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Research Article

On Approximate Euler Differential Equations

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We solve the inhomogeneous Euler differential equations of the form $x^2y''(x) + \alpha xy'(x) + \beta y(x) = \sum_{m=0}^{\infty} a_m x^m$ and apply this result to the approximation of analytic functions of a special type by the solutions of Euler differential equations.

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1. Introduction

The stability problem for functional equations starts from the famous talk of Ulam and the partial solution of Hyers to the Ulam's problem (see [1, 2]). Thereafter, Rassias [3] attempted to solve the stability problem of the Cauchy additive functional equation in a more general setting.

The stability concept introduced by Rassias' theorem significantly influenced a number of mathematicians to investigate the stability problems for various functional equations (see [4–10] and the references therein).

Assume that X and Y are a topological vector space and a normed space, respectively, and that I is an open subset of X. If for any function $f : I \to Y$ satisfying the differential inequality

$$\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) \right\| \le \varepsilon \tag{1.1}$$

for all $x \in I$ and for some $\varepsilon \ge 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0 \tag{1.2}$$

such that $\|f(x) - f_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of ε only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space X). We may apply these terminologies for other differential equations. For more detailed definition of the Hyers-Ulam stability, refer to [1, 3, 5, 6, 8–11].

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [12, 13]). Here, we will introduce a result of Alsina and Ger (see [14]): if a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(x) - y(x)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \leq 3\varepsilon$ for any $x \in I$.

This result of Alsina and Ger has been generalized by Takahasi, Miura, and Miyajima: they proved in [15] that the Hyers-Ulam stability holds for the Banach space-valued differential equation $y'(x) = \lambda y(x)$ (see also [16]).

Using the conventional power series method, the first author investigated the general solution of the inhomogeneous Hermite differential equation of the form

$$y''(x) - 2xy'(x) + 2\lambda y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{1.3}$$

under some specific condition, where λ is a real number and the convergence radius of the power series is positive. This result was applied to prove that every analytic function can be approximated in a neighborhood of 0 by a Hermite function with an error bound expressed by $Cx^2e^{x^2}$ (see [17–20]).

In Section 2 of this paper, using power series method, we will investigate the general solution of the inhomogeneous Euler (or Cauchy) differential equation

$$x^2y''(x) + \alpha xy'(x) + \beta y(x) = \sum_{m=0}^{\infty} a_m x^m, \tag{1.4}$$

where α and β are fixed complex numbers and the coefficients a_m of the power series are given such that the radius of convergence is $\rho > 0$. Moreover, using the idea from [17–19], we will approximate some analytic functions by the solutions of Euler differential equations.

In this paper, $\mathbb{N}_0$ denotes the set of all nonnegative integers.

2. General Solution of Inhomogeneous Euler Equations

The second-order Euler differential equation

$$x^2y''(x) + \alpha xy'(x) + \beta y(x) = 0, \tag{2.1}$$

which is sometimes called the second-order Cauchy differential equation, is one of the most famous differential equations and frequently appears in applications.

The quadratic equation

$$m^2 + (\alpha - 1)m + \beta = 0 \tag{2.2}$$

is called the auxiliary equation of the Euler differential equation (2.1), and every solution of (2.1) is of the form

$$y_h(x) = \begin{cases} c_1 x^{m_1} + c_2 x^{m_2} & \text{if } m_1 \text{ and } m_2 \text{ are distinct roots of (2.2)}, \\ (c_1 + c_2 \ln x) x^{(1-\alpha)/2} & \text{if } \dfrac{(1-\alpha)}{2} \text{ is a double root of (2.2)}, \end{cases} \tag{2.3}$$

where c_1 and c_2 are complex constants (see [21, Section 2.7]).

Theorem 2.1. *Let α and β be complex constants such that no root of the auxiliary equation* (2.2) *is a nonnegative integer. If the radius of convergence of power series $\sum_{m=0}^{\infty} a_m x^m$ is at least $\rho > 0$, then every solution $y : (0, \rho) \to \mathbb{C}$ of the inhomogeneous Euler differential equation* (1.4) *can be expressed by*

$$y(x) = y_h(x) + \sum_{m=0}^{\infty} \frac{a_m x^m}{m^2 + (\alpha - 1)m + \beta} \tag{2.4}$$

for all $x \in (0, \rho)$, where $y_h(x)$ is a solution of the Euler differential equation (2.1).

Proof. Assume that $y : (0, \rho) \to \mathbb{C}$ is a function given by (2.4), where $y_h(x)$ is a solution of the homogeneous Euler differential equation (2.1). We first prove that the function $y_p(x)$, defined by $y(x) - y_h(x)$, satisfies the inhomogeneous equation (1.4).

Indeed, we have

$$\begin{aligned} x^2 y_p''(x) + \alpha x y_p'(x) + \beta y_p(x) &= \sum_{m=2}^{\infty} \frac{m(m-1) a_m x^m}{m^2 + (\alpha - 1)m + \beta} + \sum_{m=1}^{\infty} \frac{\alpha m a_m x^m}{m^2 + (\alpha - 1)m + \beta} \\ &\quad + \sum_{m=0}^{\infty} \frac{\beta a_m x^m}{m^2 + (\alpha - 1)m + \beta} \\ &= \sum_{m=0}^{\infty} a_m x^m, \end{aligned} \tag{2.5}$$

which proves that $y_p(x)$ is a particular solution of the inhomogeneous equation (1.4). Moreover, each power series appearing in the above equalities has the same radius of convergence as $\sum_{m=0}^{\infty} a_m x^m$ (which can be verified by using the ratio test). Since every solution to (1.4) can be expressed as a sum of a solution $y_h(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the inhomogeneous equation, every solution of (1.4) is certainly of the form (2.4).

We will now apply the ratio test to the power series in (2.4). Indeed, by setting $c_m = a_m / [m^2 + (\alpha - 1)m + \beta]$, we get

$$\left| \frac{c_{m+1}}{c_m} \right| = \left| \frac{m^2 + (\alpha - 1)m + \beta}{(m+1)^2 + (\alpha - 1)(m+1) + \beta} \right| \left| \frac{a_{m+1}}{a_m} \right|, \tag{2.6}$$

and hence we have

$$\limsup_{m \longrightarrow \infty} \left| \frac{c_{m+1}}{c_m} \right| = \limsup_{m \longrightarrow \infty} \left| \frac{a_{m+1}}{a_m} \right|, \tag{2.7}$$

which implies that the power series given in (2.4) has the same radius of convergence as power series $\sum_{m=0}^{\infty} a_m x^m$, which is at least ρ. That is, $y(x)$ given in (2.4) is well defined on its domain $(0, \rho)$. □

3. Approximate Euler Differential Equations

In this section, assume that α and β are complex constants and ρ is a positive constant. For a given $K \geq 0$, we denote by $\mathcal{C}_K$ the set of all functions $y : (0, \rho) \to \mathbb{C}$ with the properties (a) and (b):

(a) $y(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ;

(b) $\sum_{m=0}^{\infty} |a_m x^m| \leq K | \sum_{m=0}^{\infty} a_m x^m|$ for any $x \in (0, \rho)$, where we set $a_m = [m^2 + (\alpha - 1)m + \beta] b_m$ for $m \geq 0$.

Let $\{b_m\}$ be a sequence of positive real numbers such that the radius of convergence of the series $\sum_{m=0}^{\infty} b_m x^m$ is at least ρ, and let α and β satisfy either $\alpha \geq 1$ and $\beta > 0$ or $\beta \geq (\alpha - 1)^2/4$. If a function $y : (0, \rho) \to \mathbb{R}$ is defined by $y(x) = \sum_{m=0}^{\infty} b_m x^m$, then y certainly belongs to $\mathcal{C}_K$ with $K \geq 1$. So, the set $\mathcal{C}_K$ is not empty if $K \geq 1$. In particular, if ρ is small and K is large, then $\mathcal{C}_K$ is a large class of analytic functions $y : (0, \rho) \to \mathbb{C}$.

We will now solve the approximate Euler differential equations in a special class of analytic functions, $\mathcal{C}_K$.

Theorem 3.1. *Let α and β be complex constants such that no root of the auxiliary equation* (2.2) *is a nonnegative integer. If a function $y \in \mathcal{C}_K$ satisfies the differential inequality*

$$\left| x^2 y''(x) + \alpha x y'(x) + \beta y(x) \right| \leq \varepsilon \tag{3.1}$$

for all $x \in (0, \rho)$ and for some $\varepsilon \geq 0$, then there exists a solution $y_h : (0, \rho) \to \mathbb{C}$ of the Euler differential equation (2.1) *such that*

$$\left| y(x) - y_h(x) \right| \leq \left(\sup_{k \in \mathbb{N}_0} \frac{1}{\left| k^2 + (\alpha - 1)k + \beta \right|} \right) \frac{K\rho}{\rho - x} \varepsilon \tag{3.2}$$

for any $x \in (0, \rho)$.

Proof. Since y belongs to $\mathcal{C}_K$, it follows from (a) and (b) that

$$x^2 y''(x) + \alpha x y'(x) + \beta y(x) = \sum_{m=0}^{\infty} \left[m^2 + (\alpha - 1)m + \beta \right] b_m x^m = \sum_{m=0}^{\infty} a_m x^m \tag{3.3}$$

for all $x \in (0, \rho)$. By considering (3.1) and (3.3), we get

$$\left| \sum_{m=0}^{\infty} a_m x^m \right| \leq \varepsilon \tag{3.4}$$

for any $x \in (0, \rho)$. This inequality, together with (b), yields that

$$\sum_{m=0}^{\infty} |a_m x^m| \leq K \left| \sum_{m=0}^{\infty} a_m x^m \right| \leq K\varepsilon \tag{3.5}$$

for each $x \in (0, \rho)$.

Now, suppose that an arbitrary $x \in (0, \rho)$ is given. Then we can choose an arbitrary constant $\rho_0 \in (x, \rho)$. By Abel's formula (see [22, Theorem 6.30]), we have

$$\begin{aligned} \sum_{m=0}^{n} |a_m \rho_0^m| \frac{|x/\rho_0|^m}{|m^2 + (\alpha-1)m + \beta|} &= \left(\sum_{m=0}^{n} |a_m \rho_0^m| \right) \frac{|x/\rho_0|^{n+1}}{|(n+1)^2 + (\alpha-1)(n+1) + \beta|} \\ &\quad + \sum_{k=0}^{n} \left(\sum_{m=0}^{k} |a_m \rho_0^m| \right) \frac{|x/\rho_0|^k}{|k^2 + (\alpha-1)k + \beta|} M(k; \rho_0; x), \end{aligned} \tag{3.6}$$

where we set

$$M(k; \rho_0; x) = 1 - \left| \frac{k^2 + (\alpha-1)k + \beta}{(k+1)^2 + (\alpha-1)(k+1) + \beta} \right| \left| \frac{x}{\rho_0} \right|. \tag{3.7}$$

Since $M(k; \rho_0; x) \leq 1$ for any $k \in \mathbb{N}_0$ and $\rho_0 \in (x, \rho)$, it follows from (3.5) and (3.6) that

$$\sum_{m=0}^{n} |a_m \rho_0^m| \frac{|x/\rho_0|^m}{|m^2 + (\alpha-1)m + \beta|} \leq K\varepsilon \left(\frac{|x/\rho_0|^{n+1}}{|(n+1)^2 + (\alpha-1)(n+1) + \beta|} + \sum_{k=0}^{n} \frac{|x/\rho_0|^k}{|k^2 + (\alpha-1)k + \beta|} \right) \tag{3.8}$$

If we let $n \to \infty$ in the above inequality, then we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{|a_m x^m|}{|m^2 + (\alpha-1)m + \beta|} &\leq K\varepsilon \sum_{k=0}^{\infty} \frac{1}{|k^2 + (\alpha-1)k + \beta|} \left| \frac{x}{\rho_0} \right|^k \\ &\leq \left(\sup_{k \in \mathbb{N}_0} \frac{1}{|k^2 + (\alpha-1)k + \beta|} \right) \frac{K\rho_0}{\rho_0 - |x|} \varepsilon \end{aligned} \tag{3.9}$$

for all $x \in (0, \rho)$ and for any $\rho_0 \in (x, \rho)$. Since $\rho_0/(\rho_0 - x) \downarrow \rho/(\rho - x)$ as $\rho_0 \to \rho$, we get

$$\sum_{m=0}^{\infty} \frac{|a_m x^m|}{|m^2 + (\alpha - 1)m + \beta|} \leq \left(\sup_{k \in \mathbb{N}_0} \frac{1}{|k^2 + (\alpha - 1)k + \beta|} \right) \frac{K\rho}{\rho - x} \varepsilon \tag{3.10}$$

for every $x \in (0, \rho)$.

Finally, it follows from (3.3), Theorem 2.1, and (3.10) that there exists a solution y_h of the Euler differential equation (2.1) such that

$$\begin{aligned} |y(x) - y_h(x)| &\leq \left| \sum_{m=0}^{\infty} \frac{a_m x^m}{m^2 + (\alpha - 1)m + \beta} \right| \\ &\leq \sum_{m=0}^{\infty} \frac{|a_m x^m|}{|m^2 + (\alpha - 1)m + \beta|} \\ &\leq \left(\sup_{k \in \mathbb{N}_0} \frac{1}{|k^2 + (\alpha - 1)k + \beta|} \right) \frac{K\rho}{\rho - x} \varepsilon \end{aligned} \tag{3.11}$$

for any $x \in (0, \rho)$. □

4. An Example

We fix $\alpha = 0$, $\beta = 1/4$ and suppose that a small $\varepsilon > 0$ is given. We can choose a constant $0 < \rho < 1$ such that

$$\varepsilon = \sum_{m=1}^{\infty} \left(m - 1 + \frac{1}{4m} \right) \rho^m. \tag{4.1}$$

We will consider the function $y(x) = \ln(1 - x)$ which can be expressed by the power series $\sum_{m=1}^{\infty} (-x^m/m)$, whose radius of convergence is 1.

If we set $b_0 = 0$ and $b_m = -1/m$ for any $m \in \mathbb{N}$, then it follows from (b) that

$$a_m = \begin{cases} 0 & \text{for } m = 0, \\ -m + 1 - \dfrac{1}{4m} & \text{for } m \in \mathbb{N}. \end{cases} \tag{4.2}$$

Thus, we have

$$\sum_{m=0}^{\infty} |a_m x^m| = \sum_{m=1}^{\infty} \left(m - 1 + \frac{1}{4m} \right) x^m = \left| \sum_{m=0}^{\infty} a_m x^m \right| \tag{4.3}$$

for any $x \in (0, \rho)$, which enables us to choose $K = 1$. Thus, it holds that $y \in C_1$.

Moreover, we have

$$\begin{aligned}\left|x^2y''(x)+\frac{1}{4}y(x)\right| &= \sum_{m=1}^{\infty}\left(m-1+\frac{1}{4m}\right)x^m \\ &\le \sum_{m=1}^{\infty}\left(m-1+\frac{1}{4m}\right)\rho^m \\ &= \varepsilon\end{aligned} \tag{4.4}$$

for all $x \in (0,\rho)$. It now follows from Theorem 3.1 that there exist complex numbers A and B with

$$\left|\ln(1-x)-(A+B\ln x)\sqrt{x}\,\right| \le \frac{4\rho}{\rho-x}\sum_{m=1}^{\infty}\left(m-1+\frac{1}{4m}\right)\rho^m \tag{4.5}$$

for any $x \in (0,\rho)$.

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