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*Research Article*

# A Two-Dimensional Landau-Lifshitz Model in Studying Thin Film Micromagnetics

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The paper is concerned with a two-dimensional Landau-Lifshitz equation which was first raised by A. DeSimone and F. Otto, and so fourth, when studying thin film micromagnetics. We get the existence of a local weak solution by approximating it with a higher-order equation. Penalty approximation and semigroup theory are employed to deal with the higher-order equation.



## 1. Introduction

Landau-Lifshitz equations are fundamental equations in the theory of ferromagnetism. They describe how the magnetization field inside ferromagnetic material evolves in time. The study of these equations is a very challenging mathematical problem, and is rewarded by the great amount of applications of magnetic devices, such as recording media, computer memory chips, and computer disks. The equations were first derived by Landau and Lifshitz on a phenomenological ground in [1]. They can be written as

$$\frac{\partial m}{\partial t} = -\alpha^2 m \times (m \times \mathscr{H}(m)) + \beta m \times \mathscr{H}(m), \tag{1.1}$$

where $\times$ is the vector cross product in $R^n$ $(n \geq 2), m = (m_1, m_2, \dots, m_n) : \Omega \times [0, +\infty) \to R^n$ is the magnetization and $\alpha^2$ is a Gilbert damping constant. The system (1.1) is implied by the conservation of energy and magnitude of $m.\mathscr{H}(m) = -\delta E/\delta m$ is the unconstrained first variation of the energy functional $E(m)$. The magnitude of the magnetization is finite, that is, $|m|^2 = \sum_{i=1}^{n} m_i^2 = 1$. Here

$$E = E(m) = \int_\Omega |\nabla m|^2 dx + \int_\Omega \phi(m) dx + \int_{R^n} |\nabla \Phi|^2 dx \tag{1.2}$$

is the free energy functional, and it is composed of three parts:

(i) $E_{\mathrm{ex}}(m) = \int_\Omega |\nabla m|^2 dx$ is the exchange energy. It tends to align $m$ in the same direction and prevents $m$ from being discontinuous in space;

(ii) $E_{\mathrm{an}}(m) = \int_\Omega \phi(m) dx$ is the anisotropy energy. $\phi \in C^\infty(R^3), \phi \geq 0$ depends on the crystal structure of the material. It arises from the fact that the material has some preferred magnetization direction, for example, if (1,0,0) is the preferred magnetization direction, $\phi(m) = m_2^2 + m_3^2$ for $|m| = 1$;

(iii) $E_{\mathrm{fi}}(m) = \int_{R^n} |\nabla\Phi|^2 dx$ is the energy of the stray field $\nabla\Phi$ induced by $m$. By the magnetostatics theory

$$-\Delta\Phi = \operatorname{div} m \quad \text{in } \mathfrak{D}'(R^n). \tag{1.3}$$

Equation (1.1) has been widely studied. In the case $\beta = 0, \alpha \neq 0$, (1.1) corresponds to the heat flow for harmonic maps studied in [2, 3]; if $\beta \neq 0, \alpha \neq 0$ (which implies strong damping in physics), the interested readers can refer to [2, 4–7] for mathematical theory; while in the conservative case, that is, $\beta \neq 0, \alpha = 0$, (1.1) corresponds to Schrödinger flow which represents conservation of angular momentum [8]. The numerical treatment to the problem can be found in [9, 10].

Recently, the study of the theory of ferromagnetism, especially the theory on thin film, is one of the focuses for both physicists and mathematicians. In the asymptotic regime which is readily accessible experimentally, DeSimone and Otto, and so forth, deduced a thin film micromagnetics model in which self-induced energy is the leading term of the free energy functional (see [11]). The physical consequences of the model are discussed further in [12]. The free energy functional is $E(m) = \int_{R^2} (|\xi \cdot \widehat{m\chi_\Omega}|^2 / |\xi|) d\xi$. We have $\delta E / \delta m = -\nabla(-\Delta)^{-1/2} \operatorname{div} m$ (see Section 2 for detailed computation) and the Landau-Lifshitz equation ($\beta = 0$) becomes,

$$\frac{\partial m}{\partial t} - \nabla(-\Delta)^{-1/2} \operatorname{div} m + \nabla(-\Delta)^{-1/2} \operatorname{div} m \cdot m m = 0, \tag{1.4}$$

in which $m = (m_1, m_2) : \mathbb{T}^2 \times [0, +\infty) \to R^2$ is in-plane component of the magnetization, $\mathbb{T}^2 = R^2 / (2\pi Z)^2$ is a flat torus. $u \cdot v$ is the inner product. To the best knowledge of ours, this is the first time a new model has been raised. Equation (1.4) is not easy to deal with because of lower order of differential operator with respect to $x$-variable and its strong nonlinear term. Inspired by physical prototype of the problem, we approximate it by a second-order equation,

$$\frac{\partial m^\varepsilon}{\partial t} = \varepsilon\Delta m^\varepsilon + \nabla(-\Delta)^{-1/2} \operatorname{div} m^\varepsilon + \varepsilon \left|\nabla m^\varepsilon\right|^2 m^\varepsilon - \nabla(-\Delta)^{-1/2} \operatorname{div} m^\varepsilon \cdot m^\varepsilon m^\varepsilon. \tag{1.5}$$

Equation (1.5) is the Landau-Lifshitz equation corresponding to the free energy $E(m) = \varepsilon\int_\Omega |\nabla m|^2 dx + \int_{R^2} (|\xi \cdot \widehat{m\chi_\Omega}|^2 / |\xi|) d\xi$, sum of exchange and self-induced energy. One difficulty in dealing with (1.5) lies in the nonconvex constraint $|m^\varepsilon| = 1$, which is overcame by considering a penalty approximation mimicking treatment of harmonic maps. To get existence of a unique mild solution of the penalized equation, we first give the formal solution of the corresponding linear equation, which requires special tricks and techniques. In the convergence process, a compensated compactness principle is applied.

The rest of this paper is organized as follows. Section 2 is devoted to studying (1.5). More precisely, we first study the penalized equation. In order to do this, we consider the corresponding linear equation and get its formal solution and well-posedness, then we get the existence of a unique mild solution of the penalized equation using semigroup theory. Second, we get the existence of a weak solution of (1.5) by passing to the limit in the penalized equation. The key point in the convergence process relies on a compensated compactness principle. In Section 3, we get existence of weak solution of (1.4) in Theorem 3.1 by passing to the limit in (1.5) as $\varepsilon \to 0$.

## 2. Approximation Equations

In this section, we always suppose that $\mathbb{T}^2 = R^2/(2\pi Z)^2$ is the flat torus. We prove existence of a weak solution of the following equations:

$$\frac{\partial m^\varepsilon}{\partial t} = \varepsilon\Delta m^\varepsilon + \nabla(-\Delta)^{-1/2}\mathrm{div}\, m^\varepsilon + \varepsilon\left|\nabla m^\varepsilon\right|^2 m^\varepsilon - \nabla(-\Delta)^{-1/2}\mathrm{div}\, m^\varepsilon \cdot m^\varepsilon m^\varepsilon, \quad \text{in } \mathbb{T}^2 \times (0,+\infty), \tag{2.1}$$

$$m^\varepsilon(x,0) = m_0(x), \quad \text{on } \mathbb{T}^2, \tag{2.2}$$

$$m^\varepsilon : \mathbb{T}^2 \times (0,\infty) \longrightarrow R^2, \quad \left|m^\varepsilon\right| = 1 \text{ a.e. in } \mathbb{T}^2. \tag{2.3}$$

Denote $Lm^\varepsilon = -\varepsilon\Delta m^\varepsilon - \nabla(-\Delta)^{-1/2}\mathrm{div}\, m^\varepsilon$. Note that the corresponding energy is $E(m) = \varepsilon\int_\Omega |\nabla m|^2 dx + \int_{R^2}(|\xi\cdot\widehat{m\chi_\Omega}|^2/|\xi|)d\xi$. The variation of the self-induced energy is

$$\begin{aligned}
\lim_{\eta\to 0}\int_{R^2}\frac{\left|\xi\cdot(\widehat{m\chi_\Omega+\eta v})\right|^2 - \left|\xi\cdot\widehat{m\chi_\Omega}\right|^2}{|\xi|\eta}d\xi &= \int_{R^2}\frac{2i\xi\cdot\widehat{m\chi_\Omega}}{|\xi|^{1/2}}\frac{\overline{i\xi\cdot\hat{v}}}{|\xi|^{1/2}}d\xi \\
&= 2\int_{R^2}\left((-\Delta)^{-1/4}\mathrm{div}\, m\chi_\Omega\right)\left((-\Delta)^{-1/4}\mathrm{div}\, v\right)dx \\
&= 2\int_{R^2}(-\Delta)^{-1/2}\mathrm{div}\, m\chi_\Omega\ \mathrm{div}\, v\, dx \\
&= 2\int_{R^2} - \nabla(-\Delta)^{-1/2}\mathrm{div}\, m\chi_\Omega\cdot v\, dx.
\end{aligned}$$

Equation (2.1) can be written as

$$\frac{\partial m^\varepsilon}{\partial t} = -Lm^\varepsilon + \left(Lm^\varepsilon\cdot m^\varepsilon\right)m^\varepsilon. \tag{2.4}$$

It is very easy to prove that (2.1) is equivalent to

$$m^\varepsilon \times \frac{\partial m^\varepsilon}{\partial t} + m^\varepsilon \times Lm^\varepsilon = 0. \tag{2.5}$$

The equivalence follows from the following.

**Lemma 2.1.** *In the classical sense, $m^\varepsilon$ is a solution of* (2.1)–(2.3) *if and only if $m^\varepsilon$ is a solution of* (2.5).

*Proof.* Suppose that $m^\varepsilon$ is a solution of (2.1)–(2.3). By the vector cross product formula

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c, \tag{2.6}$$

we have

$$\begin{aligned} \frac{\partial m^\varepsilon}{\partial t} &= -Lm^\varepsilon + \left(Lm^\varepsilon \cdot m^\varepsilon\right)m^\varepsilon \\ &= \left(Lm^\varepsilon \cdot m^\varepsilon\right)m^\varepsilon - \left(m^\varepsilon \cdot m^\varepsilon\right)Lm^\varepsilon \\ &= m^\varepsilon \times \left(m^\varepsilon \times Lm^\varepsilon\right). \end{aligned} \tag{2.7}$$

By the cross product of $m^\varepsilon$ and (2.7), we have

$$m^\varepsilon \times \frac{\partial m^\varepsilon}{\partial t} = m^\varepsilon \times \left(m^\varepsilon \times \left(m^\varepsilon \times Lm^\varepsilon\right)\right) = -m^\varepsilon \times Lm^\varepsilon. \tag{2.8}$$

This proves that $m^\varepsilon$ satisfies (2.5).

Suppose that $m^\varepsilon$ is a solution of (2.5). Then by the cross product of $m^\varepsilon$ and (2.5), we obtain

$$m^\varepsilon \times \left(m^\varepsilon \times \frac{\partial m^\varepsilon}{\partial t}\right) + m^\varepsilon \times \left(m^\varepsilon \times Lm^\varepsilon\right) = 0. \tag{2.9}$$

Since $|m^\varepsilon| = 1$, we have $m^\varepsilon \cdot (\partial m^\varepsilon/\partial t) = 0$. Hence (2.9) implies

$$\frac{\partial m^\varepsilon}{\partial t} = -Lm^\varepsilon + \left(Lm^\varepsilon \cdot m^\varepsilon\right)m^\varepsilon. \tag{2.10}$$

□

We define a local weak solution of (2.1) as follows.

*Definition 2.2.* A vector-valued function $m^\varepsilon(x,t)$ is said to be a local weak solution of (2.1), if $m^\varepsilon$ is defined a.e. in $\mathbb{T}^2 \times (0,T)$ such that

(1) $m^\varepsilon \in L^\infty(0,T;H^1(\mathbb{T}^2))$ and $\partial m^\varepsilon/\partial t \in L^2(\mathbb{T}^2 \times (0,T))$;

(2) $|m^\varepsilon(x,t)| = 1$ a.e. in $\mathbb{T}^2 \times (0,T)$;

(3) (2.1) holds in the sense of distribution;

(4) $m^\varepsilon(x,0) = m_0(x)$ in the trace sense.

We state our main result in this section as follows.

**Theorem 2.3.** *For every $m_0(x) \in H^1(\mathbb{T}^2)$ and $|m_0(x)| = 1$, a.e. in $\mathbb{T}^2$, there exists a weak solution of* (2.1)–(2.3).

To prove Theorem 2.3, we have to consider a penalized equation.

## *2.1. The Penalized Equation*

In the spirit of [13], we first construct weak solutions to a penalized problem, where the constraint $|m^\varepsilon| = 1$ is relaxed:

$$\frac{\partial m^k}{\partial t} + Lm^k - k^2\left(1 - \left|m^k\right|^2\right)m^k = 0, \quad \text{in } \mathbb{T}^2 \times (0, +\infty), \tag{2.11}$$

$$m^k(x,0) = m_0(x), \quad \text{on } \mathbb{T}^2, \tag{2.12}$$

$$\left|m_0(x)\right| = 1, \quad \text{on } \mathbb{T}^2. \tag{2.13}$$

Here $m^k : \mathbb{T}^2 \times (0,\infty) \to R^2$. In order to prove the existence of a mild solution of semilinear system (2.11)–(2.13), we consider the corresponding linear equation.

### *2.1.1. The Corresponding Linear Equation*

First, we consider the corresponding linear equation of (2.11)–(2.13) in the whole space:

$$\begin{gathered} \frac{\partial m}{\partial t} = \varepsilon \Delta m + \nabla(-\Delta)^{-1/2}\operatorname{div} m + k^2 m, \quad \text{in } R^2 \times (0, +\infty), \\ m(x,0) = m_0(x), \quad \text{on } R^2, \end{gathered} \tag{2.14}$$

where $m_0(x) = (m_{01}(x), m_{02}(x))$. While dealing with linear equation (2.14), we just write $m$ instead of $m^k$ unless there may be some confusion.

By Fourier transform in the $x$-variable, (2.14) are turned into

$$\begin{gathered} \widehat{m}_t + \varepsilon\left|\xi\right|^2\widehat{m} + \left(\frac{\xi \otimes \xi}{|\xi|}\right)\widehat{m} - k^2\widehat{m} = 0, \quad \text{in } R^2 \times (0, +\infty), \\ \widehat{m}(\xi, 0) = \widehat{m}_0(\xi), \quad \text{on } R^2. \end{gathered} \tag{2.15}$$

For each fixed $\xi$, the problem has a unique solution

$$\widehat{m}(\xi, t) = e^{-\mathcal{B}(\xi)t} \cdot e^{-\mathcal{A}(\xi)t}\widehat{m}_0(\xi), \tag{2.16}$$

where

$$\mathcal{A}(\xi) = \frac{1}{|\xi|}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix}, \qquad \mathcal{B}(\xi) = \begin{pmatrix} -k^2 + \varepsilon|\xi|^2 & 0 \\ 0 & -k^2 + \varepsilon|\xi|^2 \end{pmatrix}. \tag{2.17}$$

So the problem has the solution

$$m(x,t) = \frac{1}{4\pi^2}\left(e^{-\mathcal{B}(\xi)t}\right)^{\vee} * \left(e^{-\mathcal{A}(\xi)t}\right)^{\vee} * m_0(x). \tag{2.18}$$

Now the only problem left is to find the inverse Fourier transforms of $e^{-\mathcal{A}(\xi)t}$ and $e^{-\mathcal{B}(\xi)t}$. First, we need to find an orthogonal matrix $\mathcal{O}(\xi)$ such that $\mathcal{O}(\xi)\mathcal{A}(\xi)\mathcal{O}^{\tau}(\xi)$ is the Jordan normal form of $\mathcal{A}(\xi)$. In fact,

$$\mathcal{O}(\xi) = \frac{1}{|\xi|}\begin{pmatrix} \xi_2 & -\xi_1 \\ \xi_1 & \xi_2 \end{pmatrix}. \tag{2.19}$$

Now we begin to calculate the inverse Fourier transform of $e^{-\mathcal{A}(\xi)t}$

$$\begin{aligned}
\left(e^{-\mathcal{A}(\xi)t}\right)^{\vee} &= \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi} e^{-\mathcal{A}(\xi)t} d\xi \\
&= \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi} \mathcal{O}^{\tau}(\xi)\mathcal{O}(\xi) e^{-\mathcal{A}(\xi)t} \mathcal{O}^{\tau}(\xi)\mathcal{O}(\xi) d\xi \\
&= \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi} \mathcal{O}^{\tau}(\xi)\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}\left(\mathcal{O}(\xi)\mathcal{A}(\xi)\mathcal{O}^{\tau}(\xi)\right)^n\right)\mathcal{O}(\xi) d\xi \\
&= \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi} \mathcal{O}^{\tau}(\xi)\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}\begin{pmatrix} 0 & 0 \\ 0 & |\xi| \end{pmatrix}^n\right)\mathcal{O}(\xi) d\xi \\
&= \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi} \mathcal{O}^{\tau}(\xi)\left(I + \sum_{n=1}^{\infty}\frac{(-1)^n t^n}{n!}\begin{pmatrix} 0 & 0 \\ 0 & |\xi|^n \end{pmatrix}\right)\mathcal{O}(\xi) d\xi \\
&= \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi} \mathcal{O}^{\tau}(\xi)\left(I + \begin{pmatrix} 0 & 0 \\ 0 & e^{-|\xi|t}-1 \end{pmatrix}\right)\mathcal{O}(\xi) d\xi \\
&= \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi}\left(I + \frac{1}{|\xi|^2}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix}\left(e^{-|\xi|t}-1\right)\right) d\xi \\
&= \delta(x)I + \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi}\frac{1}{|\xi|^2}\left(\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix}\left(e^{-|\xi|t}-1\right)\right) d\xi.
\end{aligned} \tag{2.20}$$

Denote

$$R_{ij}(x,t) = \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi}\left(e^{-|\xi|t}-1\right)\frac{\xi_i\xi_j}{|\xi|^2} d\xi. \tag{2.21}$$

By the property of the Fourier transform, we have

$$R_{ij}(x,t) = -\partial x_i \partial x_j \left\{ \frac{1}{2\pi} \int_{R^2} e^{ix\cdot\xi} \left(e^{-|\xi|t} - 1\right) \frac{1}{|\xi|^2} d\xi \right\}. \tag{2.22}$$

Denote $(1/2\pi)\int_{R^2} e^{ix\cdot\xi}(e^{-|\xi|t}-1)(1/|\xi|^2)d\xi$ by $I(x,t)$. Obviously, we have

$$\begin{gathered} I(x,0) = 0, \qquad \frac{\partial I(x,t)}{\partial t} = -\frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi} e^{-|\xi|t} \frac{1}{|\xi|} d\xi, \\ \left.\frac{\partial I(x,t)}{\partial t}\right|_{t=0} = -\frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi} \frac{1}{|\xi|} d\xi, \qquad \frac{\partial^2 I(x,t)}{\partial t^2} = \frac{1}{2\pi}\int_{R^2} e^{ix\cdot\xi} e^{-|\xi|t} d\xi. \end{gathered} \tag{2.23}$$

By [14, page 15-16], we know that

$$I''(t) = \frac{t}{\left(t^2 + |x|^2\right)^{3/2}}. \tag{2.24}$$

In harmonic analysis, (2.24) is known as Poisson kernel.

Also by [14, page 107], we have $I'(0) = -1/|x|$.

Hence

$$\begin{aligned} \frac{\partial I(x,t)}{\partial t} &= \int_0^t \frac{\tau}{\left(\tau^2 + |x|^2\right)^{3/2}} d\tau + I'(0) \\ &= -\left(t^2 + |x|^2\right)^{-1/2} + |x|^{-1} - |x|^{-1} \\ &= -\left(t^2 + |x|^2\right)^{-1/2}. \end{aligned} \tag{2.25}$$

Therefore,

$$I(x,t) = \int_0^t \frac{\partial I(x,\tau)}{\partial \tau} d\tau = \ln|x| - \ln\left|t + \sqrt{|x|^2 + t^2}\right|. \tag{2.26}$$

We continue to compute other terms,

$$\begin{gathered} \frac{\partial I(x,t)}{\partial x_i} = \frac{x_i}{|x|^2} - \frac{x_i}{t^2 + |x|^2 + t\sqrt{t^2 + |x|^2}}, \quad i = 1,2, \\ R_{ij}(x,t) = -\frac{\partial^2 I(x,t)}{\partial x_i \partial x_j} = \frac{2x_i x_j}{|x|^4} - \frac{2x_i x_j + t\left(t^2 + |x|^2\right)^{-1/2} x_i x_j}{\left(t^2 + |x|^2 + t\sqrt{t^2 + |x|^2}\right)^2}, \end{gathered} \tag{2.27}$$

in which $i, j = 1, 2,\ i \neq j$

$$
\begin{aligned}
R_{ii}(x,t) &= -\frac{\partial^2 I(x,t)}{\partial x_i^2} \\
&= \frac{2x_i^2}{|x|^4} - \frac{2x_i^2 + t\left(t^2+|x|^2\right)^{-1/2} x_i^2}{\left(t^2+|x|^2+t\sqrt{t^2+|x|^2}\right)^2} - \frac{1}{|x|^2} + \frac{1}{t^2+|x|^2+t\sqrt{t^2+|x|^2}}, \quad i = 1, 2.
\end{aligned}
\tag{2.28}
$$

Hence we obtain

$$
\left(e^{-\mathcal{A}(\xi)t}\right)^{\vee} = \begin{pmatrix} \delta(x) + R_{11} & R_{12} \\ R_{21} & \delta(x) + R_{22} \end{pmatrix}. \tag{2.29}
$$

By standard procedure, we can get

$$
\left(e^{-\mathcal{B}(\xi)t}\right)^{\vee} = \begin{pmatrix} W(x,t) & 0 \\ 0 & W(x,t) \end{pmatrix}, \tag{2.30}
$$

where $W(x,t) = (2\varepsilon t)^{-1} e^{-(|x|^2/4\varepsilon t)+k^2 t}$. Therefore,

$$
\begin{aligned}
\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} &= \frac{1}{4\pi^2} \begin{pmatrix} W + W{*}R_{11} & W{*}R_{12} \\ W{*}R_{21} & W + W{*}R_{22} \end{pmatrix} * \begin{pmatrix} m_{01}(x) \\ m_{02}(x) \end{pmatrix} \\
&= \frac{1}{4\pi^2} \begin{pmatrix} (W + W{*}R_{11}){*}m_{01}(x) + W{*}R_{12}{*}m_{02}(x) \\ W{*}R_{21}{*}m_{01}(x) + (W + W{*}R_{22}){*}m_{02}(x) \end{pmatrix}.
\end{aligned}
\tag{2.31}
$$

**Theorem 2.4.** *Suppose that $m_0(x) \in \left(L^2(R^2)\right)^2$, then there exists a solution $m(x,t) \in (C([0,T];$ $L^2(R^2)))^2$ of* (2.14) *and*

$$
\lim_{t \to 0} \left\| m(x,t) - m_0(x) \right\|_{L^2(R^2)} = 0. \tag{2.32}
$$

*Proof.* From (2.21) and (2.30), we know $\widehat{W}\widehat{R}_{ij} \in L^{\infty}(R^2) = \mathcal{M}_2^2(R^2)$, so $R_{ij}{*}W \in L_2^2(R^2)$ and $R_{ij}{*}W{*}m_0 \in L^2(R^2)$.$\mathcal{M}_2^2$ is a Hörmander space (see [14], page 49-50). Moreover,

$$
\begin{aligned}
&\int_{R^2} \left| R_{ij}{*}W{*}m_{0i} \right|^2 dx \\
&\quad = \int_{R^2} \left|\widehat{W}\right|^2 \left|\widehat{R}_{ij}\right|^2 \left|\widehat{m_{0i}}\right|^2 d\xi \\
&\quad = \int_{\{\xi \in R^2 \mid |\xi| \le A\}} \left|\widehat{W}\right|^2 \left|\widehat{R}_{ij}\right|^2 \left|\widehat{m_{0i}}\right|^2 d\xi + \int_{\{\xi \in R^2 \mid |\xi| > A\}} \left|\widehat{W}\right|^2 \left|\widehat{R}_{ij}\right|^2 \left|\widehat{m_{0i}}\right|^2 d\xi \\
&\quad = I + II.
\end{aligned}
\tag{2.33}
$$

Notice that $|\widehat{W}||\widehat{R}_{ij}| = |e^{-|\xi|^2 t}||(e^{-|\xi|t} - 1)\xi_i \xi_j / |\xi|^2| \le C$.

For any $\varepsilon > 0$, choosing $A$ large enough such that $\int_{\{\xi\in R^2 \| \xi| > A\}} |\widehat{m_{0i}}|^2 d\xi < (\varepsilon/2C)$, we have $II < \varepsilon/2$.

For above $\varepsilon$, there exists a $t_0 > 0$ such that $|\widehat{R}_{ij}| < (\sqrt{\varepsilon}/\|\widehat{W}\|_{L^\infty}\|\widehat{m_{0i}}\|_{L^2})$ as $t < t_0$ and $|\xi| \le A$. Hence $I < (\varepsilon/2)$, that is

$$\int_{R^2} \left|R_{ij} * W * m_{0i}\right|^2 dx \longrightarrow 0, \quad \text{as } t \longrightarrow 0. \tag{2.34}$$

By standard procedure (see [14]), we can prove that

$$\lim_{t\to 0} \left\| \frac{W(\cdot,t)}{4\pi^2} * m_{0i} - m_{0i} \right\|_{L^2} = 0. \tag{2.35}$$

Therefore the proof is completely finished. □

*Remark 2.5.* Consider

$$\begin{gathered} \frac{\partial m}{\partial t} = \varepsilon \Delta m + \nabla(-\Delta)^{-1/2} \operatorname{div} m + k^2 m, \quad \text{in } \mathbb{T}^2 \times (0, +\infty), \\ m(x,0) = m_0(x), \quad \text{on } \mathbb{T}^2, \end{gathered} \tag{2.36}$$

where $m_0(x + 2\pi\vec{n}) = m_0(x)$, $\forall \vec{n} \in Z^2$ and $\mathbb{T}^2$ is a flat torus $R^2/(2\pi Z)^2$. By extending the equations periodically with respect to variable $x$ to the whole space, and using Fourier transform, we obtain

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} (\widetilde{W} + \widetilde{W*R_{11}}) * m_{01}(x) + \widetilde{W*R_{12}} * m_{02}(x) \\ \widetilde{W*R_{21}} * m_{01}(x) + (\widetilde{W} + \widetilde{W*R_{22}}) * m_{02}(x) \end{pmatrix}, \tag{2.37}$$

in which

$$\widetilde{W}(x,t) = \sum_{\vec{n}\in Z^2} W(x + 2\pi\vec{n}, t), \qquad \widetilde{W*R_{ij}} = \sum_{\vec{n}\in Z^2} (W*R_{ij})(x + 2\pi\vec{n}, t), \quad i,j = 1,2. \tag{2.38}$$

### *2.1.2. Existence of a Unique Mild Solution of the Penalized Equation*

First, let us recall a classical theorem in the theory of semigroup.

**Theorem 2.6** (see [15]). *Let $N(u) : X \to X$ be locally Lipschitz continuous in $u$. If $L$ is the infinitesimal generator of a $C_0$ semigroup $S(t)$ on $X$, then for every $u_0(x) \in X$ there is a $T \le \infty$ such that the initial value problem*

$$\begin{gathered} \frac{\partial u}{\partial t} = Lu + N(u), \quad t \in [0, \infty), \\ u(x,0) = u_0(x), \end{gathered} \tag{2.39}$$

*has a unique mild solution $u$ on $[0,T)$. Moreover, if $T < \infty$, then $\lim_{t\to T}\|u(t)\| = \infty$.*

Applying Theorem 2.6 to (2.11)–(2.13), we get the following theorem.

**Theorem 2.7.** *For every $m_0 \in H^1(\mathbb{T}^2)$, there exists a unique mild solution $m^k$ of* (2.11)–(2.13).

*Proof.* Here $Lm^k = \varepsilon\Delta m^k + \nabla(-\Delta)^{-1/2}\operatorname{div} m^k + k^2 m^k, N(m^k) = -k^2|m^k|^2 m^k$. By Theorem 2.4 and Remark 2.5, we know that $L$ is the infinitesimal generator of a $C_0$ semigroup on $H^1(\mathbb{T}^2)$. Next, we want to check the inequality

$$\|N(u) - N(v)\|_{H^1} \le C\left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right)\|u - v\|_{H^1}. \tag{2.40}$$

Letting $B(u, v, w) = k^2 uvw$, we have

$$N(u) - N(v) = B(u - v, u, u) + B(v, u - v, u) + B(v, v, u - v). \tag{2.41}$$

So it is sufficient to prove

$$\|B(u, v, w)\|_{H^1} \le C\left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right)\|w\|_{H^1}. \tag{2.42}$$

This last result is an easy consequence of Sobolev embedding theorem. Therefore, Theorem 2.6 gives us the desired result. □

### ***2.2. Existence of Weak Solution of Approximate Equation*** (2.1)–(2.3)

In this section, we establish our main results about the approximate equations (2.1)–(2.3) by passing to the limit in the penalized equation (2.11) as $k \to \infty$.

*Proof of Theorem 2.3.* Multiplying (2.11) with $\partial m^k/\partial t$, and integrating over $\mathbb{T}^2 \times (0, T)$, we have

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_{\mathbb{T}^2}\left|\nabla m^k\right|^2 dx + \frac{k^2}{4}\int_{\mathbb{T}^2}\left(\left|m^k\right|^2 - 1\right)^2 dx + \int_0^T\int_{\mathbb{T}^2}\left|\frac{\partial m^k}{\partial t}\right|^2 dx\, dt \\
&\quad \le \int_{\mathbb{T}^2}\frac{\left|(-\Delta)^{-1/4}\operatorname{div} m_0\right|^2}{2}dx + \frac{\varepsilon}{2}\int_{\mathbb{T}^2}\left|\nabla m_0\right|^2 dx.
\end{aligned} \tag{2.43}$$

We now take the limit as $k$ goes to infinite: from (2.43), we deduce that

$$\begin{aligned}
&m^k \text{ is bounded in } L^\infty(0, T; H^1(\mathbb{T}^2)), \\
&\frac{\partial m^k}{\partial t} \text{ is bounded in } L^2(0, T; L^2(\mathbb{T}^2)), \\
&\left|m^k\right|^2 - 1 \longrightarrow 0, \quad \text{in } L^\infty(0, T; L^2(\mathbb{T}^2)).
\end{aligned} \tag{2.44}$$

Therefore, up to a subsequence, we have

$$m^k \rightharpoonup m^\varepsilon \quad \text{in } L^\infty(0,T;H^1(\mathbb{T}^2)) \text{ weak}^*, \tag{2.45}$$

$$\frac{\partial m^k}{\partial t} \rightharpoonup \frac{\partial m^\varepsilon}{\partial t} \quad \text{in } L^2(0,T;L^2(\mathbb{T}^2)) \text{ weakly}, \tag{2.46}$$

$$m^k \longrightarrow m^\varepsilon \quad \text{in } L^2(0,T;L^2(\mathbb{T}^2)) \text{ strongly}, \tag{2.47}$$

$$|m^k|^2 - 1 \longrightarrow 0 \quad \text{in } L^\infty(0,T;L^2(\mathbb{T}^2)) \text{ strongly and a.e. in } \mathbb{T}^2 \times (0,T), \tag{2.48}$$

and $|m^\varepsilon| = 1$ a.e. in $\mathbb{T}^2 \times (0,T)$.

In order to pass to the limit in (2.11), let $\Phi$ be in $(C^\infty(\mathbb{T}^2 \times (0,T)))^3$, and let the test function $\varphi = m^k \times \Phi$, there holds

$$\int_0^T\int_{\mathbb{T}^2}\left(m^k \times \frac{\partial m^k}{\partial t}\right)\cdot \Phi dx\, dt + \int_0^T\int_{\mathbb{T}^2}(m^k \times Lm^k)\cdot\Phi dx\, dt = 0. \tag{2.49}$$

From (2.45), (2.46), and (2.47), as $k$ goes to infinite, we have

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^2} m^k \times \frac{\partial m^k}{\partial t}\cdot\Phi dx\, dt \longrightarrow \int_0^T\int_{\mathbb{T}^2} m^\varepsilon \times \frac{\partial m^\varepsilon}{\partial t}\cdot\Phi dx\, dt,\\
&\int_0^T\int_{\mathbb{T}^2} m^k \times \nabla(-\Delta)^{-1/2}\operatorname{div} m^k\cdot\Phi dx\, dt \longrightarrow \int_0^T\int_{\mathbb{T}^2} m^\varepsilon \times \nabla(-\Delta)^{-1/2}\operatorname{div} m^\varepsilon\cdot\Phi dx\, dt,\\
&\int_0^T\int_{\mathbb{T}^2} m^k \times \varepsilon\Delta m^k\cdot\Phi dx\, dt = -\int_0^T\int_{\mathbb{T}^2} m^k \times \varepsilon\nabla m^k\cdot\nabla\Phi dx\, dt\\
&\longrightarrow -\int_0^T\int_{\mathbb{T}^2} m^\varepsilon \times \varepsilon\nabla m^\varepsilon\cdot\nabla\Phi dx\, dt.
\end{aligned} \tag{2.50}$$

Namely, (2.49) is convergent to

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^2} m^\varepsilon \times \frac{\partial m^\varepsilon}{\partial t}\cdot\Phi dx\, dt + \int_0^T\int_{\mathbb{T}^2} m^\varepsilon \times (-\nabla(-\Delta)^{-1/2}\operatorname{div} m^\varepsilon)\cdot\Phi dx\, dt\\
&\quad - \int_0^T\int_{\mathbb{T}^2} m^\varepsilon \times \varepsilon\nabla m^\varepsilon\cdot\nabla\Phi dx\, dt = 0.
\end{aligned} \tag{2.51}$$

Hence by Lemma 2.1, we know that (2.1)–(2.3) has a weak solution. □

*Remark 2.8.* From (2.43) and Theorem 2.6, we know that the unique mild solution of the penalized equation (2.11) globally exists.

## 3. Existence of Weak Solution of (1.4)

From above section, we know that for each fixed $\varepsilon > 0$, (2.1)–(2.3) admit weak solutions $m^\varepsilon \in L^\infty(0,T;H^1(\mathbb{T}^2))$. In this section, we will prove that there exists a subsequence of $m^\varepsilon$

(still denoted by $m^\varepsilon$) strongly converging to $m$ in $L^2(0,T;L^2(\mathbb{T}^2))$, which is the weak solution of (1.4). More precisely, we state our main result of this section in the following theorem.

**Theorem 3.1.** *Suppose that* $m_0(x) \in H^1(\mathbb{T}^2), |m_0(x)| = 1$, *a.e. in* $\mathbb{T}^2$, *and* $\operatorname{div} m_0 = 0$, *there exists a weak solution* $m(x,t) \in L^\infty(0,T;H^1(\mathbb{T}^2))$ *and* $(\partial m/\partial t) \in L^2(0,T;L^2(\mathbb{T}^2))$ *of* (1.4).

*Proof.* Form (2.43), we have

$$\frac{\varepsilon}{2}\int_\Omega |\nabla m^k|^2 dx + \int_0^T\int_\Omega \left|\frac{\partial m^k}{\partial t}\right|^2 dx\,dt \le \frac{\varepsilon}{2}\int_\Omega |\nabla m_0|^2 dx. \tag{3.1}$$

Passing to the limit as $k \to \infty$ and taking (2.45), (2.46) into consideration, we have

$$\frac{\varepsilon}{2}\int_\Omega |\nabla m^\varepsilon|^2 dx + \int_0^T\int_\Omega \left|\frac{\partial m^\varepsilon}{\partial t}\right|^2 dx\,dt \le \frac{\varepsilon}{2}\int_\Omega |\nabla m_0|^2 dx. \tag{3.2}$$

So we conclude that $m^\varepsilon$ is bounded in $L^\infty(0,T;H^1(\Omega))$, and $\partial m^\varepsilon/\partial t$ is bounded in $L^2(0,T;L^2(\Omega))$.

Therefore, up to subsequence,

$$\begin{gathered} m^\varepsilon \rightharpoonup m \quad \text{in } L^\infty\left(0,T;H^1\left(\mathbb{T}^2\right)\right) \text{ weak}^*, \\ \frac{\partial m^\varepsilon}{\partial t} \rightharpoonup \frac{\partial m}{\partial t} \quad \text{in } L^2\left(0,T;L^2\left(\mathbb{T}^2\right)\right) \text{ weakly}. \end{gathered} \tag{3.3}$$

By [16, Chapter 1, Theorem 5.1, pages 56–60], we know that

$$m^\varepsilon \longrightarrow m \text{ strongly} \quad \text{in } L^2\left(\mathbb{T}^2 \times (0,T)\right), \quad \text{a.e. in } \mathbb{T}^2 \times (0,T). \tag{3.4}$$

Passing to the limit as $\varepsilon$ goes to zero in (2.51), we have,

$$\begin{gathered} \int_0^T\int_{\mathbb{T}^2} m^\varepsilon \times \frac{\partial m^\varepsilon}{\partial t}\cdot \Phi dx\,dt \longrightarrow \int_0^T\int_{\mathbb{T}^2} m \times \frac{\partial m}{\partial t}\cdot \Phi dx\,dt, \\ \int_0^T\int_{\mathbb{T}^2} m^\varepsilon \times \nabla(-\Delta)^{-1/2}\operatorname{div} m^\varepsilon \cdot \Phi dx\,dt \longrightarrow \int_0^T\int_{\mathbb{T}^2} m \times \nabla(-\Delta)^{-1/2}\operatorname{div} m \cdot \Phi dx\,dt, \\ \int_0^T\int_{\mathbb{T}^2} m^\varepsilon \times \varepsilon\nabla m^\varepsilon \cdot \nabla\Phi dx\,dt \longrightarrow 0. \end{gathered} \tag{3.5}$$

That is to say, $m$ is the weak solution of

$$m \times \frac{\partial m}{\partial t} - m \times \nabla(-\Delta)^{-1/2}\operatorname{div} m = 0. \tag{3.6}$$

By an argument analogous to Lemma 2.1, (3.6) is equivalent to (1.4). □

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