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Research Article

Solvability of a Higher-Order Three-Point Boundary Value Problem on Time Scales

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We consider a higher-order three-point boundary value problem on time scales. A new existence result is first obtained by using a fixed point theorem due to Krasnoselskii and Zabreiko. Later, under certain growth conditions imposed on the nonlinearity, several sufficient conditions for the existence of a nonnegative and nontrivial solution are obtained by using Leray-Schauder nonlinear alternative. Our conditions imposed on nonlinearity are all very easy to verify; as an application, some examples to demonstrate our results are given.

1. Introduction

We are concerned with the following even-order three-point boundary value problem on time scales $\mathbb{T}$:

$$\begin{gathered}(-1)^n y^{\Delta^{2n}}(t) = f\left(y^{\sigma}(t)\right), \quad t \in [a,b] \subset \mathbb{T}, \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), \quad 0 \le i \le n-1,\end{gathered} \tag{1.1}$$

$$\begin{gathered}(-1)^n y^{\Delta^{2n}}(t) = f\left(t, y^{\sigma}(t)\right), \quad t \in [a,b] \subset \mathbb{T}, \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), \quad 0 \le i \le n-1,\end{gathered} \tag{1.2}$$

$n \ge 1$, $a < \eta < \sigma(b)$; we assume that $\sigma(b)$ is right dense so that $\sigma^j(b) = \sigma(b)$ for $j \ge 1$ and that for each $1 \le i \le n$, α_i, β_i, γ_i coefficients satisfy the following condition:

$$(H)\ 0 \le \alpha_i < \frac{\sigma(b) - \gamma_i \eta + (\gamma_i - 1)(a - \beta_i)}{\sigma(b) - \eta}, \quad \beta_i \ge 0,\ 0 < \gamma_i < \frac{\sigma(b) - a + \beta_i}{\eta - a + \beta_i}. \tag{1.3}$$

Throughout this paper, we let $\mathbb{T}$ be any time scale and let $[a,b]$ be a subset of $\mathbb{T}$ such that $[a,b] = \{t \in \mathbb{T} : a \le t \le b\}$. Some preliminary definitions and theorems on time scales can be found in [1–5] which are excellent references for the calculus of time scales.

In recent years, there is much attention paid to the existence of positive solution for second-order multipoint and higher-order two-point boundary value problems on time scales; for details, see [6–16] and references therein. However, to the best of our knowledge, there are not many results concerning multipoint boundary value problems of higher-order on time scales; we refer the readers to [17–20] for some recent results.

We would like to mention some results of Anderson and Avery [17], Anderson and Karaca [18], Han and Liu [19], and Yaslan [20]. In [17], Anderson and Avery studied the following even-order three-point BVP:

$$\begin{gathered}(-1)^n x^{(\Delta\nabla)^n}(t) = \lambda h(t) f(x(t)), \quad t \in [a,c] \subset \mathbb{T}, \\ x^{(\Delta\nabla)^i}(a) = 0, \quad x^{(\Delta\nabla)^i}(c) = \beta x^{(\Delta\nabla)^i}(b), \quad 0 \le i \le n-1.\end{gathered} \tag{1.4}$$

They have studied the existence of at least one positive solution to the BVP (1.4) using the functional-type cone expansion-compression fixed point theorem.

In [18], Anderson and Karaca were concerned with the dynamic three-point boundary value problem (1.2) and the eigenvalue problem $(-1)^n y^{\Delta^{2n}}(t) = \lambda f(t, y^{\sigma}(t))$ with the same boundary conditions where λ is a positive parameter. Existence results of bounded solutions of a noneigenvalue problem were first established as a result of the Schauder fixed point theorem. Second, the monotone method was discussed to ensure the existence of solutions of the BVP (1.2). Third, they established criteria for the existence of at least one positive solution of the eigenvalue problem by using the Krasnosel'skii fixed point theorem. Later, they investigated the existence of at least two positive solutions of the BVP (1.2) by using the Avery-Henderson fixed point theorem.

In [19], Han and Liu studied the existence and uniqueness of nontrivial solution for the following third-order p-Laplacian m-point eigenvalue problems on time scales:

$$\begin{gathered}\left(\phi_p\left(u^{\Delta\nabla}\right)\right)^{\nabla} + \lambda f\left(t, u(t), u^{\Delta}(t)\right) = 0, \quad t \in (0,T), \\ \alpha u(0) - \beta u^{\Delta}(0) = 0, \qquad u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u^{\Delta\nabla}(0) = 0,\end{gathered} \tag{1.5}$$

where $\phi_p(s)$ is p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ \lambda > 0$ is a parameter, and $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$. They obtained several sufficient conditions of the existence and uniqueness of nontrivial solution of the BVP (1.5) when λ is in some interval. Their approach was based on the Leray-Schauder nonlinear alternative.

Very recently, Yaslan [20] investigated the existence of solutions to the nonlinear even-order three-point boundary value problem on time scales $\mathbb{T}$:

$$\begin{gathered}(-1)^n y^{\Delta^{2n}}(t) = f(t, y(\sigma(t))), \quad t \in [t_1, t_3] \subset \mathbb{T}, \\ y^{\Delta^{2i+1}}(t_1) = 0, \qquad \alpha y^{\Delta^{2i}}(\sigma(t_3)) + \beta y^{\Delta^{2i+1}}(\sigma(t_3)) = y^{\Delta^{2i+1}}(t_2)\end{gathered} \tag{1.6}$$

for $0 \leq i \leq n-1$, where $\alpha > 0$ and $\beta > 1$ are given constants. On the one hand, the author established criteria for the existence of at least one solution and of at least one positive solution for the BVP (1.6) by using the Schauder fixed point theorem and Krasnosel'skii fixed point theorem, respectively. On the other hand, the author investigated the existence of multiple positive solutions to the BVP (1.6) by using Avery-Henderson fixed point theorem and Leggett-Williams fixed point theorem.

In this paper, motivated by [21], firstly, a new existence result for (1.1) is obtained by using a fixed point theorem, which is due to KrasnoseĺskiǏ and ZabreǏko [22]. Particularly, f may not be sublinear. Secondly, some simple criteria for the existence of a nonnegative solution of the BVP (1.2) are established by using Leray-Schauder nonlinear alternative. Thirdly, we investigate the existence of a nontrivial solution of the BVP (1.2); our approach is also based on the application of Leray-Schauder nonlinear alternative. Particularly, we do not require any monotonicity and nonnegativity on f. Our conditions imposed on f are all very easy to verify; our method is motivated by [1, 21, 23, 24].

2. Preliminaries

To state and prove the main results of this paper, we need the following lemmas.

Lemma 2.1 (see [18]). *For $1 \leq i \leq n$, let $G_i(t,s)$ be Green's function for the following boundary value problem:*

$$\begin{gathered} -y^{\Delta^2}(t) = 0, \quad t \in [a,b] \subset \mathbb{T}, \\ \alpha_i y(\eta) + \beta_i y^{\Delta}(a) = y(a), \qquad \gamma_i y(\eta) = y(\sigma(b)), \end{gathered} \tag{2.1}$$

and let $d_i = (\gamma_i - 1)(a - \beta_i) + (1-\alpha_i)\sigma(b) + \eta(\alpha_i - \gamma_i)$. Then, for $1 \leq i \leq n$,

$$G_i(t,s) = \begin{cases} G_{i_1}(t,s), & a \leq s \leq \eta, \\ G_{i_2}(t,s), & \eta < s \leq b, \end{cases} \tag{2.2}$$

where

$$\begin{gathered} G_{i_1}(t,s) = \frac{1}{d_i}\begin{cases} [\gamma_i(t-\eta) + \sigma(b) - t](\sigma(s) + \beta_i - a), & \sigma(s) \leq t, \\ [\gamma_i(\sigma(s) - \eta) + \sigma(b) - \sigma(s)](t + \beta_i - a) + \alpha_i(\eta - \sigma(b))(t - \sigma(s)), & t \leq s, \end{cases} \\ G_{i_2}(t,s) = \frac{1}{d_i}\begin{cases} [\sigma(s)(1-\alpha_i) + \alpha_i\eta + \beta_i - a](\sigma(b) - t) + \gamma_i(\eta - a + \beta_i)(t - \sigma(s)), & \sigma(s) \leq t, \\ [t(1-\alpha_i) + \alpha_i\eta + \beta_i - a](\sigma(b) - \sigma(s)), & t \leq s. \end{cases} \end{gathered} \tag{2.3}$$

Lemma 2.2 (see [18]). *Under condition (H), for $1 \leq i \leq n$, Green's function $G_i(t,s)$ in* (2.2) *possesses the following property:*

$$G_i(t,s) > 0, \quad (t,s) \in (a,\sigma(b)) \times (a,b). \tag{2.4}$$

Lemma 2.3 (see [18]). *Assume that* (H) *holds. Then, for* $1 \leq i \leq n$*, Green's function* $G_i(t,s)$ *in* (2.2) *satisfies*

$$G_i(t,s) \leq \max\left\{G_i(a,s), G_i(\sigma(s),s), \frac{1}{d_i}(\eta-a+\beta_i)(\sigma(b)-\sigma(s))\right\},$$
$$t,s \in [a,\sigma(b)]\times[a,b],\ 0<\gamma_i\leq 1,$$
$$G_i(t,s) \leq \max\{G_i(\sigma(b),s), G_i(\sigma(s),s)\}, \quad t,s \in [a,\sigma(b)]\times[a,b],\ 1<\gamma_i<\frac{\sigma(b)-a+\beta_i}{\eta-a+\beta_i}. \tag{2.5}$$

Lemma 2.4 (see [18]). *Assume that condition* (H) *is satisfied. For* G *as in* (2.2)*, take* $H_1(t,s) := G_1(t,s)$ *and recursively define*

$$H_j(t,s) = \int_a^{\sigma(b)} H_{j-1}(t,r)G_j(r,s)\Delta r \tag{2.6}$$

for $2 \leq j \leq n$*. Then* $H_n(t,s)$ *is Green's function for the homogeneous problem:*

$$(-1)^n y^{\Delta^{2n}}(t) = 0, \quad t \in [a,b] \subset \mathbb{T},$$
$$\alpha_{i+1}y^{\Delta^{2i}}(\eta) + \beta_{i+1}y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1}y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), \quad 0 \leq i \leq n-1. \tag{2.7}$$

Lemma 2.5 (see [18]). *Assume that* (H) *holds. If one defines* $K = \prod_{j=1}^{n-1} K_j$*, then the Green function* $H_n(t,s)$ *in Lemma 2.4 satisfies the following inequalities:*

$$0 \leq H_n(t,s) \leq K\|G_n(\cdot,s)\|, \quad (t,s) \in [a,\sigma(b)]\times[a,b], \tag{2.8}$$

where

$$K_j = \int_a^{\sigma(b)} \|G_j(\cdot,s)\|\Delta s > 0, \quad 1 \leq j \leq n. \tag{2.9}$$

Lemma 2.6 (see [22]). *LetX be a Banach space and let* $F : X \to X$ *be completely continuous. If there exists a bounded and linear operator* $A : X \to X$ *such that* 1 *is not an eigenvalue of* A *and*

$$\lim_{\|u\|\to\infty} \frac{\|F(u)-A(u)\|}{\|u\|} = 0, \tag{2.10}$$

then F *has a fixed point in* X*.*

Lemma 2.7 (see [25]). *Let* X *be a real Banach space, let* Ω *be a bounded open subset of* X*,* $0 \in \Omega$*, and let* $F : \overline{\Omega} \to X$ *be a completely continuous operator. Then either there exist* $x \in \partial\Omega$*,* $\lambda > 1$ *such that* $F(x) = \lambda x$ *or there exists a fixed point* $x^* \in \overline{\Omega}$*.*

Suppose that $\mathcal{B}$ denotes the Banach space $C[a,\sigma(b)]$ with the norm $\|y\| = \sup_{t\in[a,\sigma(b)]}|y(t)|$.

3. Existence Results

In this section, we apply Lemmas 2.6 and 2.7 to establish some existence criteria for (1.1) and (1.2).

Theorem 3.1. *Suppose that condition* (H) *holds, and* $f : \mathbb{R} \to \mathbb{R}$ *is continuous with* $\lim_{s\to\infty}(f(s)/s) = m$*. If*

$$|m| < d = \left[\prod_{j=1}^{n} K_j\right]^{-1}, \tag{3.1}$$

then the BVP (1.1) *has a solution* y^**, and* $y^* \neq 0$ *when* $f(0) \neq 0$*.*

Proof. Define the integral operator $F : \mathcal{B} \to \mathcal{B}$ by

$$Fy(t) = \int_a^{\sigma(b)} H_n(t,s) f(y^\sigma(s))\Delta s \tag{3.2}$$

for $t \in [a,\sigma(b)]$. Obviously, the solutions of the BVP (1.1) are the fixed points of operator F. From the proof of Theorem 3.1 of [18], we can know that $F : \mathcal{B} \to \mathcal{B}$ is completely continuous.

In order to apply Lemma 2.6, we consider the following BVP:

$$\begin{gathered} (-1)^n y^{\Delta^{2n}}(t) = m y^\sigma(t), \quad t \in [a,b] \subset \mathbb{T}, \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), \quad 0 \leq i \leq n-1. \end{gathered} \tag{3.3}$$

Define the integral operator $A : \mathcal{B} \to \mathcal{B}$ by

$$Ay(t) = m\int_a^{\sigma(b)} H_n(t,s) y^\sigma(s)\Delta s \tag{3.4}$$

for $t \in [a,\sigma(b)]$. Then it is easy to check that $A : \mathcal{B} \to \mathcal{B}$ is completely continuous (so bounded) linear operator and that solutions of the BVP (3.3) are the fixed points of operator A and conversely.

First, we claim that 1 is not an eigenvalue of A.

In fact, if $m = 0$, then it is obvious that the BVP (3.3) has no nontrivial solution.

If $m \neq 0$ and the BVP (3.3) has a nontrivial solution y, then $\|y\| > 0$, and so

$$\begin{aligned}
\|y\| &= \|Ay\| \\
&= \sup_{t\in[a,\sigma(b)]}\left| m\int_a^{\sigma(b)} H_n(t,s)y^\sigma(s)\Delta s\right| \\
&= |m| \sup_{t\in[a,\sigma(b)]}\left| \int_a^{\sigma(b)} H_n(t,s)y^\sigma(s)\Delta s\right| \\
&\le |m|\int_a^{\sigma(b)} K\|G_n(\cdot,s)\|\left|y^\sigma(s)\right|\Delta s \\
&\le |m|\left(\prod_{j=1}^{n} K_j\right)\|y\| \\
&< d\cdot\frac{1}{d}\|y\| = \|y\|,
\end{aligned} \tag{3.5}$$

which is impossible. So, 1 is not an eigenvalue of A.

Next, we will show that

$$\lim_{\|y\|\to\infty}\frac{\|F(y)-A(y)\|}{\|y\|} = 0. \tag{3.6}$$

In fact, for any $\epsilon > 0$, since $\lim_{s\to\infty}(f(s)/s) = m$, there must exist a number $\Upsilon_1 > 0$ such that

$$\left|f(s) - ms\right| < \epsilon|s|, \quad |s| > \Upsilon_1. \tag{3.7}$$

Let

$$M = \max_{|s|\le\Upsilon_1}\left|f(s)\right|. \tag{3.8}$$

Then for any $y \in \mathcal{B}$ and $\|y\| > \Upsilon$ ($\Upsilon > 0$), we distinguish the following two cases.

Case 1 ($\Upsilon < \Upsilon_1$). In this case, choose Υ such that

$$\frac{M + |m|\Upsilon_1}{\Upsilon} < \epsilon. \tag{3.9}$$

Thus, when $s \in [a,\sigma(b)]$ and $|y^\sigma(s)| \le \Upsilon_1$, we have

$$\left|f(y^\sigma(s)) - my^\sigma(s)\right| \le \left|f(y^\sigma(s))\right| + |m|\left|y^\sigma(s)\right| \le M + |m|\Upsilon_1 < \epsilon\Upsilon < \epsilon\|y\|, \tag{3.10}$$

which together with (3.7) implies that

$$\left|f\left(y^{\sigma}(s)\right) - my^{\sigma}(s)\right| < \epsilon\|y\|, \quad \|y\| > \Upsilon. \tag{3.11}$$

Case 2 ($\Upsilon \geq \Upsilon_1$). In this case, when $s \in [a, \sigma(b)]$, from (3.7), we see that

$$\left|f\left(y^{\sigma}(s)\right) - my^{\sigma}(s)\right| < \epsilon\left|y^{\sigma}(s)\right| \leq \epsilon\|y\|. \tag{3.12}$$

Thus, we can deduce from (3.11) and (3.12) that for any $y \in \mathcal{B}$ and $\|y\| > \Upsilon$

$$\left|f\left(y^{\sigma}(s)\right) - my^{\sigma}(s)\right| < \epsilon\|y\|, \quad \forall s \in [a, \sigma(b)]. \tag{3.13}$$

From (3.13), we have

$$\begin{aligned}
\|F(y) - A(y)\| &= \sup_{t\in[a,\sigma(b)]}\left|\int_a^{\sigma(b)} H_n(t,s)\left[f\left(y^{\sigma}(s)\right) - my^{\sigma}(s)\right]\Delta s\right| \\
&\leq \sup_{t\in[a,\sigma(b)]}\int_a^{\sigma(b)} H_n(t,s)\left|f\left(y^{\sigma}(s)\right) - my^{\sigma}(s)\right|\Delta s \\
&\leq \epsilon\|y\|\left(\prod_{j=1}^{n} K_j\right) \\
&= \frac{\epsilon}{d}\|y\|.
\end{aligned} \tag{3.14}$$

that is to say,

$$\lim_{\|y\|\to\infty}\frac{\|F(y) - A(y)\|}{\|y\|} = 0. \tag{3.15}$$

Then, it follows from Lemma 2.6 that F has a fixed point $y^* \in \mathcal{B}$. In other words, y^* is a solution of the BVP (1.1). Moreover, we can assert that y^* is nontrivial when $f(0) \neq 0$. In fact, if $f(0) \neq 0$, then

$$(-1)^n(0)^{\Delta^{2n}} = 0 \neq f(0), \tag{3.16}$$

that is, 0 is not a solution of the BVP (1.1). □

Corollary 3.2. *Assume that condition* (H) *holds, and* $f : [0, +\infty) \to [0, +\infty)$ *is continuous with* $\lim_{s\to+\infty}(f(s)/s) = 0$. *Then the BVP* (1.1) *has a nonnegative solution.*

Proof. Let

$$f^*(s) = \begin{cases} f(s), & s \geq 0, \\ f(-s), & s < 0, \end{cases} \tag{3.17}$$

then $f^* : \mathbb{R} \to [0, +\infty)$ is continuous, and from $\lim_{s\to+\infty}(f(s)/s) = 0$, we know that

$$\lim_{s\to+\infty} \frac{f^*(s)}{s} = 0. \tag{3.18}$$

Consider the following BVP:

$$\begin{gathered} (-1)^n y^{\Delta^{2n}}(t) = f^*(y^\sigma(t)), \quad t \in [a,b] \subset \mathbb{T}, \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), \quad 0 \leq i \leq n-1. \end{gathered} \tag{3.19}$$

It follows from Theorem 3.1 that the BVP (3.19) has a solution y^*, that is,

$$\begin{gathered} (-1)^n (y^*)^{\Delta^{2n}}(t) = f^*((y^*)^\sigma(t)), \quad t \in [a,b] \subset \mathbb{T}, \\ \alpha_{i+1} (y^*)^{\Delta^{2i}}(\eta) + \beta_{i+1} (y^*)^{\Delta^{2i+1}}(a) = (y^*)^{\Delta^{2i}}(a), \quad \gamma_{i+1} (y^*)^{\Delta^{2i}}(\eta) = (y^*)^{\Delta^{2i}}(\sigma(b)), \quad 0 \leq i \leq n-1. \end{gathered} \tag{3.20}$$

Since $H_n(t,s)$ and f^* are nonnegative, we can get that $y^* \geq 0$ on $[a, \sigma(b)]$. Consequently, from the definition of f^*, we have

$$(-1)^n (y^*)^{\Delta^{2n}}(t) = f((y^*)^\sigma(t)), \quad t \in [a,b] \subset \mathbb{T}. \tag{3.21}$$

It follows from the boundary conditions of (3.20) and (3.21) that y^* is a nonnegative solution of the BVP (1.1). □

Remark 3.3. In Corollary 3.2, we only need that $\lim_{s\to+\infty}(f(s)/s) = 0$. Thus, f may not be sublinear.

Theorem 3.4. *Assume that condition (H) holds, and*

$$\begin{gathered} f : [a, \sigma(b)] \times [0, +\infty) \longrightarrow [0, +\infty) \text{ is continuous with} \\ f(t,u) > 0 \quad \text{for } (t,u) \in [a, \sigma(b)] \times (0, +\infty), \end{gathered} \tag{3.22}$$

$$\begin{gathered} f(t,u) \leq \varphi(t) g(u) \quad \text{on} \quad [a, \sigma(b)] \times [0, +\infty) \text{ with } g \geq 0 \text{ continuous,} \\ \text{and nondecreasing on } [0, +\infty) \text{ and } \varphi : [a, \sigma(b)] \to (0, \infty) \text{ continuous} \end{gathered} \tag{3.23}$$

$$\exists r > 0 \ \text{ with } r > g(\|r\|) \int_a^{\sigma(b)} \varphi(s) K \|G_n(\cdot, s)\| \Delta s, \tag{3.24}$$

where K and $G_n(\cdot, s)$ are defined in Lemmas 2.5 and 2.1, respectively. Then the BVP (1.2) *has a nonnegative solution y_1 with $\|y_1\| < r$.*

Proof. We consider the following boundary value problem:

$$\begin{gathered}(-1)^n y^{\Delta^{2n}}(t) = \lambda f^*(t, y^\sigma(t)), \quad t \in [a,b] \subset \mathbb{T}, \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), \quad 0 \le i \le n-1.\end{gathered} \tag{3.25}$$

where $0 < \lambda < 1$, and

$$f^*(t,u) = \begin{cases} f(t,u), & u \ge 0, \\ f(t,0), & u < 0. \end{cases} \tag{3.26}$$

Let y be any solution of (3.25). Then

$$y(t) = \lambda \int_a^{\sigma(b)} H_n(t,s) f^*(s, y^\sigma(s)) \Delta s \tag{3.27}$$

for $t \in [a, \sigma(b)]$. We note that $y(t) \ge 0$ for $t \in [a, \sigma(b)]$. It follows from condition (3.23) in Theorem 3.4 that for $t \in [a, \sigma(b)]$

$$\begin{aligned} y(t) &\le \int_a^{\sigma(b)} H_n(t,s) \varphi(s) g(|y^\sigma(s)|) \Delta s \\ &\le g(\|y\|) \int_a^{\sigma(b)} \varphi(s) K \|G_n(\cdot, s)\| \Delta s. \end{aligned} \tag{3.28}$$

Consequently,

$$\|y\| \le g(\|y\|) \int_a^{\sigma(b)} \varphi(s) K \|G_n(\cdot, s)\| \Delta s, \tag{3.29}$$

which together with the condition (3.24) in Theorem 3.4 implies that $\|y\| \ne r$.

Let $N : \mathcal{B} \to \mathcal{B}$ be given by

$$Ny(t) = \int_a^{\sigma(b)} H_n(t,s) f^*(s, y^\sigma(s)) \Delta s. \tag{3.30}$$

It is easy to show that $N : \mathcal{B} \to \mathcal{B}$ is completely continuous.

Let

$$U = \{u \in \mathcal{B} : \|u\| < r\}. \tag{3.31}$$

Since $\|y\| \neq r$, any solution $y \in \partial U$ of $y = \lambda N y$ with $0 < \lambda < 1$ cannot occur. Lemma 2.7 guarantees that N has a fixed point y_1 in $\overline{U}$. In other words, the BVP (1.2) has a solution $y_1 \in \mathcal{B}$ with $\|y_1\| < r$. □

Theorem 3.5. *Assume that condition (H) is satisfied. Suppose that $f(t,0) \not\equiv 0$, $t \in [a,\sigma(b)]$, $f : [a,\sigma(b)] \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist nonnegative integrable functions k, h such that*

$$\begin{gathered}
|f(t,y)| \le k(t)|y| + h(t), \quad (t,y) \in [a,\sigma(b)] \times \mathbb{R}, \\
K\int_a^{\sigma(b)} \|G_n(\cdot,s)\| k(s)\Delta s < 1.
\end{gathered} \tag{3.32}$$

Then, the BVP (1.2) *has at least one nontrivial solution $y^* \in \mathcal{B}$.*

Proof. Let

$$\begin{gathered}
A = K\int_a^{\sigma(b)} \|G_n(\cdot,s)\| h(s)\Delta s, \\
B = K\int_a^{\sigma(b)} \|G_n(\cdot,s)\| k(s)\Delta s.
\end{gathered} \tag{3.33}$$

By hypothesis $B < 1$. Since $f(t,0) \not\equiv 0$, there exists $[m,n] \subset [a,\sigma(b)]$ such that $\min_{t\in[m,n]}|f(t,0)| > 0$. On the other hand, from the condition $h(t) \ge |f(t,0)|$, a.e. $t \in [a,\sigma(b)]$, we know that $A > 0$.

Let $d = A(1-B)^{-1}$, $\Omega_d = \{y \in \mathcal{B} : \|y\| < d\}$. For $t \in [a,\sigma(b)]$, the operator T is defined by

$$Ty(t) = \int_a^{\sigma(b)} H_n(t,s) f(s, y^\sigma(s))\Delta s, \tag{3.34}$$

from the proof of Theorems 3.1 and 3.4, we have known that $T : \mathcal{B} \to \mathcal{B}$ is a completely continuous operator, and the BVP (1.2) has at least one nontrivial solution $y^* \in \mathcal{B}$ if and only if y^* is a fixed point of T in $\mathcal{B}$.

Suppose $y \in \partial\Omega_d$, $\lambda > 1$ such that $Ty = \lambda y$, then

$$\begin{aligned}
\lambda d = \lambda\|y\| = \|Ty\| &= \sup_{t\in[a,\sigma(b)]}|Ty(t)| \\
&= \sup_{t\in[a,\sigma(b)]}\left|\int_a^{\sigma(b)} H_n(t,s)f(s,y^\sigma(s))\Delta s\right| \\
&\le K\int_a^{\sigma(b)} \|G_n(\cdot,s)\|\left|f(s,y^\sigma(s))\right|\Delta s \\
&\le K\int_a^{\sigma(b)} \|G_n(\cdot,s)\|\left[k(s)\left|y^\sigma(s)\right| + h(s)\right]\Delta s \\
&= K\int_a^{\sigma(b)} \|G_n(\cdot,s)\|k(s)\left|y^\sigma(s)\right|\Delta s + K\int_a^{\sigma(b)} \|G_n(\cdot,s)\|h(s)\Delta s \\
&\le B\|y\| + A = Bd + A.
\end{aligned} \tag{3.35}$$

Therefore

$$(\lambda - 1)d \le A - (1 - B)d = A - A = 0, \tag{3.36}$$

which contradicts $\lambda > 1$. By Lemma 2.7, T has a fixed point $y^* \in \overline{\Omega}_d$. Noting $f(t,0) \not\equiv 0$, the BVP (1.2) has at least one nontrivial solution $y^* \in \mathcal{B}$. This completes the proof. □

Corollary 3.6. *Assume that condition* (H) *is satisfied. Suppose that* $f(t,0) \not\equiv 0$, $t \in [a,\sigma(b)]$, $f : [a,\sigma(b)] \times \mathbb{R} \to \mathbb{R}$ *is continuous, and there exist nonnegative integrable functions* k, h *such that*

$$\begin{gathered}
\left|f(t,y)\right| \le k(t)\left|y\right| + h(t), \quad (t,y) \in [a,\sigma(b)] \times \mathbb{R}, \\
k(t) < e = \left[\prod_{j=1}^{n} K_j\right]^{-1}, \quad t \in [a,\sigma(b)].
\end{gathered} \tag{3.37}$$

Then, the BVP (1.2) *has at least one nontrivial solution* $y^* \in \mathcal{B}$.

Proof. In this case, we have

$$K\int_a^{\sigma(b)} \|G_n(\cdot,s)\|k(s)\Delta s < Ke\int_a^{\sigma(b)} \|G_n(\cdot,s)\|\Delta s = K\left[\prod_{j=1}^{n} K_j\right]^{-1}\int_a^{\sigma(b)} \|G_n(\cdot,s)\|\Delta s = 1. \tag{3.38}$$

By Theorem 3.5, this completes the proof. □

Corollary 3.7. *Assume that condition* (H) *is satisfied. Suppose that* $f(t,0)\not\equiv 0$, $t \in [a,\sigma(b)]$, $f : [a,\sigma(b)] \times \mathbb{R} \to \mathbb{R}$ *is continuous, and there exist nonnegative integrable functions* k, h *such that*

$$\left|f(t,y)\right| \leq k(t)\left|y\right| + h(t), \quad (t,y) \in [a,\sigma(b)] \times \mathbb{R},$$
$$\overline{\lim_{|l|\to\infty}} \max_{t\in[a,\sigma(b)]} \left|\frac{f(t,l)}{l}\right| < \left[\prod_{j=1}^{n} K_j\right]^{-1}. \tag{3.39}$$

Then, the BVP (1.2) *has at least one nontrivial solution* $y^* \in \mathcal{B}$.

Proof. Let

$$\epsilon_1 = \frac{1}{2}\left[\left(\prod_{j=1}^{n} K_j\right)^{-1} - \overline{\lim_{|l|\to\infty}} \max_{t\in[a,\sigma(b)]} \left|\frac{f(t,l)}{l}\right|\right], \tag{3.40}$$

then, there exists $c > 0$ such that

$$\left|f(t,l)\right| \leq \left[\left(\prod_{j=1}^{n} K_j\right)^{-1} - \epsilon_1\right]|l|, \quad (t,l) \in [a,\sigma(b)] \times \mathbb{R} \setminus (-c,c). \tag{3.41}$$

Set

$$\widetilde{M} = \max\left\{f(t,l) \mid (t,l) \in [a,\sigma(b)] \times [-c,c]\right\}, \tag{3.42}$$

and it follows from (3.41) and (3.42) that

$$\left|f(t,l)\right| \leq \left[\left(\prod_{j=1}^{n} K_j\right)^{-1} - \epsilon_1\right]|l| + \widetilde{M}, \quad (t,l) \in [a,\sigma(b)] \times \mathbb{R}. \tag{3.43}$$

By Corollary 3.6, we can deduce that Corollary 3.7 is true. □

4. Two Examples

In the section, we present two examples to explain our results.

Example 4.1. Let $\mathbb{T} = \mathbb{R}[0,1) \cup \{1,2,3,4,5\} \cup [6,7]$, and consider the following BVP:

$$-y^{\Delta^2}(t) = f\left(y^{\sigma}(t)\right), \quad t \in [0,5] \subset \mathbb{T},$$
$$\frac{3}{5}y\left(\frac{2}{3}\right) + \frac{2}{3}y^{\Delta}(0) = y(0), \qquad \frac{1}{6}y\left(\frac{2}{3}\right) = y(\sigma(5)), \tag{4.1}$$

where $f(y) = (17737/89902)y + 1/101$. It is easy to check that

$$
\begin{gathered}
0 < \frac{3}{5} = \alpha_1 < \frac{\sigma(b) - \gamma_1\eta + (\gamma_1 - 1)(a - \beta_1)}{\sigma(b) - \eta} = \frac{6 - (1/6)\cdot(2/3) + (1/6 - 1)(0 - 2/3)}{6 - 2/3} = \frac{29}{24}, \\
\beta_1 = \frac{2}{3} > 0, \quad 0 < \frac{1}{6} = \gamma_1 < \frac{\sigma(b) - a + \beta_1}{\eta - a + \beta_1} = \frac{6 - 1 + (2/3)}{2/3 - 0 + 2/3} = 5,
\end{gathered}
\tag{4.2}
$$

and therefore, the condition (H) is satisfied. By computation, we can get that

$$
\begin{gathered}
d_1 = (\gamma_1 - 1)(a - \beta_1) + (1 - \alpha_1)\sigma(b) + \eta(\alpha_1 - \gamma_1) = \frac{146}{45}, \\
G_1(t,s) = \begin{cases} G_{1_1}(t,s), & 0 \le s \le \dfrac{2}{3}, \\ G_{1_2}(t,s), & \dfrac{2}{3} < s \le 5, \end{cases}
\end{gathered}
\tag{4.3}
$$

where

$$
\begin{gathered}
G_{1_1}(t,s) = \frac{45}{146}\begin{cases} \left[\dfrac{1}{6}\left(t - \dfrac{2}{3}\right) + \sigma(5) - t\right]\left(\sigma(s) + \dfrac{2}{3}\right), & \sigma(s) \le t, \\ \left[\dfrac{1}{6}\left(\sigma(s) - \dfrac{2}{3}\right) + 6 - \sigma(s)\right]\left(t + \dfrac{2}{3}\right) + \dfrac{3}{5}\left(\dfrac{2}{3} - 6\right)(t - \sigma(s)), & t \le s, \end{cases} \\
G_{1_2}(t,s) = \frac{45}{146}\begin{cases} \left[\sigma(s)\left(1 - \dfrac{3}{5}\right) + \dfrac{3}{5} \times \dfrac{2}{3} + \dfrac{2}{3}\right](6 - t) + \dfrac{1}{6}\left(\dfrac{2}{3} + \dfrac{2}{3}\right)(t - \sigma(s)), & \sigma(s) \le t, \\ \left[t\left(1 - \dfrac{3}{5}\right) + \dfrac{3}{5} \times \dfrac{2}{3} + \dfrac{2}{3}\right](6 - \sigma(s)), & t \le s. \end{cases}
\end{gathered}
\tag{4.4}
$$

From the proof of Lemma 2.5 in [18], we can get that

$$
\|G_1(\cdot,s)\| = \max\left\{G_1(0,s), G_1(\sigma(s),s), \frac{1}{d_1}(\eta - a + \beta_1)(\sigma(b) - \sigma(s))\right\} = G_1(\sigma(s),s), \tag{4.5}
$$

and for $\gamma_1 = 1/6 \in (0,1]$ and $\alpha_1 = 3/5 \in (0,1)$, we have

$$
\begin{aligned}
K_1 &= \frac{45}{146}\int_0^{2/3}\left[\frac{1}{6}\left(s - \frac{2}{3}\right) + 6 - s\right]\left(s + \frac{2}{3} - 0\right)ds + \frac{45}{146}\int_{2/3}^{1}\left[s\left(1 - \frac{3}{5}\right) + \frac{3}{5} \times \frac{2}{3} + \frac{2}{3} - 0\right](6 - s)ds \\
&\quad + \sum_{s=1}^{5}\frac{45}{146}\left[(s+1)\left(1 - \frac{3}{5}\right) + \frac{3}{5} \times \frac{2}{3} + \frac{2}{3}\right][6 - (s+1)] \\
&= \frac{89902}{17739}.
\end{aligned}
\tag{4.6}
$$

Thus, we can obtain that

$$\lim_{y\to\infty}\frac{f(y)}{y}=\lim_{y\to\infty}\frac{(17737/89902)y+1/101}{y}=\frac{17737}{89902}<d=K_1^{-1}=\frac{17739}{89902}. \tag{4.7}$$

By Theorem 3.1, it is easy to get that the BVP (4.1) has a solution y^*.

Example 4.2. Let us introduce an example to illustrate the usage of Theorem 3.5. Let $n=2,\mathbb{T}=\{(2/3)^n : n\in\mathbb{N}_0\}\cup\{0\}\cup[1,2]$, $a=8/27$, $\eta=4/9, b=2/3$, $\alpha_1=1/2$, $\beta_1=1/9,\alpha_2=1/10$, $\beta_2=7/27,\gamma_1=3/2$, and $\gamma_2=2$. Then condition (H) is satisfied. Green's function $G_1(t,s)$ in Lemma 2.1 is

$$G_1(t,s)=\begin{cases} G_{1_1}(t,s), & \dfrac{8}{27}\le s\le\dfrac{4}{9},\\[2mm] G_{1_2}(t,s), & \dfrac{4}{9}< s\le\dfrac{2}{3}, \end{cases} \tag{4.8}$$

where

$$\begin{gathered} G_{1_1}(t,s)=\frac{27}{4}\begin{cases} \left[\dfrac{3}{2}\left(t-\dfrac{4}{9}\right)+1-t\right]\left[\dfrac{3}{2}s+\dfrac{1}{9}-\dfrac{8}{27}\right], & \dfrac{3}{2}s\le t,\\[2mm] \left[\dfrac{3}{2}\left(\dfrac{3}{2}s-\dfrac{4}{9}\right)+1-\dfrac{3}{2}s\right]\left(t+\dfrac{1}{9}-\dfrac{8}{27}\right)+\dfrac{1}{2}\left(\dfrac{4}{9}-1\right)\left(t-\dfrac{3}{2}s\right), & t\le s, \end{cases}\\ G_{1_2}(t,s)=\frac{27}{4}\begin{cases} \left[\dfrac{3}{2}s\left(1-\dfrac{1}{2}\right)+\dfrac{1}{2}\times\dfrac{4}{9}+\dfrac{1}{9}-\dfrac{8}{27}\right](1-t)+\dfrac{3}{2}\left(\dfrac{4}{9}-\dfrac{8}{27}+\dfrac{1}{9}\right)\left(t-\dfrac{3}{2}s\right), & \dfrac{3}{2}s\le t,\\[2mm] \left[t\left(1-\dfrac{1}{2}\right)+\dfrac{1}{2}\times\dfrac{4}{9}+\dfrac{1}{9}-\dfrac{8}{27}\right]\left(1-\dfrac{3}{2}s\right), & t\le s. \end{cases} \end{gathered} \tag{4.9}$$

Green's function $G_2(t,s)$ in Lemma 2.1 is

$$G_2(t,s)=\begin{cases} G_{2_1}(t,s), & \dfrac{8}{27}\le s\le\dfrac{4}{9},\\[2mm] G_{2_2}(t,s), & \dfrac{4}{9}< s\le\dfrac{2}{3}, \end{cases} \tag{4.10}$$

where

$$G_{2_1}(t,s) = \frac{54}{5}\begin{cases} \left[2\left(t-\frac{4}{9}\right)+1-t\right]\left[\frac{3}{2}s+\frac{7}{27}-\frac{8}{27}\right], & \frac{3}{2}s \le t, \\ \left[2\left(\frac{3}{2}s-\frac{4}{9}\right)+1-\frac{3}{2}s\right]\left(t+\frac{7}{27}-\frac{8}{27}\right)+\frac{1}{10}\left(\frac{4}{9}-1\right)\left(t-\frac{3}{2}s\right), & t \le s, \end{cases}$$

$$G_{2_2}(t,s) = \frac{54}{4}\begin{cases} \left[\frac{3}{2}s\left(1-\frac{1}{10}\right)+\frac{1}{10}\times\frac{4}{9}+\frac{7}{27}-\frac{8}{27}\right](1-t)+2\left(\frac{4}{9}-\frac{8}{27}+\frac{7}{27}\right)\left(t-\frac{3}{2}s\right), & \frac{3}{2}s\le t, \\ \left[t\left(1-\frac{1}{10}\right)+\frac{1}{10}\times\frac{4}{9}+\frac{7}{27}-\frac{8}{27}\right]\left(1-\frac{3}{2}s\right), & t \le s. \end{cases} \tag{4.11}$$

Since $s \in [a, (\gamma_1(\eta - a + \beta_1) - \alpha_1\eta - \beta_1 + a)/(1-\alpha_1)) = [8/27, 19/27)$, by using the cases in the proof of Lemma 2.5 of [18], we can know that $\|G_1(\cdot,s)\| = G_1(\sigma(2/3), s)$. Therefore, we have $K_1 = 133/324$.

Set $f(t,y) = t|y| \sin y + t^3 - 2\sin t$, $k(t) = t$, and $h(t) = t^3 + 2\sin t$. Then it is easy to prove that

$$f(t,y) \le k(t)|y| + h(t), \quad (t,y) \in [a,\sigma(b)] \times \mathbb{R}. \tag{4.12}$$

On the other hand, since $s \in [a, (\gamma_2(\eta - a + \beta_2) - \alpha_2\eta - \beta_2 + a)/(1-\alpha_2)) = [8/27, 218/243)$, we can know that $\|G_2(\cdot,s)\| = G_2(1,s)$ by using the cases in the proof of Lemma 2.5 of [18]. Therefore, we have

$$\int_a^{\sigma(b)} \|G_2(\cdot,s)\| k(s)\Delta s = \frac{6\cdot 11\cdot 16\cdot 47}{5\cdot 27^3}. \tag{4.13}$$

Thus

$$K\int_a^{\sigma(b)} \|G_2(\cdot,s)\| k(s)\Delta s = \left(\prod_{j=1}^{2-1} K_j\right)\int_a^{\sigma(b)} \|G_2(\cdot,s)\| k(s)\Delta s = \frac{133}{324}\cdot\frac{6\cdot 11\cdot 16\cdot 47}{5\cdot 27^3} < 1. \tag{4.14}$$

Hence, by Theorem 3.5, the BVP (1.2) has at least one nontrivial solution y^*.

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