

which is a contradiction. Therefore $q_1 = q_2$. This shows that $\{x_n\}$ converges weakly to $q_1 \in F(T)$. □

Remark 3.2. (1) In [3, Theorem 2], Bruck et al. proved that if $T : C \to C$ is asymptotically nonexpansive in the weak sense and T is τ-asymptotically regular, then Picard's iterated sequence $\{T^n x\}$ is τ-convergent to a fixed point of T. Since every asymptotically nonexpansive in the weak sense is a generalized asymptotically nonexpansive mapping and if its fixed point set is nonempty, then it is a generalized quasiasymptotically nonexpansive mapping. So we can apply Theorem 3.1 with a mapping T which is asymptotically nonexpansive in the weak sense when its fixed point set is nonempty and obtain that the sequence $\{x_n\}$ generated by the modified Mann iteration process

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad \forall n \in \mathbb{N} \tag{3.7}$$

converges weakly to a fixed point of T without asymptotically regularity condition of T.

(2) In [3, Theorem 4], Bruck et al. showed that if C is a bounded convex subset of a uniformly convex Banach space and C is sequentially τ-compact, $T : C \to C$ is asymptotically nonexpansive in the intermediate sense and $\sum_{i=1}^{\infty} c_{n_i} < \infty$ where $c_n = \max\{0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|)\}$ for some sequence of nonnegative integers $\{n_i\}$, then the sequence $\{x_i\}$ generated by

$$x_{i+1} = \alpha_i x_i + (1-\alpha_i)T^{n_i} x_i \tag{3.8}$$

is τ-convergent to a fixed point of T. Note that every asymptotically nonexpansive mapping in the intermediate sense is a generalized asymptotically nonexpansive mapping. Hence, Theorem 3.1 can be applied to the class of asymptotically nonexpansive mappings in the intermediate sense to obtain weak convergence of the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad \forall n \in \mathbb{N} \tag{3.9}$$

without the boundedness and compactness conditions on C.

Note that the modified Mann's iteration in Theorem 3.1 has only weak convergence.

Question 1. How can we modify the modified Mann's iteration in order to obtain strong convergence?

In the following theorem, we introduce a monotone hybrid method with the modified Mann's iteration to obtain a strong convergence theorem for an infinite family of uniformly Lipschitzian and generalized asymptotically quasi-nonexpansive mappings.

Theorem 3.3. *Let C be a closed and convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of uniformly L_i-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings of C into itself with nonnegative real sequences $\{k_{i,n}\}$, $\{c_{i,n}\}$ such that $k_{i,n} \geq 1$, $\lim_{n\to\infty} k_{i,n} = 1$, $\lim_{n\to\infty} c_{i,n} = 0$, for all $i \in N$. Assume that $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and the sequence $\alpha_{i,n} \in [0,1)$, for all $i, n \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by Algorithm 1.3 converges strongly to $P_F x_0$.*

Proof. We split the proof into six steps.

Step 1. Show that $P_F x_0$ is well defined for every $x_0 \in C$.

By Lemma 2.8, we obtain that $F(T_i)$ is a closed and convex subset of C for every $i \in \mathbb{N}$. Hence, $F := \bigcap_{i=1}^{\infty} F(T_i)$ is a nonempty, closed, and convex subset of C; consequently, $P_F x_0$ is well defined for every $x_0 \in C$.

Step 2. Show that $P_{C_n \cap Q_n} x_0$ is well defined.

From the definition of C_n and Q_n, it is obvious that Q_n is closed and convex for each $n \in \mathbb{N}$. It follows from Lemma 2.7 that $C_{i,n}$ is closed and convex for all $i, n \in \mathbb{N}$. This implies that C_n is closed and convex for each $n \in \mathbb{N}$. Next, we will show that $F \subset C_n \cap Q_n$, for all $n \geq 0$. First, we prove that $F \subset C_{i,n}$ for all $n \geq 0$ and $i \in \mathbb{N}$. Since T_i is a generalized asymptotically quasi-nonexpansive mapping for all $i \in \mathbb{N}$, we have that for any $p \in F$,

$$\begin{aligned}
\|y_{i,n} - p\|^2 &= \|\alpha_{i,n} x_n + (1-\alpha_{i,n}) T_i^n x_n - p\|^2 \\
&= \alpha_{i,n} \|x_n - p\|^2 + (1-\alpha_{i,n}) \|T_i^n x_n - p\|^2 - \alpha_{i,n}(1-\alpha_{i,n}) \|T_i^n x_n - x_n\|^2 \\
&\leq \alpha_{i,n} \|x_n - p\|^2 + (1-\alpha_{i,n}) (k_{i,n} \|x_n - p\| + c_{i,n})^2 \\
&\quad - \alpha_{i,n}(1-\alpha_{i,n}) \|T_i^n x_n - x_n\|^2 \\
&= \|x_n - p\|^2 - \alpha_{i,n}(1-\alpha_{i,n}) \|T_i^n x_n - x_n\|^2 \\
&\quad + (1-\alpha_{i,n}) \left(\left(k_{i,n}^2 - 1 \right) \|x_n - p\|^2 + 2k_{i,n} c_{i,n} \|x_n - p\| + c_{i,n}^2 \right) \\
&\leq \|x_n - p\|^2 - \alpha_{i,n}(1-\alpha_{i,n}) \|T_i^n x_n - x_n\|^2 + (1-\alpha_{i,n}) \theta_{i,n}.
\end{aligned} \tag{3.10}$$

Hence, $p \in C_{i,n}$ for all $n \geq 0$ and $i \in \mathbb{N}$. This proves that $F \subset C_{i,n}$ for all $n \geq 0$ and $i \in \mathbb{N}$. Hence, $F \subset C_n$ for all $n \geq 0$.

As shown in Marino and Xu [12], by induction, we can show that $F \subset Q_n$ for all $n \geq 0$. Hence $F \subset C_n \cap Q_n$, for all $n \geq 0$, and so $P_{C_n \cap Q_n} x_0$ is well defined.

Step 3. Show that $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

From $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 0. \tag{3.11}$$

On the other hand, as $F \subset Q_n$, we obtain

$$\|x_n - x_0\| \leq \|z - x_0\|, \quad \forall n \geq 0,\ \forall z \in F. \tag{3.12}$$

So we have that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

Step 4. Show that $x_n \to q$, where $q \in C$.

For $m > n$, by the definition of Q_n, we see that $Q_m \subset Q_n$. Noting that $x_m = P_{Q_m} x_0$ and $x_n = P_{Q_n} x_0$, by Lemma 2.5, we conclude that

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{3.13}$$

It follows from Step 3 that $\{x_n\}$ is a Cauchy. So, we can assume that $x_n \to q$ as $n \to \infty$ for some $q \in C$. In particular, we have that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Step 5. Show that $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$, for all $i \in \mathbb{N}$.

Let $i, n \in \mathbb{N}$. Since $x_{n+1} \in C_n$, it follows from (3.10) that

$$\left\|y_{i,n} - x_{n+1}\right\|^2 \leq \|x_n - x_{n+1}\|^2 - \alpha_{i,n}(1-\alpha_{i,n})\left\|T_i^n x_n - x_n\right\|^2 + (1-\alpha_{i,n})\theta_{i,n}. \tag{3.15}$$

Moreover, by Lemma 2.6, we have

$$\begin{aligned} \left\|y_{i,n} - x_{n+1}\right\|^2 &= \left\|\alpha_{i,n}x_n + (1-\alpha_{i,n})T_i^n x_n - x_{n+1}\right\|^2 \\ &= \alpha_{i,n}\|x_n - x_{n+1}\|^2 + (1-\alpha_{i,n})\left\|T_i^n x_n - x_{n+1}\right\|^2 \\ &\quad - \alpha_{i,n}(1-\alpha_{i,n})\left\|T_i^n x_n - x_n\right\|^2. \end{aligned} \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$(1-\alpha_{i,n})\left\|T_i^n x_n - x_{n+1}\right\|^2 \leq (1-\alpha_{i,n})\|x_n - x_{n+1}\|^2 + (1-\alpha_{i,n})\theta_{i,n}. \tag{3.17}$$

Since $\theta_{i,n} \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$, we have by (3.14) and (3.17) that

$$\lim_{n\to\infty}\left\|T_i^n x_n - x_{n+1}\right\| = 0. \tag{3.18}$$

This implies that

$$\left\|T_i^n x_n - x_n\right\| \leq \left\|T_i^n x_n - x_{n+1}\right\| + \|x_{n+1} - x_n\| \longrightarrow 0 \tag{3.19}$$

for all $i \in \mathbb{N}$. By Lemma 2.3 and (3.14), we get that $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i \in \mathbb{N}$.

Step 6. Show that $q = P_F x_0$.

Since $x_n \to q$ and $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i \in \mathbb{N}$, we have $q = T_i q$ for all $i \in \mathbb{N}$. Hence $q \in F$. By Lemma 2.4, we obtain

$$\langle z - x_n, x_0 - x_n\rangle \leq 0 \tag{3.20}$$

for all $z \in Q_{n-1} \cap C_{n-1}$. Since $F \subset Q_{n-1} \cap C_{n-1}$, we have

$$\langle z - q, x_0 - q\rangle \leq 0 \tag{3.21}$$

for all $z \in F$. Again by Lemma 2.4, we obtain that $q = P_F x_0$. This completes the proof. □

Theorem 3.4. *Let C be a closed and convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of uniformly L_i-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings of C into itself with nonnegative real sequences $\{k_{i,n}\}$, $\{c_{i,n}\}$ such that $k_{i,n} \geq 1$, $\lim_{n\to\infty}k_{i,n} = 1$, $\lim_{n\to\infty}c_{i,n} = 0$, for all $i \in \mathbb{N}$. Assume that $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and the sequence $\{\alpha_{i,n}\} \subset [0,1)$, for all $i, n \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by Algorithm 1.4 converges strongly to $P_F x_0$.*

Proof. We divide our proof into four steps.

Step 1. Show that C_n is closed and convex and $F \subset C_n$ for all $n \geq 1$.

It follows from Lemma 2.7 that $C_{i,n}$ is closed and convex for all $i, n \in \mathbb{N}$. This implies that C_n is closed and convex for each $n \in \mathbb{N}$. Next, we will show that $F \subset C_{i,n}$ for all $n \in \mathbb{N}$. For $n = 1$, $F \subset C_{i,1} = C$. Assume that $F \subset C_{i,n}$ for $n \in \mathbb{N}$. It follows from (3.10) and the definition of $C_{i,n+1}$ that $F \subset C_{i,n+1}$.

Step 2. Show that $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

From $x_n = P_{C_n}x_0$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_n$, for all $n \geq 0$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 0. \tag{3.22}$$

On the other hand, as $F \subset C_n$, we obtain

$$\|x_n - x_0\| \leq \|z - x_0\|, \quad \forall n \geq 0, \ \forall z \in F. \tag{3.23}$$

So we have that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

Step 3. Show that $x_n \to q$, where $q \in C$.

For $m > n$, by the definition of C_n, we see that $x_m = P_{C_m}x_0 \in C_m \subset C_n$. By Lemma 2.5, we obtain that

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{3.24}$$

From Step 2, we obtain that $\{x_n\}$ is Cauchy. Hence $x_n \to q$ as $n \to \infty$ for some $q \in C$ and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. By using the same proof as in Step 5 of Theorem 3.3, we can show that $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$, for all $i \in \mathbb{N}$.

Step 4. Show that $q = P_F x_0$.

Since $x_n \to q$ and $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$, for all $i \in \mathbb{N}$, we have $q = T_i q$, for all $i \in \mathbb{N}$. Hence $q \in F$. Since $x_n = P_{C_n}x_0$, by Lemma 2.4, we have

$$\langle z - x_n, x_0 - x_n \rangle \leq 0 \tag{3.25}$$

for all $z \in C_n$, and hence,

$$\langle z - q, x_0 - q \rangle \leq 0 \tag{3.26}$$

for all $z \in F$. This shows that $q = P_F x_0$, which completes the proof. □

Since a generalized asymptotically quasi-nonexpansive mapping is to unify various classes of mappings associated with the class of generalized asymptotically nonexpansive mapping, we have the following.

Corollary 3.5. *Let C be a closed and convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of uniformly L_i-Lipschitzian and generalized asymptotically nonexpansive mappings of C into itself with nonnegative real sequences $\{k_{i,n}\}$, $\{c_{i,n}\}$ such that $k_{i,n} \geq 1$,*

$\lim_{n\to\infty} k_{i,n} = 1$, $\lim_{n\to\infty} c_{i,n} = 0$, *for all* $i \in N$. *Assume that* $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ *and the sequence* $\{\alpha_{i,n}\} \subset [0,1)$, *for all* $i, n \in \mathbb{N}$. *Let a sequence* $\{x_n\}$ *be generated by the following manner:*

$$
\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily,} \\
y_{i,n} = \alpha_{i,n} x_n + (1-\alpha_{i,n}) T_i^n x_n, \\
C_{i,n} = \left\{ z \in C : \left\| y_{i,n} - z \right\|^2 \leq \|x_n - z\|^2 - \alpha_{i,n}(1-\alpha_{i,n}) \left\| T_i^n x_n - x_n \right\|^2 + (1-\alpha_{i,n})\theta_{i,n} \right\}, \\
C_n = \bigcap_{i=1}^{\infty} C_{i,n}, \\
Q_0 = C, \\
Q_n = \{ z \in Q_{n-1} : \langle z - x_n, x_0 - x_n \rangle \leq 0 \}, \quad n \geq 1, \\
x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0,
\end{gathered}
\tag{3.27}
$$

where $\theta_{i,n} = (k_{i,n}^2 - 1)\nabla_n^2 + 2k_{i,n} c_{i,n} \nabla_n + c_{i,n}^2$, $\nabla_n = \sup_{n\in\mathbb{N}} \{\|x_n - z\| : z \in F\} < \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_F x_0$.

Corollary 3.6. *Let* C *be a closed and convex subset of a real Hilbert space* H. *Let* $\{T_i\}_{i=1}^{\infty}$ *be an infinitely countable family of uniformly* L_i*-Lipschitzian and generalized asymptotically nonexpansive mappings of* C *into itself with nonnegative real sequences* $\{k_{i,n}\}$, $\{c_{i,n}\}$ *such that* $k_{i,n} \geq 1$, $\lim_{n\to\infty} k_{i,n} = 1$, $\lim_{n\to\infty} c_{i,n} = 0$, *for all* $i \in N$. *Assume that* $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ *and the sequence* $\{\alpha_{i,n}\} \subset [0,1)$, *for all* $i, n \in \mathbb{N}$. *Let a sequence* $\{x_n\}$ *be generated by the following manner:*

$$
\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily,} \\
C_{i,0} = C, \\
y_{i,n} = \alpha_{i,n} x_n + (1-\alpha_{i,n}) T_i^n x_n, \\
C_{i,n+1} = \left\{ z \in C_{i,n} : \left\| y_{i,n} - z \right\|^2 \leq \|x_n - z\|^2 - \alpha_{i,n}(1-\alpha_{i,n}) \left\| T_i^n x_n - x_n \right\|^2 + (1-\alpha_{i,n})\theta_{i,n} \right\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0,
\end{gathered}
\tag{3.28}
$$

where $\theta_{i,n} = (k_{i,n}^2 - 1)\nabla_n^2 + 2k_{i,n} c_{i,n} \nabla_n + c_{i,n}^2$, $\nabla_n = \sup_{n\in\mathbb{N}} \{\|x_n - z\| : z \in F\} < \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_F x_0$.

Remark 3.7. (i) Corollaries 3.5 and 3.6 improve and extend the main result in [10].

(ii) If we take $T_i = T$, $k_{i,n} = 1$ and $c_{i,n} = 0$ for all $i, n \in \mathbb{N}$ where T is a nonexpansive mapping, then Corollary 3.5 reduces to [9, Theorem 3.4].

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